Fixed Point Theory, 14(2013), No. 2, 263-268
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SOME FIXED POINT RESULTS IN TVS-CONE METRIC SPACES

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Abstract. Every TVS-cone metric space is topologically isomorphic to a topological metric space. In this paper, by using a nonlinear scalarization, we give some fixed point results with nonlinear contractive conditions on TVS-cone metric spaces.
Key Words and Phrases: Cone metric, fixed point, topological vector space.
2010 Mathematics Subject Classification: 47H10, 46T99, 54H25.

1. Introduction

In the workshop on real and complex singularities which held in Brazil, August 1992, Rabelo introduced the sentence of cone metrics ([26]). In 1996, Jeffres wrote her ph.d. thesis entitled "Kahler-Einstein cone metrics" in State University of New York at Stony Brook ([18]). Then, she published two papers about cone metrics in the new century ([19] and [20]). In 2005, Schumacher and Trapani published a paper on variation of cone metrics on Riemann surfaces ([33]). In 2006, Parker published a paper about cone metrics on the sphere and Livne's lattices ([25]). But, all of them used cone metrics by using geometric sights. Although there are some works which have used the notion of cone metric spaces (see for example, [24], [35] and [36]), but Huang and Zhang used the notion of cone metric spaces (CMS) systemically in 2007 ([17]). In fact in cone metric spaces, distance of two elements is an element of a topological vector space. Huang and Zhang supposed that the distance is an element of a real Banach space and they used the notion of normality for obtaining their fixed point results. In 2008, Rezapour and Hamlbarani Haghi showed that there are non-normal cones and in most results the assumption of normality is redundant([28]). After this time, many authors published a lot of papers about fixed point theory in cone metric spaces (see for example, [1], [2], [3], [5], [11], [21], [27], [31], [32] and [34]). But, some authors liked to investigate exactly the structures of CMS. In this way, Wei-Shih Du was the first author did it by publishing [15]. Later, Abdeljawad and Karapinar showed that there is a gap in his proofs ([6]). Immediately, Amini-Harandi and Fakhar and on the other hand Rezapour, Khandani and Vaezpour investigated the structure separately by using different methods and considering topological vector spaces instead Banach spaces ([7] and [29]). Later by using the structure of normality, Khamsi defined metric type structures and gave some results in this field ([22] and [23]). Of course, it proved later that his some investigations are not true for non-normal cone metric spaces ([30]). In some papers, normality, regularity and strongly minihedral cones have been used in the assumptions. However, some results have been carried from metric spaces to CMS without any extra assumptions on the cone. Also, some authors investigated the topological structures in CMS (see for example, [3], [32] and [34]). For example, it has been proved that CMS are first countable paracompact topological spaces provided that the cone has no empty interior. Also, each TVS-CMS is a first countable paracompact topological space (see [4], [8] and [34]). The proofs based on that every TVS-CMS is topologically isomorphic to a usual metric space. It can be described that how one can construct such comparable metric topologies to the TVS-cone metric topologies by using the system of the seminorms generating the topology of the locally convex topological space E (for more details, basic notions and some topological properties of TVS-CMS, see [4]). Throughout this paper, we suppose that E is a locally convex topological vector space and P is a cone in E with nonempty interior. Also, for each $e \in intP$, we define the nonlinear scalarization function $\xi_e : E \to \mathbb{R}$ by $\xi_e(y) = \inf\{t \in \mathbb{R} : y \in te - P\}$ for all $y \in E$ (see [10], [14], [15]). We appeal the following key Lemma in our results.

