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GENERALIZED NONEXPANSIVE MULTIVALUED MAPPINGS IN STRICTLY CONVEX BANACH SPACES

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Abstract. In this paper, we present some common fixed point results for a commuting pair of mappings, including a quasi-nonexpansive single valued mapping and a generalized nonexpansive multivalued mapping in strictly convex Banach spaces, as well as for a pointwise asymptotically nonexpansive mapping and a generalized nonexpansive multivalued mapping in uniformly convex Banach spaces. The results we obtain extend and improve some known results due to Garcia-Falset et al. (2011), Kirk and Massa (1990), Espinola et al. (2011), Kaewcharoen and Panyanak (2011) as well as that of Abkar and Eslamian (2010).
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1. Introduction

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [17] and Nadler [18]. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Using the Edelstein's method of asymptotic centers, Lim [16] proved the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces. Kirk and Massa [15] extended Lim's theorem to Banach spaces for which the asymptotic center of a bounded sequence in a bounded closed convex subset is nonempty and compact.

On the other hand, in 2008, Suzuki [19] introduced a condition on mappings, called condition (C), which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. He then proved some fixed point and convergence theorems for such mappings. Motivated by this result, J. Garcia-Falset, E. Llorens-Fuster and T. Suzuki in [6], introduced two kinds of generalization for the condition (C) and studied

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both the existence of fixed points and their asymptotic behavior. Recently, the current authors used a modified condition for multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying this condition in Banach spaces [1], as well as in CAT(0) spaces [2]. Very recently, Kaewcharoen and Panyanak [12] and Espinola, Lorenzo and Nicolae [5] generalized Kirk and Massa's theorem for continuous generalized nonexpansive multivalued mappings.

In this paper, we present some new common fixed point theorems for a commuting pair of mappings, including a quasi-nonexpansive single valued mapping and a generalized nonexpansive multivalued mapping in a strictly convex Banach space, as well as for a pointwise asymptotically nonexpansive mapping and a generalized nonexpansive multivalued mapping in a uniformly convex Banach space. Our result improves a number of known results; including that of Lim [16], Kirk and Massa [15], Suzuki [19], Garcia et al. [6], Abkar and Eslamian [1], Kaewcharoen and Panyanak [12], Espinola et al. [5] and of Dhompongsa et al. [4].

2. Preliminaries

Let X be a Banach space. X is said to be strictly convex if $\|x+y\| < 2$ for all $x, y \in X$, $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that a Banach space X is said to be *uniformly convex in every direction* (UCED, for short) provided that for every $\epsilon \in (0,2]$ and $z \in X$ with $\|z\| = 1$, there exists a positive number δ (depending on ϵ and z) such that for all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $x - y \in \{tz : t \in [-2,-\epsilon] \cup [\epsilon, 2]\}$ we have $\|x+y\| \leq 2(1-\delta)$. X is said to be *uniformly convex* if X is UCED and $inf\{\delta(\epsilon, z) : \|z\| = 1\} > 0$ for all $\epsilon \in (0,2]$. It is rather obvious that uniform convexity implies UCED, and UCED implies strict convexity.

Definition 2.1 ([14]). *Let D be a nonempty subset of a Banach space X. A mapping $T : D \to D$ is called pointwise asymptotically nonexpansive if there exists a sequence of functions α_n with $\lim_{n\to\infty} \alpha_n(x) = 1$ and $\alpha_n(x) \geq 1$ such that*

$$\|T^n(x) - T^n(y)\| \leq \alpha_n(x)\|x - y\|, \qquad x, y \in D.$$

Definition 2.2 ([14]). *Let D be a bounded closed convex subset of a uniformly convex Banach space X and $T : D \to D$ be a pointwise asymptotically nonexpansive mapping. Then the set of fixed points of T is nonempty, closed and convex.*

The following definition is due to Suzuki [19].

Definition 2.3 ([19]). *Let T be a mapping on a subset D of a Banach space X. T is said to satisfy condition (C) if*

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in D.$$

In [6], J. Garcia-Falset et al. introduced two generalizations of the condition (C) in a Banach space:

Definition 2.4 *Let T be a mapping on a subset D of a Banach space X and $\mu \geq 1$. T is said to satisfy condition (E_μ) if*

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|, \quad x, y \in D.$$

We say that T satisfies condition (E) whenever T satisfies the condition (E_μ) for some $\mu \geq 1$.

