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ON THE SET OF SOLUTIONS FOR THE DARBOUX PROBLEM FOR FRACTIONAL ORDER PARTIAL HYPERBOLIC FUNCTIONAL DIFFERENTIAL INCLUSIONS

SAÏD ABBAS* AND MOUFFAK BENCHOHRA**

*2320, Rue de Salaberry, apt 10
Montréal, QC H3M 1K9, Canada
E-mail: abbasmsaid@yahoo.fr

**Laboratoire de Mathématiques, Université de Sidi Bel-Abbès,
B.P. 89, 22000, Sidi Bel-Abbès, Algérie
E-mail: benchohra@univ-sba.dz

Abstract. In this paper we prove the arcwise connectedness of the solution set for the initial value problems (IVP for short), for a class of nonconvex and nonclosed functional hyperbolic differential inclusions of fractional order.

Key Words and Phrases: Partial hyperbolic differential inclusion; fractional order; left-sided mixed Riemann-Liouville integral; Caputo fractional derivative; solution set, arcwise connected.

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1. Introduction

The subject of fractional calculus is as old as the differential calculus since, starting from some speculations of G.W. Leibniz (1697) and L. Euler (1730), it has been developed up to nowadays. The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers. Fractional calculus techniques are widely used in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [15, 18, 22, 24, 26]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas *et al.* [19], Miller and Ross [25], Podlubny [27], Samko *et al.* [29], the papers of Abbas and Benchohra [1, 2, 3], Abbas *et al.* [4, 5], Belarbi *et al.* [6], Benchohra *et al.* [7, 8, 9], Diethelm [14], Kilbas and Marzan [20], Mainardi [22], Podlubny *et al* [28], Vityuk and Golushkov [30], Yu and Gao [31], Zhang [32] and the references therein.

Qualitative properties and structure of the set of solutions of the Darboux problem for hyperbolic partial integer order differential equations and inclusions have been studied by many authors; see for instance [10, 11, 12, 13, 16]. In the present article we are concerning by the arcwise connectedness of the solution set for the IVP, for the system

$$ {}^{c}D_{\theta}^{r}u(x,y) \in F\Big(x,y,u(x,y),G\Big(x,y,u(x,y)\Big)\Big);\ \text{if } (x,y)\in J:=[0,a]\times[0,b], \quad (1.1)$$

$$\begin{cases} u(x,0)=\varphi(x);\ x\in[0,a],\\ u(0,y)=\psi(y);\ y\in[0,b],\\ \varphi(0)=\psi(0), \end{cases} \quad (1.2)$$

where $a,b>0$, $\theta=(0,0)$, ${}^{c}D_{\theta}^{r}$ is the standard Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $F:J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$, $G:J\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ are given multifunctions, $\mathcal{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$, $\varphi:[0,a]\to\mathbb{R}^n$ and $\psi:[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions.

The paper is organized as follows: in *Section 2* we recall some preliminary results about the theory of Banach spaces, and we recall some basic definitions and facts on partial fractional calculus theory. We give also some properties of set-valued maps. *Section 3* is devoted to present some auxiliary lemmas that we use in the sequel. In *Section 4* we prove our main result for the problem (1.1)-(1.2). The present result extends those considered with integer order derivative [10, 11, 16].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_{\infty}=\sup_{(x,y)\in J}\|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebesgue-integrable functions $w:J\to\mathbb{R}^n$ with the norm

$$\|w\|_1=\int_0^a\int_0^b\|w(x,y)\|dydx.$$

By $\mathcal{M}$ we mean the linear subspace of $C(J)$ consisting of all $\mu\in C(J)$ such that, there exist functions $\varphi\in AC([0,a])$ and $\psi\in AC([0,b])$ with $\varphi(0)=\psi(0)$, satisfying

$$\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0),\ \text{for every } (x,y)\in J.$$

It is clear that $(\mathcal{M},\|.\|_{\infty})$ is a separable Banach space.