Lemma 1.1. *([10], [14] and [15]) For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:*

(i) $\xi_e(y) \leq t \Leftrightarrow y \in te - P$,
(ii) $\xi_e(y) > t \Leftrightarrow y \notin te - P$,
(iii) $\xi_e(y) \geq t \Leftrightarrow y \notin te - intP$,
(iv) $\xi_e(y) < t \Leftrightarrow y \in te - intP$,
(v) $\xi_e(y)$ *is positively homogeneous and continuous on* E,
(vi) *if* $y_1 \in y_2 + P$, *then* $\xi_e(y_2) \leq \xi_e(y_1)$,
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$, *for all* $y_1, y_2 \in E$.
(viii) *if* $y_1 \in y_2 + int(P)$, *then* $\xi_e(y_2) < \xi_e(y_1)$

Let (X, p) be a TVS-CMS. Du showed that $d_p = \phi_e \circ p$ is a metric on X ([15]).

2. Main Results

Now, we are ready to state and prove our main results. The aim of this section is to show that there are many fixed point results which we can generalize them to TVS-CMS. We do it by providing some ones.

Theorem 2.1. *Let (X, p) be a TVS-CMS, $e \in intP$, $d_p = \xi_e \circ p$, T a selfmap on X such that $p(Tx, Ty) \leq \varphi(p(x, y))$ for all $x, y \in X$, where $\varphi : P \to P$ is an increasing mapping such that $\varphi(re) \leq r\varphi(e)$ for all $r \geq 0$. Then, there exists a linear, increasing and continuous mapping $\phi : [0, \infty) \to [0, \infty)$ such that $d_p(Tx, Ty) \leq \phi(d_p(x, y))$ for all $x, y \in X$.*

Proof. Define $\phi(t) = \xi_e(\varphi(e))t$ for all $t \geq 0$. It is clear that ϕ is linear, continuous, increasing, $\phi(0) = 0$ and $\phi(\xi_e(e)) = \phi(1) = \xi_e(\varphi(e))$. Let $\varepsilon > 0$. Choose a positive real number r such that $0 \leq d_p(x,y) < r < d_p(x,y)+\varepsilon$, where $p(x,y) \in re - P$. Since φ is increasing, by using Lemma 1.1, we have

$$d_p(Tx,Ty) = \xi_e(p(Tx,Ty)) \leq \xi_e(\varphi(p(x,y)))$$
$$\leq \xi_e(\varphi(re)) \leq r\xi_e(\varphi(e)) = r\phi(1) = \phi(r) \leq \phi(d_p(x,y)) + \phi(\varepsilon).$$

Since ε is arbitrary and $\phi(0) = 0$, we get $d_p(Tx,Ty) \leq \phi(d_p(x,y))$. □

In 1969, Boyd and Wong proved the following result ([9]).

Theorem 2.2. *Let (X,d) be a complete metric space and T a selfmap on X such that $d(Tx,Ty) \leq \varphi(d(x,y))$ for all $x,y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous from the right such that $\phi(t) < t$ for all $t > 0$. Then, T has a unique fixed point u and $T^n x \to u$ for all $x \in X$.*

By using Theorem 2.1, we obtain the following generalization of Theorem 2.2.

Proposition 2.3. *Let (X,p) be a complete TVS-CMS, T a selfmap on X such that*

$$p(Tx,Ty) \leq \varphi(p(x,y))$$

for all $x,y \in X$, where $\varphi : P \to P$ is increasing, $\varphi(re) \leq r\varphi(e)$ for all $r \geq 0$ and $\xi_e(\varphi(e)) < 1$. Then, T has a unique fixed point u and $T^n x \to u$ for all $x \in X$.

In 2010, Djafari Rouhani and Moradi proved the following result ([13]).

Theorem 2.4. *Let (X,d) be a complete metric space, $T : X \to CB(X)$ a multifunction and f a selfmap on X such that $H(\{fx\},Ty) \leq M(x,y) - \varphi(M(x,y))$ for all $x,y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is lower semicontinuous with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and*

$$M(x,y) = \max\{d(x,y), d(x,fx), d(y,Ty), \frac{d(x,Ty)+d(y,fx)}{2}\}.$$

Then, there exists a unique point $x \in X$ such that $fx = x \in Tx$.