Definition 2.5 *Let T be a mapping on a subset D of a Banach space X and $\lambda \in (0, 1)$. T is said to satisfy condition (C_λ) if*

$$\lambda\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in D.$$

Notice that if $0 < \lambda_1 < \lambda_2 < 1$ then the condition (C_{λ_1}) implies the condition (C_{λ_2}). We recall that a mapping $T : D \to D$ is said to be quasi-nonexpansive provided that $Fix(T) \neq \emptyset$ and for each $x \in D$ and $y \in Fix(T)$ we have

$$\|T(x) - y\| \leq \|x - y\|.$$

It is clear that every mapping T with nonempty fixed point set that satisfies the condition (C_λ) is quasi-nonexpansive.

Theorem 2.6 ([6]). *Let D be a nonempty bounded convex subset of a Banach space X. Let $T : D \to D$ satisfy the condition (C_λ) on D for some $\lambda \in (0, 1)$. For $r \in [\lambda, 1)$ define a sequence $\{x_n\}$ in D by taking $x_1 \in D$ and*

$$x_{n+1} = rT(x_n) + (1 - r)x_n \qquad \textit{for} \quad n \geq 1,$$

then $\{x_n\}$ is an approximate fixed point sequence for T, that is

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$$

Lemma 2.7 ([11]). *Let T be a quasi nonexpansive mapping defined on a closed subset E of a Banach space X. Then Fix(T) is closed. Moreover, if X is strictly convex and E is convex, then Fix(T) is also convex.*

Let D be a nonempty subset of a Banach space X. For $x \in X$, we write

$$dist(x, D) = inf\{\| x - z \|: z \in D\}.$$

We denote by $CB(D)$ and $KC(D)$ the collection of all nonempty closed bounded subsets, and nonempty compact convex subsets of D, respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(X)$.

Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in T(x)$.

It is rather obvious that if D is a convex subset of a strictly convex Banach space X, then for $x \in X$, if there exist $y, z \in D$ such that

$$\| x - y \|= dist(x, D) =\| x - z \|$$

then $y = z$.

Definition 2.8 *A multivalued mapping $T : X \to CB(X)$ is said to be nonexpansive provided that*

$$H(Tx, Ty) \le \|x - y\|, \quad x, y \in X.$$

Definition 2.9 ([3]). *A multivalued mapping $T : X \to CB(X)$ is said to satisfy the condition (C) provided that*

$$\frac{1}{2} dist(x, Tx) \le \|x - y\| \implies H(Tx, Ty) \le \|x - y\|, \quad x, y \in X.$$

We now state the multivalued analogs of the conditions (E) and (C_λ) in the following manner (see also [2]):

Definition 2.10 *A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (E_μ) provided that*

$$dist(x, Ty) \le \mu\, dist(x, Tx) + \|x - y\|, \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \ge 1$.

Definition 2.11 *A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ provided that*

$$\lambda\, dist(x, Tx) \le \|x - y\| \implies H(Tx, Ty) \le \|x - y\|, \quad x, y \in X.$$

It is rather easy to see that every multivalued nonexpansive mapping satisfies the condition (E_1).

Lemma 2.12 *Let $T : X \to CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (E_1).*

We now provide an example of a generalized nonexpansive multivalued mapping which satisfies the conditions (C_λ) and (E), but it is not nonexpansive.

Example 2.13 We define T on the closed interval $[0, 5]$ by

$$T(x) = \begin{cases} [0, \frac{x}{5}], & x \ne 5 \\ \{1\} & x = 5. \end{cases}$$

It is not difficult to verify that T has the required properties (for details, see [2]).

Finally, we recall the following lemma from [8].