Given a continuous function $d:J\to(0,\infty)$, we denote by L^1 the Banach space of all (equivalence class of) Lebesgue measurable functions $w:J\to\mathbb{R}^n$, endowed with the norm

$$\|w\|_{L^1}=\int_0^a\int_0^b d(x,y)\|w(x,y)\|dydx. \quad (2.1)$$

Consider $H_d : \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(A,v)\right\},$$

where $d(A,v) = \inf_{u\in A}\|u-v\|,\ d(u,B) = \inf_{v\in B}\|u-v\|$. Then $(\mathcal{P}_{cl}(\mathbb{R}^n), H_d)$ is a generalized metric space (see [21]). Here $\mathcal{P}_{cl}(\mathbb{R}^n) = \{Y \in \mathcal{P}(\mathbb{R}^n) : Y \text{ closed}\}$.

Definition 2.1 A multivalued map $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is closed valued if $T(u)$ is closed for all $u \in \mathbb{R}^n$. T is called Hausdorff continuous if for any $u_0 \in \mathbb{R}^n$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $u \in \mathbb{R}^n,\ \|u-u_0\| < \delta$ implies $H_d(T(u), T(u_0)) < \epsilon$. A multivalued map $G : \mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $v \in \mathbb{R}^n$, the function $u \longmapsto d(v, G(u)) = \inf\{\|v-z\| : z \in G(u)\}$ is measurable. The range of G is the set $G(\mathbb{R}^n) = \bigcup_{u\in\mathbb{R}^n} G(u)$.

Definition 2.2 Let $\mathcal{A}$ be a subset of $J\times\mathbb{R}^n$. $\mathcal{A}$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if $\mathcal{A}$ belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{N}\otimes\mathcal{D}$ where $\mathcal{N}$ is Lebesgue measurable in J and $\mathcal{D}$ is Borel measurable in $\mathbb{R}^n$. A subset $\mathcal{I}$ of $L^1(J,\mathbb{R}^n)$ is decomposable if for all $u, v \in \mathcal{I}$ and $N \subset J$ measurable, $u\chi_{\mathcal{I}} + v\chi_{J-\mathcal{N}} \in \mathcal{I}$, where χ_J stands for the characteristic function of J.

Definition 2.3 A space X is said to be arcwise connected if any two distinct points can be joined by an arc, that is a path f which is a homeomorphism between the unit interval $[0,1]$ and its image $f([0,1])$. A metric space Z is called absolute retract if, for any metric space X and any $X_0 \in \mathcal{P}_{cl}(X)$, every continuous function $g : X_0 \to Z$ has a continuous extension $g : X \to Z$ over X.

Every continuous image of an absolute retract is an arcwise connected space.

Now, we introduce notations and definitions concerning to partial fractional calculus theory.

Definition 2.4 [28, 30] Let $\theta = (0,0),\ r_1, r_2 \in (0,\infty)$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds,$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi > 0$.

In particular,

$$(I_\theta^\theta f)(x,y) = f(x,y),\ (I_\theta^\sigma f)(x,y) = \int_0^x\int_0^y f(s,t)\,dt\,ds; \text{ for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r f$ exists for all $r_1, r_2 \in (0,\infty)$, when $f \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r f) \in \mathcal{C}$, moreover

$$(I_\theta^r f)(x,0) = (I_\theta^r f)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 2.5 Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2}; \text{ for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.6 [28, 30] Let $r \in (0,1] \times (0,1]$ and $f \in L^1(J)$. The Caputo fractional-order derivative of order r of f is defined by the expression

$${}^cD_\theta^r f(x,y) = (I_\theta^{1-r} D_{xy}^2 f)(x,y) = \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y \frac{D_{st}^2 f(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dtds.$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma f)(x,y) = (D_{xy}^2 f)(x,y); \text{ for almost all } (x,y) \in J.$$

Example 2.7 Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$${}^cD_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}; \text{ for almost all } (x,y) \in J.$$

3. Auxiliary Results

For the prove of our main result for the problem (1.1)-(1.2) we need the following preliminary lemmas.

Lemma 3.1 [1] *Let $f \in E$. A function $u \in AC(J)$ such that its mixed derivative D_{xy}^2 exists and is integrable on J is a solution of problem*

$$\begin{cases} ({}^cD_\theta^r u)(x,y) = f(x,y); \ (x,y) \in J, \\ u(x,0) = \varphi(x); \ x \in [0,a], \\ u(0,y) = \psi(y); \ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if u satisfies

$$u(x,y) = \mu(x,y) + (I_\theta^r f(x,y); \ (x,y) \in J,$$

where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Let S be a separable Banach space. Set $E := L^1(J)$.

