A HALPERN-LIONS-REICH-LIKE ITERATIVE METHOD FOR NONEXPANSIVE MAPPINGS

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Abstract. We prove strong convergence of a Halpern-Lions-Reich-like iterative algorithm for approximating fixed points of nonexpansive mappings in a uniformly smooth Banach space. The idea of this algorithm is then applied to solve a quadratic minimization problem in a Hilbert space.

Key Words and Phrases: Halpern-Lions-Reich-like iterative algorithm, nonexpansive mapping, fixed point, uniformly smooth Banach space, quadratic minimization problem.

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1. Introduction

In 1965, Halpern [4] invented an iterative algorithm for finding a fixed point of a nonexpansive mapping in the framework of Hilbert spaces. To state Halpern’s algorithm, recall that a self-mapping of a closed convex subset $C$ of a real Banach space $H$ is nonexpansive if

\[ \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C. \]  

(1.1)

The set of fixed points of $T$ is denoted $Fix(T)$ and suppose that $Fix(T) \neq \emptyset$.

Halpern’s algorithm [4] then generates a sequence $\{x_n\}$ by the recursive process:

\[ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0 \]  

(1.2)

where $u \in C$ is called an anchor, $x_0 \in C$ is an initial guess, and $\{\alpha_n\} \subset (0, 1)$ is a sequence of iteration parameters.

Halpern called a sequence $\{\alpha_n\} \subset (0, 1)$ acceptable if the sequence $\{x_n\}$ generated by (1.2) always converges in norm to a fixed point of $T$ irrespective of the choice of Hilbert space $H$, closed convex subset $C$ of $H$, nonexpansive mapping $T : C \to C$ such that $Fix(T) \neq \emptyset$, anchor $u \in C$, and starting point $x_0 \in C$. He proved that the following conditions (H1) and (H2) are necessary for $\{\alpha_n\}$ to be acceptable:

(H1) $\lim_{n \to \infty} \alpha_n = 0$.

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(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Halpern [4] also proved that the conditions (H1), (H2) and (H3) are sufficient for \( \{\alpha_n\} \) to be acceptable, where

(H3) there is a strictly increasing sequence of positive integers, \( \{n_j\} \), such that

$$\left\{ \begin{array}{l} \frac{\alpha_{j+n_j}}{\alpha_j} \to 1, \quad \text{as } j \to \infty, \\ n_j\alpha_j \to \infty, \quad \text{as } j \to \infty. \end{array} \right. \quad (1.3)$$

He observed that $\alpha_n = (n+1)^{-\alpha}$ for all $n$, where $0 < \alpha < 1$, satisfies (H1), (H2) and (H3), hence acceptable.

In 1977, Lions [5] proved that the conditions (H1), (H2) and (L1) are sufficient for \( \{\alpha_n\} \) to be acceptable, where

(L1) $\lim_{n \to \infty} |\alpha_n+1 - \alpha_n|/\alpha_{n+1}^2 = 0$.

Note that Lions [5] is the first to extend the algorithm (1.2) to find a common fixed point of a finite family of (firmly) nonexpansive mappings.

Many researchers made contributions to the Halpern-Lions algorithm (1.2) by finding a third condition which, together with (H1) and (H2), is sufficient for \( \{\alpha_n\} \) to be acceptable; each of the following conditions is such a third condition:

(W1) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ (Wittmann [15]),

(R1) $\{\alpha_n\}$ is decreasing (Reich [9]),

(X1) $\lim_{n \to \infty} |\alpha_{n+1} - \alpha_n|/\alpha_{n+1} = 0$ or equivalently, $\lim_{n \to \infty} (\alpha_n/\alpha_{n+1}) = 1$ (Xu [16, 17]).