Lemma 2.14 *Let $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in a Banach space X, and let $0 < \lambda < 1$. If for every natural number n we have $z_{n+1} = \lambda w_n + (1 - \lambda) z_n$ and $\|w_{n+1} - w_n\| \le \|z_{n+1} - z_n\|$, then $\lim_{n\to\infty} \|w_n - z_n\| = 0$.*

3. Common Fixed Point Theorems

Let D be a nonempty bounded closed convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X. We use $r(x, \{x_n\})$ and $A(D, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in D, respectively, i.e.

$$r(D, \{x_n\}) = \inf\{\limsup_{n\to\infty} \|x_n - x\| : x \in D\},$$
$$A(D, \{x_n\}) = \{x \in D : \limsup_{n\to\infty} \|x_n - x\| = r(D, \{x_n\})\}.$$

Obviously, the convexity of D implies that $A(D, \{x_n\})$ is convex. It is also known that in a UCED Banach space X, the asymptotic center of a sequence with respect to a weakly compact convex set is a singleton; the same is true for a sequence in a bounded closed convex subset of a uniformly convex Banach space X (see [13]).

Definition 3.1 *A bounded sequence $\{x_n\}$ is said to be* regular *with respect to D if for every subsequence $\{x'_n\}$ we have*

$$r(D, \{x_n\}) = r(D, \{x'_n\});$$

further, $\{x_n\}$ is called asymptotically uniform relative to D if

$$A(D, \{x_n\}) = A(D, \{x'_n\}).$$

The following lemma was proved by Goebel and Lim.

Lemma 3.2 (see [7] and [16]). *Let $\{x_n\}$ be a bounded sequence in X and let D be a nonempty closed convex subset of X.*

(i) *then there exists a subsequence of $\{x_n\}$ which is regular relative to D.*
(ii) *if D is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to D.*

As a consequence of Remark 2 and Theorem 8 in [6], we obtain the following result.

Theorem 3.3 *Let D be a nonempty closed convex bounded subset of a Banach space X. Let $T : D \to D$ be a single valued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. Suppose the asymptotic center relative to D of each sequence in D is nonempty and compact. Then T has a fixed point.*

Definition 3.4 *Let D be a nonempty subset of a Banach space X. Two mappings $t : D \to D$ and $T : D \to CB(D)$ are said to be commuting if $t(T(x)) \subset T(t(x))$ for all $x \in D$.*

We now state and prove the first main result of this paper.

Theorem 3.5 *Let D be a nonempty closed convex bounded subset of a strictly convex Banach space X, $t : D \to D$ be a quasi-nonexpansive single valued mapping, and $T : D \to KC(D)$ be a continuous multivalued mapping satisfying the condition (C_λ) for some $\lambda \in (0, 1)$, and that t, T commute. If the asymptotic center relative to $Fix(t)$*

of each sequence in $Fix(t)$ is nonempty and compact, then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.

Proof. According to Lemma 2.7, it follows that $Fix(t)$ is a closed convex subset of X. We show that for $x \in Fix(t)$, $T(x) \cap Fix(t) \neq \emptyset$. To see this, let $x \in Fix(t)$ and let $y \in T(x)$ be the unique nearest point to x. Since t and T commute, we have $t(y) \in T(t(x)) = T(x)$. Since t is quasi nonexpansive, we have $\|t(y) - x\| \leq \|y - x\|$. Now by the uniqueness of y as the nearest point to x, we get $t(y) = y$. Therefore $T(x) \cap Fix(t) \neq \emptyset$ for $x \in Fix(t)$.

Now we find an approximate fixed point sequence for T in $Fix(t)$. Take $x_0 \in Fix(t)$, since $T(x_0) \cap Fix(t) \neq \emptyset$, we can choose $y_0 \in T(x_0) \cap Fix(t)$. Define

$$x_1 = (1 - \lambda)x_0 + \lambda y_0.$$

Since $Fix(t)$ is a convex set, we have $x_1 \in Fix(t)$. Let $y_1 \in T(x_1)$ be chosen in such a way that

$$\|y_0 - y_1\| = dist(y_0, T(x_1)).$$

We see that $y_1 \in Fix(t)$. Indeed, Since t is quasi-nonexpansive, we get

$$\|y_0 - t(y_1)\| \leq \|y_0 - y_1\|$$

which contradicts the uniqueness of y_1 as the unique nearest point to y_0 (note that $t(y_1) \in T(x_1)$). Similarly, put

$$x_2 = (1 - \lambda)x_1 + \lambda y_1,$$

again we choose $y_2 \in T(x_2)$ in such a way that

$$\|y_1 - y_2\| = dist(y_1, T(x_2)).$$

By the same argument, we get $y_2 \in Fix(t)$. In this way we will find a sequence $\{x_n\}$ in $Fix(t)$ such that