Lemma 3.2 [23] *Assume that $G : S \times E \to \mathcal{P}_{cl}(E)$ and $F : S \times E \times E \to \mathcal{P}_{cl}(E)$ are Hausdorff continuous multifunctions with decomposable values, satisfying the following conditions:*

a) *There exists $L \in [0,1)$ such that, for every $s \in S$ and every $u, u' \in E$,*

$$H_d(G(s,u), G(s,u')) \leq L\|u-u'\|_1.$$

b) *There exists* $M \in [0,1)$ *such that* $L+M<1$ *and for every* $s \in S$ *and every* $u,v,w,z \in E$.

$$H_d(F(s,u,z),F(s,v,w))\| \leq M(\|u-v\|_1 + \|z-w\|_1).$$

Set $Fix(\Gamma(s,.)) = \{u \in E : u \in \Gamma(s,u)\}$, *where*

$$\Gamma(s,u) = F(s,u,G(s,u));\ (s,u) \in S \times E.$$

Then

1) *For every* $s \in S$ *the set* $Fix(\Gamma(s,.))$ *is nonempty and arcwise connected.*
2) *For any* $s_i \in S$, *and any* $u_i \in Fix(\Gamma(s,.))$; $i = 1,\dots,p$ *there exists a continuous function* $\gamma(s) \in Fix(\Gamma(s,.))$ *for all* $s \in S$ *and* $\gamma(s_i) = u_i$; $i = 1,\dots,p$.

Lemma 3.3 [23] *Let* $U : J \to \mathcal{P}_{cl}(\mathbb{R}^n)$ *and* $V : J \times \mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ *be two multifunctions, satisfying the following conditions:*

a) U *is measurable and there exists* $\rho \in E$ *such that* $H_d(U(x,y)),\{0\}) \leq \rho(x,y)$ *for almost all* $(x,y) \in J$,
b) *The multifunction* $(x,y) \to V(x,y,u)$ *is measurable for every* $u \in E$,
c) *The multifunction* $u \to V(x,y,u)$ *is Hausdorff continuous for all* $(x,y) \in J$.

Let $v : J \to \mathbb{R}^n$ *be a measurable selection from* $(x,y) \to V(x,y,U(x,y))$. *Then there exists a selection* $u \in E$ *such that* $v(x,y) \in V(x,y,u(x,y))$; $(x,y) \in J$.

Lemma 3.4 [11] *Let* $\zeta \in (0,1)$ *and let* $N : J \to [0,\infty)$ *be integrable function. Then there exists a continuous function* $d : J \to (0,\infty)$ *which, for every* $(x,y) \in J$, *satisfies*

$$\int_x^a \int_y^b N(s,t)d(s,t)dtds = \zeta(d(x,y)-1).$$

4. The Main Result

Let us start by defining what we mean by a solution of the problem (1.1)-(1.2).

Definition 4.1 A function $u \in C(J)$ is said to be a solution of (1.1)-(1.2) if there exists $\sigma \in L^1$ such that

$$\sigma(x,y) \in F(x,y,u(x,y),G(x,y,u(x,y)));\ a.e.(x,y) \in J,$$

$$u(x,y) = \mu(x,y) + I_\theta^r \sigma(x,y);\ a.e.(x,y) \in J,$$

where $F(x,y,u,G(x,y,u)) = \bigcup_{v \in G(x,y,u)} F(x,y,u,v)$, and u satisfies conditions (1.2) on J.

Further, we present conditions for some meaningful properties of the solution set of problem (1.1)-(1.2).