A question gives rise to whether or not the conditions (H1) and (H2) are sufficient for \( \{\alpha_n\} \) to be acceptable. This question was answered negatively by Suzuki [13]. However, it is still an open question: What conditions are necessary and sufficient for \( \{\alpha_n\} \) to be acceptable. If we narrow the class of nonexpansive mappings down to the class of so-called averaged nonexpansive mappings, then the conditions (H1) and (H2) are not only necessary but sufficient for \( \{\alpha_n\} \) to be acceptable. Recall that a mapping $T : C \to C$ is said to be averaged nonexpansive if $T = (1-\lambda)I + \lambda V$, where $\lambda \in (0, 1)$ and $V : C \to C$ is nonexpansive.

On the other hand, it is interesting to extend the algorithm (1.2) to the setting of Banach spaces. In this regard, Reich [8] was the first to prove that the sequence \( \{x_n\} \) generated by the algorithm (1.2) in a uniformly smooth Banach space with the choice of parameters $\alpha_n = (1+n)^{-\alpha}$ for all $n$, where $0 < \alpha < 1$, converges in norm to a fixed point of $T$. Due to this reason, we will refer the algorithm (1.2) to as the Halpern-Lions-Reich algorithm throughout the rest of this paper.

While searching new iterative algorithms, Yao, et al [20] introduced an iterative algorithm that generates a sequence \( \{x_n\} \) through the recursion:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Tx_n, \quad n \geq 0, \quad (1.4)$$

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n$. We shall call it a Halpern-Lions-Reich-like algorithm. Yao, et al [20] proved that if, in addition, there hold the conditions:

(i) $\alpha_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\gamma_n \to 0$ as $n \to \infty$, 

then $\{x_n\}$ converges weakly to a common fixed point of T.
then the sequence \( \{x_n\} \) generated by (1.4) converges in norm to a fixed point of \( T \).

Nevertheless, it is recently pointed out in [10] that Yao, et al’s result above is false, that is, the conditions (i) and (ii) are insufficient to guarantee the strong convergence of the sequence \( \{x_n\} \). It is proved in [10] that if, in addition to the condition (i), there hold the conditions:

(iii) \( \beta_n \to 0 \) as \( n \to \infty \),
(iv) \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \),
(v) \( \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty \),

then the sequence \( \{x_n\} \) generated by (1.4) does converge in norm to a fixed point of \( T \).

It is of interest to investigate the strong convergence of the Halpern-Lions-Reich-like algorithm (1.4) under appropriate conditions to be imposed on the sequences \( \{\alpha_n\} \), \( \{\beta_n\} \), \( \{\gamma_n\} \). The purpose of this paper is twofold. First, we will prove a strong convergence theorem for the Halpern-Lions-Reich-like algorithm (1.4) under different conditions from those of Sangago [10]. Secondly, we will apply our convergence result to solve a quadratic minimization problem.

2. Preliminaries

Let \( X \) be a real uniformly smooth Banach space and \( C \) a closed convex subset of \( X \). Let \( J : X \to X^* \) be the (normalized) duality map defined by

\[
J(x) \in X^*, \quad \|J(x)\| = \|x\|, \quad \langle x, J(x) \rangle = \|x\|^2.
\]

Note that the uniform smoothness of \( X \) implies that \( J \) is uniformly continuous on bounded sets in the norm-to-norm topology.

Let \( T : C \to C \) be a nonexpansive mapping such that \( \text{Fix}(T) \neq \emptyset \). For each fixed anchor \( u \in C \) and \( t \in (0, 1) \), let \( x_t \in C \) be the unique fixed point of the contraction

\[
T_t x := tu + (1-t)Tx, \quad x \in C.
\] (2.1)

The following theorem is known, the Hilbert space counterpart of which is proved by Browder [1].