First we should emphasize that by using the results of [16], one can see that this result hold as the following form.

Theorem 2.5. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ a multifunction such that $H(x,Ty) \leq M(x,y) - \varphi(M(x,y))$ for all $x,y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is lower semicontinuous with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and*

$$M(x,y) = \max\{d(x,y), d(y,Ty), \frac{d(x,Ty)+d(y,x)}{2}\}.$$

Then, there exists a unique point $x \in X$ such that $x \in Tx$.

Now by using a similar proof of Theorem 2.1, we can obtain the following generalization of Theorem 2.4.

Proposition 2.6. *Let (X,p) be a complete TVS-CMS, $T : X \to CB(X)$ a multifunction and f a selfmap on X such that for each $x,y \in X$ there exit $z \in Ty$ and $u \in \{p(x,y), p(y,z), \frac{p(x,z)+p(y,x)}{2}\}$ so that $p(x,z) \leq u - \varphi(u)$, where $\varphi : P \to P$ is increasing and $\xi_e(\varphi(e)) > 0$. Then, there exists a unique point $x \in X$ such that $x \in Tx$.*

In [12], Choudhury introduced a generalization of C-contractions. Let (X, d) be a metric space. A mapping $T : X \to X$is said to be weakly C-contractive (or a weak C-contraction) whenever

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \psi((d(x, Ty), d(y, Tx)))$$

for all $x, y \in X$, where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. In [12], the author proved that if (X, d) is complete, then each weak C-contraction has a unique fixed point.

Theorem 2.7. *Let (X, p) be a TVS-CMS, $e \in intP$, $d_p = \xi_e \circ p$, T a selfmap on X such that $p(Tx, Ty) \leq \frac{1}{2}(p(x, Ty) + p(y, Tx)) - \varphi((p(x, Ty), p(y, Tx)))$ for all $x, y \in X$, where $\varphi : P \times P \to P$ is a mapping such that $\xi_e(\varphi(e)) > 0$. Then, there exists a continuous mapping $\phi : [0, \infty)^2 \to [0, \infty)$ such that*

$$d_p(Tx, Ty) \leq \frac{1}{2}(d_p(x, Ty) + d_p(y, Tx)) - \phi((d_p(x, Ty), d_p(y, Tx)))$$

for all $x, y \in X$ and $\phi(t, s) = 0$ if and only if $t = s = 0$.

Proof. Define $\phi(t, s) = \xi_e(\varphi(e))t + \xi_e(\varphi(e))s$ for all $t, s \geq 0$. It is clear that ϕ is continuous and $\phi(t, s) = 0$ if and only if $t = s = 0$. Also, an easy computation shows that $d_p(Tx, Ty) \leq \frac{1}{2}(d_p(x, Ty) + d_p(y, Tx)) - \phi((d_p(x, Ty), d_p(y, Tx)))$ holds for all $x, y \in X$. □

Thus, by using the result of [12], we obtain the following result.

Proposition 2.8. *Let (X, p) be a TVS-CMS, $e \in intP$, $d_p = \xi_e \circ p$, T a selfmap on X such that $p(Tx, Ty) \leq \frac{1}{2}(p(x, Ty) + p(y, Tx)) - \varphi((p(x, Ty), p(y, Tx)))$ for all $x, y \in X$, where $\varphi : P \times P \to P$ is a mapping such that $\xi_e(\varphi(e)) > 0$. Then, T has a unique fixed point.*

Remark 2.1. *By using similar proofs, one can also generalize some quasi-contraction type fixed point results to TVS-CMS. In fact, there are many fixed point results which we can generalize to TVS-CMS, but we should note that many published results are not so valuable in this way. Of course, verification of some fixed point results in TVS-CMS are considerable yet. For example, can we generalize Meir-Keeler type fixed point results to TVS-CMS ?*

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Received: October 13, 2011; Accepted: January 12, 2012.