Theorem 4.2 *Let* $G : J \times \mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ *and* $F : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ *be two set-valued maps, satisfying the following assumptions:*

(H_1) *The set-valued maps* $(x,y) \to F(x,y,u,v)$ *and* $(x,y) \to G(x,y,u)$ *are measurable for all* $u,v \in \mathbb{R}^n$,

(H_2) *There exists a positive integrable function* $l : J \to \mathbb{R}$ *such that, for every* $u,u' \in \mathbb{R}^n$,

$$H_d(G(x,y,u),G(x,y,u')) \leq l(x,y)\|u-u'\|;\ a.e.\ (x,y) \in J,$$

(H_3) *There exists a positive integrable function* $m : J \to \mathbb{R}$ *and* $\eta \in [0,1]$ *such that, for every* $u, v, u', v' \in \mathbb{R}^n$,

$$H_d(F(x,y,u,v), G(x,y,u',v)) \leq m(x,y)\|u-u'\| + \eta(\|v-v'\|; \ a.e. \ (x,y) \in J,$$

(H_4) *There exist positive integrable functions* $f, g : J \to \mathbb{R}$ *such that,*

$$H_d(\{0\}, F(x,y,\{0\},\{0\})) \leq f(x,y), \ \ a.e. \ (x,y) \in J,$$

and

$$H_d(\{0\}, G(x,y,\{0\})) \leq g(x,y); \ \ a.e. \ (x,y) \in J.$$

Then,

1) *For every* $\mu \in \mathcal{M}$, *the solution set* $S(\mu)$ *of problem (1.1)-(1.2) is nonempty and arcwise connected in the space* $C(J)$.
2) *For any* $\mu_i \in \mathcal{M}$ *and any* $u_i \in S(\mu)$; $i = 1, \dots, p$, *there exists a continuous function* $s : \mathcal{M} \to C(J)$ *such that* $s(\mu) \in S(\mu)$ *for any* $\mu \in \mathcal{M}$ *and* $s(\mu_i) = u_i$; $i = 1, \dots, p$.
3) *The set* $S = \cup_{\mu \in \mathcal{M}} S(\mu)$ *is arcwise connected in the space* $C(J)$.

Proof. In what follows

$$N(x,y) = max\{l(x,y), m(x,y); \ (x,y) \in J\}$$

and take $\zeta \in (0,1)$ such that $2\zeta + \eta < 1$ and $d : J \to (0,\infty)$ in (2.1) is the corresponding mapping found in Lemma 3.4.

1) For $\mu \in \mathcal{M}$ and $u \in L^1$, set

$$u_\mu(x,y) = \mu(x,y) + (I_\theta^r u)(x,y); \ (x,y) \in J.$$

Define the multifunctions $\alpha : \mathcal{M} \times L^1 \to \mathcal{P}(L^1)$ and $\beta : \mathcal{M} \times L^1 \times L^1 \to \mathcal{P}(L^1)$ by

$$\alpha(\mu,u) = \{v \in L^1 : v(x,y) \in G(x,y,u_\mu(x,y)); \ a.e. \ (x,y) \in J\},$$

$$\beta(\mu,u,v) = \{w \in L^1 : w(x,y) \in F(x,y,u_\mu(x,y),v(x,y)); \ a.e. \ (x,y) \in J\},$$

where $\mu \in \mathcal{M}$ and $u, v \in L^1$. We prove that α and β satisfy the hypotheses of Lemma 3.2.

Since u_μ is measurable and G satisfies hypotheses (H_1) and (H_2), the multifunction $G_\mu : (x,y) \to G(x,y,u_\mu(x,y))$ is measurable and $G_\mu \in \mathcal{P}_{cl}(L^1)$, G_μ has a measurable selection. Therefore due to hypothesis (H_4), we get $\alpha(\mu,u) \neq \emptyset$. Also, by simple computation, it follows that the set $\alpha(\mu,u)$ is closed and decomposable. In the same way we obtain that $\alpha(\mu,u) \in \mathcal{P}_{cl}(L^1)$ is a decomposable set.