**Theorem 2.1** [8] If \( X \) is a uniformly smooth Banach space, then \( \{x_t\} \) converges in norm, as \( t \to 0 \), to a fixed point of \( T \); moreover, the operator \( Q : C \to \text{Fix}(T) \) defined by

\[
Q(u) := \| \cdot \| - \lim_{t \to 0} x_t, \quad u \in C
\] (2.2)

defines the unique sunny nonexpansive retraction from \( C \) onto \( \text{Fix}(T) \); that is, \( Q \) satisfies the properties:

(i) \( \langle Qu - u, J(p - u) \rangle \geq 0, \quad u \in C, \quad p \in \text{Fix}(T) \).
(ii) \( \|Qu - Qv\|^2 \leq \langle u - v, J(Qu - Qv) \rangle, \quad u, v \in C \).

To prove our main result in the next section, we need the following two lemmas.

**Lemma 2.2** [16] Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 0,
\]

where \( \{\gamma_n\} \) is a sequence in \((0,1)\) and \( \{\delta_n\} \) is a sequence in \( \mathbb{R} \) such that
Lemma 2.3 [12] Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( X \) such that
\[
x_{n+1} = \gamma_n x_n + (1 - \gamma_n) y_n, \quad n \geq 0 \tag{2.3}
\]
where \( \{\gamma_n\} \) is a sequence in \([0,1]\) such that
\[
0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1.
\]
Assume
\[
\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \tag{2.4}
\]
Then \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).

The following lemma is straightforward, but convenient in use.

Lemma 2.4 In a real smooth Banach space, there holds the inequality for all \( x, y \):
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle.
\]

3. Convergence of a Halpern-Lions-Reich-like Algorithm

Recall that our Halpern-Lions-Reich-like algorithm generate a sequence \( \{x_n\} \) through the recursion:
\[
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0, \tag{3.1}
\]
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \([0,1]\) such that \( \alpha_n + \beta_n + \gamma_n = 1 \) for all \( n \).

Yao, et al [20] claimed that the conditions
(a) \( \alpha_n \to 0 \) as \( n \to \infty \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \),
(b) \( \gamma_n \to 0 \) as \( n \to \infty \),
were sufficient to guarantee the strong convergence of the sequence \( \{x_n\} \) generated by (1.4). But the fact is that their conclusion is incorrect, as the counterexamples of Sangago [10] showed. Sangago [10] did not figure out the cause of the incorrectness in the proof given in Yao, et al [20]. So let us briefly review the main points of the proof of Yao, et al [20]. Let \( t \in (0,1) \) and \( n \geq 1 \) be given and let \( z_{t,n} \) be the unique fixed point of the contraction
\[
T_{t,n} z := \frac{(1 - \alpha_n) t}{\gamma_n + t \beta_n} u + \frac{(1 - t) \gamma_n}{\gamma_n + t \beta_n} T z, \quad z \in C. \tag{3.2}
\]
Then one has that
\[
\lim_{t \to 0} z_{t,n} = p \in Fix(T), \quad n \geq 1. \tag{3.3}
\]
Indeed, \( p = Qu \), where \( Q : C \to Fix(T) \) is the unique sunny nonexpansive retraction from \( C \) onto \( Fix(T) \) as defined in Theorem 2.1.

The key step of the proof of Yao, et al [20] is the following inequality
\[
\limsup_{n \to \infty} \langle u - p, J(x_n - p) \rangle \leq 0. \tag{3.4}
\]
Moreover, the relation (3.4) fails to hold; indeed, we have
\[ \lim_{t \to 0} \lim_{n \to \infty} \langle u - z_{t,n}, J(x_n - z_{t,n}) \rangle \leq 0 \]  
\[ (3.5) \]
by using the uniform smoothness of the space \( X \) (equivalently, the norm-to-norm uniform continuity over bounded sets of the normalized duality map \( J \)). This however requires that the limit in (3.3) be uniformly over \( n \geq 1 \), which fails to be true, in general, under the conditions (a) and (b) of Yao, et al [20]. To illustrate this, we use the counterexample in [10].