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n,$$

where $y_n \in T(x_n) \cap Fix(t)$ and

$$\|y_{n-1} - y_n\| = dist(y_{n-1}, T(x_n)).$$

Thus for every natural number $n \geq 1$ we have

$$\lambda\|x_n - y_n\| = \|x_n - x_{n+1}\|$$

from which it follows that

$$\lambda\, dist(x_n, T(x_n)) \leq \lambda\|x_n - y_n\| = \|x_n - x_{n+1}\|, \quad n \geq 1.$$

Our assumption now gives

$$H(T(x_n), T(x_{n+1})) \leq \|x_n - x_{n+1}\|, \quad n \geq 1,$$

and hence for each $n \geq 1$,

$$\begin{aligned}\|y_n - y_{n+1}\| = dist(y_n, T(x_{n+1})) &\leq H(T(x_n), T(x_{n+1})) \\ &\leq \|x_n - x_{n+1}\|.\end{aligned}$$

We now apply Lemma 2.13 to conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $y_n \in T(x_n)$. From Lemma 3.2, by passing to a subsequence, we may assume that $\{x_n\}$ is regular asymptotically uniform relative to $Fix(t)$. Let $r = r(Fix(t), \{x_n\})$. Now, we show that $Tx \cap A(Fix(t), \{x_n\}) \neq \emptyset$ for $x \in A(Fix(t), \{x_n\})$. If $r = 0$, then we have $x_n \to x$. Then by the continuity of T we have

$$dist(x, Tx) \leq \|x - x_n\| + dist(x_n, Tx_n) + H(Tx_n, Tx) \to 0$$

which implies that $x \in T(x)$. In the other case, if $r > 0$, there exists a natural number n_0 such that for every $n \geq n_0$,

$$\lambda\, dist(x_n, Tx_n) \leq \|x_n - x\|$$

and hence from our assumption we have

$$H(T(x_n), T(x)) \leq \|x_n - x\|, \qquad \forall n \geq n_0.$$

The compactness of $T(x)$ implies that for each $n \geq 1$ we can find $z_n \in T(x)$ such that

$$\|y_n - z_n\| = dist(y_n, T(x)).$$

Also we have

$$\|y_n - z_n\| = dist(y_n, T(x)) \leq H(T(x_n), T(x)) \leq \|x_n - x\|, \qquad \forall n \geq n_0.$$

Since $T(x)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = z \in T(x)$. Note that

$$\begin{aligned}\|x_{n_k} - z\| &\leq \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - z\| \\ &\leq \|x_{n_k} - y_{n_k}\| + \|x_{n_k} - x\| + \|z_{n_k} - z\|,\end{aligned}$$

for $n_k \geq n_0$. This implies that

$$\limsup_{k\to\infty} \|x_{n_k} - z\| \leq \limsup_{k\to\infty} \|x_{n_k} - x\| \leq r.$$

Since $\{x_n\}$ is regular asymptotically uniform relative to $Fix(t)$, it follows that

$$z \in A(Fix(t), \{x_{n_k}\}) = A(Fix(t), \{x_n\}),$$

therefore

$$z \in T(x) \cap A(Fix(t), \{x_n\}),$$

which in turn implies that $T(x) \cap A(Fix(t), \{x_n\}) \neq \emptyset$ for $x \in A(Fix(t), \{x_n\})$. Now we define the mapping

$$\tilde{T} : A(Fix(t), \{x_n\}) \to KC(A(Fix(t), \{x_n\}))$$

by $\tilde{T}(z) = A(Fix(t), \{x_n\}) \cap T(z)$. From Proposition 2.45 in [10] we know that the mapping $\tilde{T}$ is upper semicontinuous. Since $A(Fix(t), \{x_n\}) \cap T(z)$ is a compact convex set, we can apply the Kakutani-Bohnenblust-Karlin Theorem to obtain a fixed point v for $\tilde{T}$ (see [9]). This means that v is a common fixedpoint of T and t.

As a result, we obtain the following theorem.