Set $d := \int_0^a \int_0^b d(x,y)dydx$. Pick $(\mu,u), (\mu_1,u_1) \in \mathcal{M} \times L^1$ and choose $v \in \alpha(\mu,u)$. For each $\epsilon > 0$ there exists $v_1 \in \alpha(\mu_1,u_1)$ such that, for every $(x,y) \in J$ one has

$$\begin{aligned}
&\|v(x,y) - v_1(x,y)\| \\
\leq\ & H_d\Big(G(x,y,u_\mu(x,y)), G(x,y,u_{\mu_1}(x,y))\Big) + \epsilon \\
\leq\ & N(x,y)\Big(\|\mu(x,y) - \mu_1(x,y)\| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t) - u_1(s,t)\|dtds\Big) + \epsilon.
\end{aligned}$$

Hence, for any $\epsilon > 0$,

$$\begin{aligned}\|v-v_1\|_{L^1} &\leq \|\mu-\mu_1\|_\infty \int_0^a\int_0^b d(x,y)N(x,y)dydx \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b d(x,y)N(x,y) \\ &\times\Big(\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t)-u_1(s,t)\|dtds\Big)dydx+\epsilon d \\ &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b \|u(s,t)-u_1(s,t)\| \\ &\times\Big(\int_s^a\int_t^b (x-s)^{r_1-1}(y-t)^{r_2-1}d(x,y)N(x,y)dydx\Big)dtds+\epsilon d \\ &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b \|u(s,t)-u_1(s,t)\| \\ &\times\Big(\int_s^a\int_t^b (x-s)^{\frac{r_1-1}{1-r_3}}(y-t)^{\frac{r_2-1}{1-r_3}}dydx\Big)^{1-r_3} \\ &\times\Big(\int_s^a\int_t^b d^{\frac{1}{r_3}}(x,y)N^{\frac{1}{r_3}}(x,y)dydx\Big)^{r_3}dtds+\epsilon d,\end{aligned}$$

where $0 < r_3 < \min\{r_1, r_2\}$. Then, for any $\epsilon > 0$,

$$\begin{aligned}\|v-v_1\|_{L^1} &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty+\frac{a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \\ &\times\int_0^a\int_0^b \zeta^{r_3}\Big(d^{\frac{1}{r_3}}(s,t)N^{\frac{1-r_3}{r_3}}(s,t)-1\Big)^{r_3}\|u(s,t)-u_1(s,t)\|dtds+\epsilon d,\end{aligned}$$

where $\omega_1 = \frac{r_1-1}{1-r_3}$, $\omega_2 = \frac{r_2-1}{1-r_3}$. Thus, for any $\epsilon > 0$,

$$\begin{aligned}\|v-v_1\|_{L^1} &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty+\frac{a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \\ &\times\zeta^{r_3}N^*\|u-u_1\|_{L^1}+\epsilon d,\end{aligned}$$

where $N^* = \sup\limits_{(x,y)\in J} N^{\frac{1-r_3}{r_3}}(x,y)$. Hence, for any $\epsilon > 0$,

$$\begin{aligned}\|v-v_1\|_{L^1} &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty \\ &+\frac{\zeta^{r_3}N^*a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\|u-u_1\|_{L^1}+\epsilon d.\end{aligned}$$

This implies that,

$$\begin{aligned}d_{L^1}(v,\alpha(\mu_1,u_1)) &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty \\ &+\frac{\zeta^{r_3}N^*a^{(\omega_1+1)(1-r_3)}b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}\|u-u_1\|_{L^1},\end{aligned}$$

for all $v \in \alpha(\mu, u)$. Therefore,

$$\begin{aligned} d_{L^1}(\alpha(\mu,u),\alpha(\mu_1,u_1)) &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty \\ &\quad + \frac{\zeta^{r_3} N^* a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}. \end{aligned}$$

Consequently,

$$\begin{aligned} H_d(\alpha(\mu,u),\alpha(\mu_1,u_1)) &\leq \zeta(d(a,b)-1)\|\mu-\mu_1\|_\infty \\ &\quad + \frac{\zeta^{r_3} N^* a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}, \end{aligned}$$

which shows that α is Hausdorff continuous and satisfies the assumptions of Lemma 3.2. Also, by the same method, we obtain that the multifunction β is Hausdorff continuous and satisfies the assumptions of Lemma 3.2.
Define $\Gamma(\mu,u) = \beta(\mu,u,\alpha(\mu,u))$; $(\mu,u) \in \mathcal{M} \times L^1$. According to Lemma 3.2, the set $Fix(\Gamma(s,.) = \{u \in E : u \in \Gamma(s,u)\}$ is nonempty and arcwise connected in L^1. Moreover, for fixed $\mu_i \in \mathcal{M}$ and $u_i \in Fix(\Gamma(\mu_i,.))$; $i = 1,...,p$, there exists a continuous function $\gamma : \mathcal{M} \to L^1$ such that