**Example 3.1** [10] Take \( X = \mathbb{R} \) to be the real line equipped with the absolute value as norm, \( C = [-1,1] \), and \( T : C \to C \) to be the reflection: \( Tx = -x \) for \( x \in C \). Then \( T \) is nonexpansive and \( x = 0 \) is the unique fixed point of \( T \). Furthermore, take \( u = 1 \) and \( x_0 = \frac{1}{2} \), and take \( \alpha_n = \gamma_n \in (0, \frac{1}{3}) \) for all \( n \) so that \( \beta_n = 1 - 2\alpha_n \in (\frac{2}{3},1) \). It is then easily seen that the sequence \( \{x_n\} \) generated by the algorithm (3.1) is a constant:
\[ x_n = \frac{1}{3}, \quad n \geq 1. \]
Hence, the sequence \( \{x_n\} \) fails to converge to a fixed point of \( T \).

In this case, it is not hard to find that the unique fixed point \( z_{t,n} \) of the contraction \( T_{t,n} \) defined in (3.2) is given by
\[ z_{t,n} = \frac{(1 - \alpha_n)t}{2\alpha_n + t(\beta_n - \alpha_n)}. \]
It is immediately clear that
\[ \lim_{t \to 0} z_{t,n} = 0 \in \text{Fix}(T), \quad \lim_{n \to \infty} z_{t,n} = 1 \notin \text{Fix}(T). \]
This shows that the limit in (3.3) fails to be uniform over \( n \geq 1 \), and consequently, the order of the iterated limits in (3.5) cannot be interchanged. As a matter of fact, we have
\[ \lim_{n \to \infty} \lim_{t \to 0} \lim_{n \to \infty} \langle u - z_{t,n}, J(x_n - z_{t,n}) \rangle = \lim_{n \to \infty} \lim_{t \to 0} \lim_{n \to \infty} (1 - z_{t,n})(\frac{1}{3} - z_{t,n}) = 0, \]
\[ \lim_{n \to \infty} \lim_{t \to 0} \lim_{n \to \infty} \langle u - z_{t,n}, J(x_n - z_{t,n}) \rangle = \lim_{n \to \infty} \lim_{t \to 0} \lim_{n \to \infty} (\frac{1}{3} - z_{t,n}) = \frac{1}{3}. \]
Moreover, the relation (3.4) fails to hold; indeed, we have
\[ \lim_{n \to \infty} \lim_{n \to \infty} \langle u - p, J(x_n - p) \rangle = \lim_{n \to \infty} \sup_{n \to \infty} x_n = \frac{1}{3}. \]

We will provide with a new selection of the sequences \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\gamma_n\} \) by avoiding usage of the sequence \( \{z_{t,n}\} \). Below is our convergence result on the algorithm (3.1).

**Theorem 3.2** Let \( X \) be a real uniformly smooth Banach space, \( C \) a closed convex subset of \( X \), and \( T : C \to C \) a nonexpansive mapping such that \( \text{Fix}(T) \neq \emptyset \). Let \( \{x_n\} \) be generated by the Halpern-Lions-Reich-like algorithm (3.1). Assume that the sequences \( \{\alpha_n\} \), \( \{\beta_n\} \), and \( \{\gamma_n\} \) satisfy the following conditions:

(i) \( \alpha_n, \beta_n, \gamma_n \in (0,1) \) are such that \( \alpha_n + \beta_n + \gamma_n = 1 \) for all \( n \geq 0 \).

(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \),
(iii) \( \lim_{n \to \infty} \beta_n = \beta \in (0, 1) \).

Then \( (x_n) \) converges in norm to \( Qu \), where \( Q \) is the sunny nonexpansive retraction from \( C \) onto \( Fix(T) \) defined by (2.2).