Theorem 3.6 *Let D be a nonempty closed convex bounded subset of a strictly convex Banach space X. Let $t : D \to D$, and $T : D \to KC(D)$ be two nonexpansive mappings. Assume that t, T commute. Suppose the asymptotic center relative to $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and compact. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. Since T is nonexpansive, we conclude that T satisfies the condition (C_λ) for all $\lambda \in (0,1)$. Hence the result follows from Theorem 3.5.

Theorem 3.7 *Let D be a nonempty compact convex subset of a strictly convex Banach space X, $t : D \to D$ be a single valued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$, and $T : D \to KC(D)$ be a continuous multivalued mapping satisfying the condition (C_λ) for some $\lambda \in (0,1)$, and that t, T commute. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. By Theorem 3.3 the mapping t has a nonempty fixed point set $Fix(t)$ which is a closed convex subset of X (by Lemma 2.7). Since D is compact, we conclude that $Fix(t)$ is compact too. Since X is strictly convex, we infer that the asymptotic center relative to $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and compact. Therefore, by Theorem 3.5, T and t have a common fixed point.

By the same argument as in the proof of Theorem 3.4 in [2] we obtain the following theorem.

Theorem 3.8 *Let D be a nonempty closed convex bounded subset of a strictly convex Banach space X, $t : D \to D$ be a quasi-nonexpansive single valued mapping, and $T : D \to KC(D)$ be a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$, and that t, T commute. If the asymptotic center relative to $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and singleton, then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Theorem 3.9 *Let D be a nonempty weakly compact convex subset of a UCED Banach space X. Let $t : D \to D$ be a single valued maping, and $T : D \to KC(D)$ be a multivalued mapping, both of them satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If t, T commute, then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. By Theorem 3.3, t has a nonempty fixed point set $Fix(t)$ which is a closed convex subset of X (by Lemma 2.7). Since D is weakly compact, it follows that $Fix(t)$ is weakly compact as well. Since X is UCED, we conclude that the asymptotic center relative to $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and singleton. Therefore, by Theorem 3.8, T and t have a common fixed point.

Corollary 3.10 *Let D be a nonempty closed convex bounded subset of a uniformly convex Banach space X, $t : D \to D$ be a single valued, and $T : D \to KC(D)$ be a multivalued mapping, both satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If t, T commute, then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Finally, we state the second main result of this paper.

Theorem 3.11 *Let D be a nonempty closed convex bounded subset of a uniformly convex Banach space X. Let $f : D \to D$ be a pointwise asymptotically nonexpansive mapping, and let $T : D \to KC(D)$ be a multivalued mapping satisfying the conditions*

(E) and (C_λ) for some $\lambda \in (0,1)$. If f and T are commuting, then they have a common fixed point, i.e. there exists a point $z \in D$ such that $z = f(z) \in T(z)$.

Proof. Using Theorem 2.2, it follows that $Fix(f)$ is a nonempty closed convex subset of D. We show that for each $x \in Fix(f)$, $T(x) \cap Fix(f) \neq \emptyset$. To see this, let $x \in Fix(f)$, since f and T are commuting, we have $f(y) \in T(x)$ for each $y \in T(x)$. Therefore $T(x)$ is invariant under f for each $x \in Fix(f)$. Since $T(x)$ is a bounded closed convex subset of the uniformly convex Banach space X, by Theorem 2.2 we conclude that f has a fixed point in $T(x)$ and therefore $T(x) \cap Fix(f) \neq \emptyset$ for $x \in Fix(f)$.

Now we find an approximate fixed point sequence for T in $Fix(f)$. Take $x_0 \in Fix(f)$, since $T(x_0) \cap Fix(f) \neq \emptyset$, we can choose $y_0 \in T(x_0) \cap Fix(f)$. Define

$$x_1 = (1-\lambda)x_0 + \lambda y_0.$$

Since $Fix(f)$ is convex, we have $x_1 \in Fix(f)$. Let $y_1 \in T(x_1)$ be chosen in such a way that

$$\|y_0 - y_1\| = dist(y_0, T(x_1)).$$

Next we show that $y_1 \in Fix(f)$. Indeed, we consider the sequence $\{f^n(y_1)\}$. Since T and f commute, we know that $f^n(y_1) \in T(x_1)$ for any n. Since $T(x_1)$ is compact, the sequence $\{f^n(y_1)\}$ has a convergent subsequence with

$$\lim_{k\to\infty} f^{n_k}(y_1) = z \in T(x_1),$$

so that

$$\begin{aligned}\|z - y_0\| = \lim_{k\to\infty} \|f^{n_k}(y_1) - y_0\| &= \lim_{k\to\infty} \|f^{n_k}(y_1) - f^{n_k}(y_0)\| \\ &\leq \lim_{k\to\infty} \alpha_{n_k}(y_0)\|y_1 - y_0\| \leq dist(y_0, T(x_1)) = \|y_0 - y_1\|.\end{aligned}$$