$$\gamma(\mu) \in Fix(\Gamma(\mu,.); \ for\ all\ \mu \in \mathcal{M}, \tag{4.1}$$

$$\gamma(\mu) = u_i; \ i = 1,...,p. \tag{4.2}$$

We shall prove that

$$\begin{aligned} &Fix(\Gamma(\mu,.)) \\ &= \{u \in L^1 : u(x,y) \in F(x,y,u_\mu(x,y),G(x,y,u_\mu(x,y))); \ a.e.\ (x,y) \in J\}. \end{aligned} \tag{4.3}$$

Denote by $A(\mu)$ the right-hand side of (4.3). If $u \in Fix(\Gamma(\mu,.))$ then there is $v \in \alpha(\mu,v)$ such that $u \in \beta(\mu,u,v)$. Therefore, $v(x,y) \in G(x,y,u_\mu(x,y))$ and

$$u(x,y) \in F(x,y,u_\mu(x,y),v(x,y)) \subset F(x,y,u_\mu(x,y),G(x,y,u_\mu(x,y))); \ a.e.\ (x,y) \in J,$$

so that $Fix(\Gamma(\mu,.)) \subset A(\mu)$. Let now $u \in A(\mu)$. By Lemma 3.3, there exists a selection $v \in L^1$ of the multifunction $(x,y) \to G(x,y,u_\mu(x,y))$ satisfying

$$u(x,y) \in F(x,y,u_\mu(x,y),v(x,y)); \ a.e.\ (x,y) \in J.$$

Hence $v \in \alpha(\mu,v)$ and $u \in \beta(\mu,u,v)$ and thus $u \in \Gamma(\mu,u)$, which implies that $A(\mu) \subset Fix(\Gamma(\mu,.))$ and so that (4.3).

We next note that the function $T : L^1 \to C(J)$,

$$T(u)(x,y) = I_\theta^r u(x,y); \ (x,y) \in J$$

is continuous and one has

$$S(\mu) = \mu + T(Fix(\Gamma(\mu,.))); \ \mu \in \mathcal{M}. \tag{4.4}$$

Since $Fix(\Gamma(\mu,.))$ is nonempty and arcwise connected in L^1, the set $S(\mu)$ has the same properties in $C(J)$.

2) Let $\mu_i \in \mathcal{M}$ and let $u_i \in S(\mu_i)$; $i = 1,...,p$ be fixed. By (4.4) there exists $v_i \in Fix(\Gamma(\mu_i,.))$ such that

$$u_i = \mu_i + T(v_i); \ i = 1,...,p.$$

If $\gamma : \mathcal{M} \to L^1$ is a continuous function satisfying (4.1) and (4.2) we define, for every $\mu \in \mathcal{M}$,

$$S(\mu) = \mu + T(\gamma(\mu)).$$

Obviously, the function $s : \mathcal{M} \to C(J)$ is continuous, $s(\mu) \in S(\mu)$ for all $\mu \in \mathcal{M}$, and

$$S(\mu_i) = \mu_i + T(\gamma(\mu_i)) = \mu_i + T(v_i) = u_i;\ i = 1, ..., p.$$

3) Let $u_1, u_2 \in S = \bigcup_{\mu\in\mathcal{M}} S(\mu)$ and choose $\mu_i \in \mathcal{M};\ i = 1, 2$ such that $u_i \in S(\mu_i);\ i = 1, 2$. From the conclusion of **2)** we deduce the existence of a continuous function $s : \mathcal{M} \to S(J)$ satisfying $s(\mu_i) = u_i;\ i = 1, 2$ and $s(\mu) \in S(\mu);\ \mu \in \mathcal{M}$. Let $h : [0, 1] \to \mathcal{M}$ be a continuous function such that $h(0) = \mu_1$ and $h(1) = \mu_2$. Then the function $s \circ h : [0, 1] \to C(J)$ is continuous and verifies

$$s \circ h(0) = u_1,\ s \circ h(1) = u_2,$$

$$s \circ h(\mu) \in S(h(\mu)) \subset S;\ \mu \in \mathcal{M}.$$

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