**Proof.** 1. \( \{x_n\} \) is bounded. Indeed, take a fixed point \( p \) of \( T \) to get

\[
\|x_{n+1} - p\| = \|\alpha_n(u - p) + \beta_n(x_n - p) + \gamma_n(Tx_n - p)\|
\leq \alpha_n\|u - p\| + (\beta_n + \gamma_n)\|x_n - p\|
\leq \max\{\|u - p\|, \|x_n - p\|\}.
\]

So an induction gives

\[
\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\}, \quad n \geq 0.
\]

2. \( \|x_n - Tx_n\| \to 0 \). To see this we put

\[
y_n = \alpha_n u + \gamma_n T x_n
\]

so that we have

\[
x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n.
\]

It is not hard to find that

\[
y_{n+1} - y_n = \frac{\alpha_{n+1}u + \gamma_{n+1}Tx_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n T x_n}{1 - \beta_n}
\]

\[
= \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(Tx_{n+1} - Tx_n)
\]

\[
+ \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) u + \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) Tx_n.
\]

It turns out that

\[
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - 1 \right| M + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| M
\]

\[
+ \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| M,
\]

where \( M > 0 \) is selected so that \( M \geq \max\{\|u\|, 2\|x_j\|, \|Tx_j\|\} \) for all \( j \). Since \( \alpha_n \to 0 \) and \( \beta_n \to \beta \in (0, 1) \), \( \gamma_n \to 1 - \beta \in (0, 1) \) and we get

\[
\limsup_{n \to \infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

Due to (3.8) together with the assumption \( \beta_n \to \beta \in (0, 1) \), we can apply Lemma 2.3 to the relation (3.7) to get

\[
\lim_{n \to \infty} \|y_n - x_n\| = 0.
\]

Noticing also from (3.6)

\[
\|y_n - Tx_n\| = \frac{\alpha_n}{1 - \beta_n} \|u - T x_n\| \leq \frac{2\alpha_n M}{1 - \beta_n} \to 0,
\]

we obtain

\[
\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Tx_n\| \to 0.
\]
In particular, since $z_t \in C$, it turns out that:

$$\langle u - q, J(x_n - q) \rangle \leq 0, \text{ with } q = Qu.$$ 

Let $t \in (0, 1)$ and let $z_t \in C$ solves the fixed point equation:

$$z_t = tu + (1 - t)Tz_t.$$ 

Then $q = \lim_{t \to 0} z_t$ in the norm topology.

We have

$$\|z_t - x_n\|^2 = \|(1 - t)(Tz_t - x_n) + t(u - x_n)\|^2$$

\begin{align*}
\leq & (1 - t)^2 \|Tz_t - x_n\|^2 + 2t\|u - x_n, J(z_t - x_n)\| \\
\leq & (1 - t)^2 (\|Tz_t - Tx\| + \|Tx_n - x_n\|)^2 \\
& + 2t\|(u - z_t, J(z_t - x_n))\| + \|z_t - x\|^2 \\
\leq & (1 + t^2)\|z_t - x_n\|^2 + M\|Tx_n - x_n\| \\
& + 2t\|u - z_t, J(z_t - x_n)\|.
\end{align*}

Here $M$ is such that

$$M \geq 2\|z_t - x_n\| + \|Tx_n - x_n\| \text{ for all } n \text{ and } t \in (0, 1).$$

It turns out that

$$\langle u - z_t, J(x_n - z_t) \rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{M}{2t}\|Tx_n - x_n\|. \quad (3.9)$$

Let $K$ be a bounded set such that

$$\{x_n - z_t, x_n - q, z_t - u\} \subset K \text{ for all } n \text{ and } t \in (0, 1)$$

and let $d := \sup\{\|u\| : u \in K\} < \infty$. Since the duality map $J$ is uniformly continuous in the norm topology, there exists, given $\varepsilon > 0$, an $\delta > 0$ (we assume also that $\delta < \varepsilon$) such that

$$u, v \in K, \quad \|u - v\| < \delta \Rightarrow \|J(u) - J(v)\| < \varepsilon.$$ 

In particular, since $z_t \to q$ in norm, there exists $t_0 > 0$ small enough so that

$$\|z_t - q\| < \delta < \varepsilon \text{ for all } 0 < t < t_0.$$ \quad (3.10)