Now by the uniqueness of y_1 as the nearest point to y_0, we have $z = y_1$, consequently $\lim_{k\to\infty} f^{n_k}(y_1) = y_1$ and so $f(y_1) = y_1$. In this way we will find a sequence $\{x_n\}$ in $Fix(f)$ such that $x_{n+1} = (1-\lambda)x_n + \lambda y_n$ where $y_n \in T(x_n) \cap Fix(f)$ and

$$\|y_{n-1} - y_n\| = dist(y_{n-1}, T(x_n)).$$

Therefore for every natural number $n \geq 1$ we have

$$\lambda\|x_n - y_n\| = \|x_n - x_{n+1}\|$$

from which it follows that

$$\lambda\, dist(x_n, T(x_n)) \leq \lambda\|x_n - y_n\| = \|x_n - x_{n+1}\|, \quad n \geq 1.$$

Our assumption now gives

$$H(T(x_n), T(x_{n+1})) \leq \|x_n - x_{n+1}\|, \quad n \geq 1,$$

and hence for each $n \geq 1$,

$$\begin{aligned}\|y_n - y_{n+1}\| = dist(y_n, T(x_{n+1})) &\leq H(T(x_n), T(x_{n+1})) \\ &\leq \|x_n - x_{n+1}\|.\end{aligned}$$

We now apply Lemma 2.13 to conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $y_n \in T(x_n)$. From Lemma 3.2, by passing to a subsequence, we may assume that $\{x_n\}$ is regular with respect to $Fix(f)$. Since $Fix(f)$ is a closed convex bounded subset of a uniformly convex Banach space X, it follows that the asymptotic center of the sequence $\{x_n\}$ with respect to $Fix(f)$ is singleton. Let $A(Fix(f), \{x_n\}) = \{z\}$. Since $T(z)$ is compact, for each $n \geq 1$, we can choose $z_n \in T(z)$ such that $\|x_n - z_n\| = dist(x_n, T(z))$. Moreover $z_n \in Fix(f)$ for all natural numbers $n \geq 1$. Indeed, for any $n \geq 1$, we consider the sequence $\{f^m(z_n)\}$. Since T and f commute, and $z \in Fix(f)$ we have $f(z_n) \in f(T(z)) \subset T(f(z)) = T(z)$, and hence $f^m(z_n) \in T(z)$ for any m. Since $T(z)$ is compact, the sequence $\{f^m(z_n)\}$ has a convergent subsequence with

$$\lim_{k\to\infty} f^{m_k}(z_n) = v \in T(z),$$

so that

$$\begin{aligned}\|v - x_n\| = \lim_{k\to\infty} \|f^{m_k}(z_n) - x_n\| &= \lim_{k\to\infty} \|f^{m_k}(z_n) - f^{m_k}(x_n)\| \\ &\leq \lim_{k\to\infty} \alpha_{m_k}(x_n)\|z_n - x_n\| \leq dist(x_n, T(z)) = \|x_n - z_n\|.\end{aligned}$$

Now by the uniqueness of z_n as the nearest point to x_n, we have $v = z_n$, consequently $\lim_{k\to\infty} f^{m_k}(z_n) = z_n$ and so $f(z_n) = z_n$, i.e., $z_n \in Fix(f)$. Since $T(z)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = w \in Tz$. Since $z_{n_k} \in Fix(f)$ for all k, and $Fix(f)$ is closed, we obtain $w \in Fix(f)$. By assumption there exists $\mu \geq 1$ such that

$$dist(x_{n_k}, Tz) \leq \mu\, dist(x_{n_k}, T(x_{n_K})) + \|x_{n_k} - z\|.$$

Note that

$$\begin{aligned}\|x_{n_k} - w\| &\leq \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - w\| \\ &\leq \mu\, dist(x_{n_k}, T(x_{n_k})) + \|x_{n_k} - z\| + \|z_{n_k} - w\|.\end{aligned}$$

This entails

$$\limsup_{k\to\infty} \|x_{n_k} - w\| \leq \limsup_{k\to\infty} \|x_{n_k} - z\|.$$

We conclude that $z = w$, hence $z = f(z) \in T(z)$.