It turns out that

$$\|J(x_n - z_t) - J(x_n - q)\| < \varepsilon \text{ for all } n \text{ and } 0 < t < t_0.$$ \quad (3.11)

It follows from (3.9)-(3.11) that, for all $n$ and $0 < t < t_0$,

$$\langle u - q, J(x_n - q) \rangle = \langle u - z_t, J(x_n - z_t) \rangle + \langle z_t - q, J(x_n - q) \rangle$$

$$+ \langle u - z_t, J(x_n - q) - J(x_n - z_t) \rangle$$

$$\leq \langle u - z_t, J(x_n - z_t) \rangle + 2\varepsilon$$

$$\leq \frac{td^2}{2} + \frac{M}{2t}\|Tx_n - x_n\| + 2d\varepsilon.$$ 

Consequently, for $0 < t < t_0$,

$$\lim_{n \to \infty} \sup\{u - q, J(x_n - q)\} \leq \frac{td^2}{2} + 2d\varepsilon. \quad (3.12)$$
Letting $t \to 0$ in (3.12) yields immediately that
\begin{equation}
\limsup_{n \to \infty} (u - q, J(x_n - q)) \leq 0. \tag{3.13}
\end{equation}

4. $x_n \to q$ in norm. Applying Lemma 2.4, we get
\begin{align*}
\|x_{n+1} - q\|^2 & \leq \|\beta_n(x_n - q) + \gamma_n(Tx_n - q) + \alpha_n(u - q)\|^2 \\
& \leq \|\beta_n(x_n - q) + \gamma_n(Tx_n - q)\|^2 + 2\alpha_n(u - q, J(x_n + 1 - q)) \\
& \leq (\beta_n\|x_n - q\| + \gamma_n\|x_n - q\|)^2 + 2\alpha_n(u - q, J(x_n + 1 - q)) \\
& \leq (1 - \alpha_n)\|x_n - q\|^2 + 2\alpha_n(u - q, J(x_n + 1 - q)). \tag{3.14}
\end{align*}
By applying Lemma 2.2 to (3.14) we conclude that $\|x_n - q\|^2 \to 0$, as required. \hfill \Box

4. A Quadratic Minimization Problem

Consider the quadratic minimization problem in a real Hilbert space $H$:
\begin{equation}
\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle \tag{4.1}
\end{equation}
where $C$ is the fixed point set $Fix(T)$ of a nonexpansive mapping $T$ on $H$ and $u$ is a given point in $H$. Assume $Fix(T)$ is nonempty. Assume also $A$ is strongly positive; that is, there is a constant $\gamma > 0$ with the property
\begin{equation}
\langle Ax, x \rangle \geq \gamma\|x\|^2 \text{ for all } x \in H. \tag{4.2}
\end{equation}
Then the minimization (4.1) has a unique solution $x^* \in C$ which satisfies the optimality condition
\begin{equation}
\langle Ax^* - u, x - x^* \rangle \geq 0, \quad x \in C. \tag{4.3}
\end{equation}
In [19, 7] it is proved that the sequence $\{x_n\}$ generated by the algorithm
\begin{equation}
x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0 \tag{4.4}
\end{equation}
converges in norm to the solution $x^*$ of (4.1) provided the sequence $\{\alpha_n\}$ in (0,1) satisfies the conditions (H1) and (H2), and additionally, either condition (W1) or (X1) stated in Section 1. Below following the idea presented in Section 3, we will demonstrate a new algorithm that generates a sequence strongly converging to the solution $x^*$ of (4.1) under the conditions (H1) and (H2) only. Given an anchor $u \in H$ and a starting point $x_0 \in H$. Let $\{\alpha_n\} \subset (0,1)$ be given. Let a sequence $\{\beta_n\}$ be also given in (0,1) such that $\beta \leq \beta_n \leq \overline{\beta}$ for all $n$ and some $0 < \beta \leq \overline{\beta} < 1$. Define a sequence $\{x_n\}$ by the algorithm
\begin{equation}
x_{n+1} = (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)Tx_n) + \alpha_n u, \quad n \geq 0. \tag{4.5}
\end{equation}

**Lemma 4.1** [7] Assume $A$ is a strongly positive linear bounded operator on a real Hilbert space $H$ with coefficient $\gamma > 0$ (i.e., $\langle Ax, x \rangle \geq \gamma\|x\|^2$ for all $x \in H$) and $0 < \alpha \leq \|A\|^{-1}$. Then $\|I - \alpha A\| \leq 1 - \alpha \gamma$.

**Lemma 4.2** [3] Let $H$ be a Hilbert space, $K$ a closed convex subset of $H$, and $T : K \to K$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in $K$ weakly converging to $x$ and if $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.

**Theorem 4.3** Suppose $A$ is a strongly positive linear bounded operator with coefficient $\gamma > 0$ as given in (4.2). Suppose the sequence $\{\alpha_n\}$ of parameters satisfies the
conditions \((H1)\) and \((H2)\). Then the sequence \(\{x_n\}\) generated by the algorithm \((4.5)\) converges in norm to the unique solution \(x^*\) of the minimization problem \((4.1)\).

**Proof.** First we claim that \(\{x_n\}\) is bounded. As a matter of fact, take a \(p \in Fix(T)\) and use Lemma 4.1 to deduce (as \(\alpha_n \to 0\) we assume, with no loss of generality, that \(\alpha_n < \|A\|^{-1}\) for all \(n)\)

\[
\|x_{n+1} - p\| = \|(I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)T x_n - p) + \alpha_n (u - Ap)\| \\
\leq (1 - \gamma \alpha_n)\|x_n - p\| + \alpha_n \|u - Ap\| \\
\leq \max\{\|x_n - p\|, (1/\gamma)\|u - Ap\|\}.
\]

By induction we can get

\[
\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{1}{\gamma} \|u - Ap\|\right\}, \quad n \geq 0.
\]

Hence, \(\{x_n\}\) is bounded. Next rewrite \(x_{n+1}\) in the form:

\[
x_{n+1} = (1 - \gamma_n) x_n + \gamma_n y_n, \quad (4.6)
\]

where

\[
\gamma_n = 1 - (1 - \alpha_n) \beta_n \quad (4.7)
\]

and

\[
y_n = \frac{\alpha_n \beta_n}{\gamma_n} (I - A)x_n + \frac{1 - \beta_n}{\gamma_n} (I - \alpha_n A)Tx_n + \frac{\alpha_n}{\gamma_n} u. \quad (4.8)
\]

Since \(\alpha_n \to 0\), it is easily seen that \(\lim \inf_{n \to \infty} \gamma_n \geq 1 - \beta > 0\) and \(\lim \sup_{n \to \infty} \gamma_n \leq 1 - \beta < 1\). Since also \(\{x_n\}\) is bounded, equation \((4.8)\) shows that \(\{y_n\}\) is bounded. Set

\[
z_n := \frac{1}{\gamma_n}(\beta_n(I - A)x_n - (1 - \beta_n)AT x_n + u).
\]

Then \(\{z_n\}\) is bounded and from \((4.8)\), \(y_n\) can be rewritten as

\[
y_n = \alpha_n z_n + \frac{1 - \beta_n}{\gamma_n} Tx_n = \alpha_n z_n + \left(1 - \frac{\alpha_n \beta_n}{\gamma_n}\right) T x_n \quad (4.9)
\]

since \(\frac{1 - \beta_n}{\gamma_n} = 1 - \frac{\alpha_n \beta_n}{\gamma_n}\) due to \((4.7)\). We can now compute