Finally, we supply an example to illustrate the main result of this paper.

Example 3.12. Suppose that $X = \mathbb{R}$ and $D = [0, \frac{7}{2}]$. Define T and f by

$$T(x) = \begin{cases} [0, \frac{x}{7}], & x \neq \frac{7}{2} \\ \{1\} & x = \frac{7}{2}. \end{cases}$$

and

$$f(x) = \begin{cases} 0, & x \in [0, 3] \\ 4x - 12, & x \in [3, \frac{13}{4}] \\ -4x + 14, & x \in [\frac{13}{4}, \frac{7}{2}]. \end{cases}$$

First we show that f is a pointwise asymptotically nonexpansive mapping. To this end, we put $\alpha_1(x) = 4$ and $\alpha_n(x) = 1$ for $n \geq 2$. Since $f^n(x) = 0$ for all $n \geq 2$, it suffices to show that $|f(x) - f(y)| \leq 4|x - y|$, for all $x, y \in D$.

Let $x \in [0, 3]$ and $y \in [3, \frac{13}{4}]$, then we must have

$$|f(x) - f(y)| = 4y - 12 \leq 4y - 4x = 4|x - y|.$$

Let $x \in [0, 3]$ and $y \in [\frac{13}{4}, \frac{7}{2}]$, then we need to have

$$|fx - fy| = -4y + 14 \leq 4y - 4x,$$

or equivalently $14 + 4x \leq 8y$ which holds. Now if $x \in [3, \frac{13}{4}]$ and $y \in [\frac{13}{4}, \frac{7}{2}]$, we have to verify that

$$|4(x + y) - 26| \leq 4y - 4x.$$

If $|4(x + y) - 26| = 4(x + y) - 26$, then

$$4(x + y) - 26 \leq 4y - 4x$$

is equivalent to $8x \leq 26$ which holds. If $|4(x + y) - 26| = 26 - 4(x + y)$, then

$$26 - 4(x + y) \leq 4y - 4x$$

is equivalent to $26 \leq 8y$ which holds. Therefore for all $x, y \in D$ we have $|f(x)-f(y)| \leq 4|x - y|$. But f is not nonexpansive. Indeed, let $x = 3$ and $y = \frac{13}{4}$ then we have

$$|f(x) - f(y)| = 1 > \frac{1}{4} = |x - y|.$$

Now we show that T satisfies the conditions (E) and (C_λ). Let $x, y \in [0, \frac{7}{2})$, then we have

$$H(Tx, Ty) = |\frac{x - y}{7}| \leq |x - y|.$$

If $x \in (0, \frac{5}{2}]$ and $y = \frac{7}{2}$, we have

$$H(Tx, Ty) = 1 \leq \frac{7}{2} - x.$$

In case that $x \in (\frac{5}{2}, \frac{7}{2})$ and $y = \frac{7}{2}$, we have $dist(x, Tx) = \frac{6x}{7}$. Therefore

$$\frac{1}{2}dist(x, Tx) = \frac{6x}{14} > \frac{30}{28} > 1 > |x - y|.$$

Moreover

$$\frac{1}{2}dist(y, Ty) = \frac{5}{4} > 1 > |x - y|.$$

These inequalities show that the mapping T satisfies the condition (C_λ) for $\lambda = \frac{1}{2}$. Also for all $x, y \in D$ we have

$$dist(x, Ty) \leq 3d(x, Tx) + |x - y|,$$

so that T satisfies the condition (E). Now we shall see that the mapping T is not nonexpansive. To see this we take $x = \frac{7}{2}$ and $y = 3$. Then we have

$$H(Tx, Ty) = 1 > \frac{1}{2} = |x - y|.$$

It is not difficult to see that T and f commute. It then follows from Theorem 3.11 that T and f have a common fixed point. We observe that 0 is a common fixed point of f and T.

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