\[
y_{n+1} - y_n = \alpha_{n+1} z_{n+1} - \alpha_n z_n + \left(1 - \frac{\alpha_{n+1} \beta_{n+1}}{\gamma_{n+1}}\right) T x_{n+1} - \left(1 - \frac{\alpha_n \beta_n}{\gamma_n}\right) T x_n \\
= \alpha_{n+1} z_{n+1} - \alpha_n z_n + \left(1 - \frac{\alpha_{n+1} \beta_{n+1}}{\gamma_{n+1}}\right) (T x_{n+1} - T x_n) \\
+ \left(1 - \frac{\alpha_{n+1} \beta_{n+1}}{\gamma_{n+1}} - \left(1 - \frac{\alpha_n \beta_n}{\gamma_n}\right)\right) T x_n \\
= \left(1 - \frac{\alpha_{n+1} \beta_{n+1}}{\gamma_{n+1}}\right) (T x_{n+1} - T x_n) \\
+ \left(\frac{\alpha_n \beta_n}{\gamma_n} - \frac{\alpha_{n+1} \beta_{n+1}}{\gamma_{n+1}}\right) T x_n + \alpha_{n+1} z_{n+1} - \alpha_n z_n.
\]
It turns out that
\[
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}} \|x_{n+1} - x_n\| + \alpha_{n+1}\|z_{n+1}\|
\]
\[
+ \left|\frac{\alpha_n\beta_n}{\gamma_n} - \frac{\alpha_{n+1}\beta_{n+1}}{\gamma_{n+1}}\right| \|Tx_n\| + \alpha_n\|z_n\|. \tag{4.10}
\]
Since \(\alpha_n \to 0\) as \(n \to \infty\), it is immediately clear from (4.10) that
\[
\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]
Consequently, we can apply Lemma 2.3 to assert that
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{4.11}
\]
By (4.9) we have
\[
\|y_n - Tx_n\| = \alpha_n \left\|z_n - \frac{\beta_n}{\gamma_n}Tx_n\right\| \to 0. \tag{4.12}
\]
This together with (4.11) yields
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0. \tag{4.13}
\]
Lemma 4.3 then implies that \(\omega_w(x_n) \subset Fix(T) = C\). Here
\[
\omega_w(x_n) = \{z : \exists x_n \to z \text{ weakly}\}
\]
is the set of weak \(\omega\)-limit points of the sequence \(\{x_n\}\).

Let \(x^*\) be the unique solution to the minimization (4.1). Then by the definition of the algorithm (4.5), we can write
\[
x_{n+1} - x^* = (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)Tx_n - x^*) + \alpha_n(u - Ax^*).
\]
Apply Lemma 2.4 (as \(J\) is the identity in a Hilbert space) and use Lemma 4.1 to get (noting \(\|\beta_n x_n + (1 - \beta_n)Tx_n - x^*\| \leq \|x_n - x^*\|\))
\[
\|x_{n+1} - x^*\|^2 \leq \|(I - \alpha_n A)(\beta_n x_n + (1 - \beta_n)Tx_n - x^*)\|^2
\]
\[
+ 2\alpha_n(u - Ax^*, x_{n+1} - x^*)
\]
\[
\leq (1 - \gamma \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - Ax^*, x_{n+1} - x^*\rangle. \tag{4.14}
\]
However, we can take a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) such that
\[
\limsup_{n \to \infty} \langle u - Ax^*, x_n - x^*\rangle = \lim_{j \to \infty} \langle u - Ax^*, x_{n_j} - x^*\rangle
\]
and also \(x_{n_j} \to p\) weakly. Then, since \(p \in Fix(T) = C\), we get from the optimality condition (4.3),
\[
\limsup_{n \to \infty} \langle u - Ax^*, x_n - x^*\rangle = \langle u - Ax^*, p - x^*\rangle \leq 0. \tag{4.15}
\]
Therefore, applying Lemma 2.2 to (4.14) and noticing (4.15), we conclude that \(\|x_n - x^*\|^2 \to 0\). \(\square\)
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