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NEARLY INVOLUTIONS ON BANACH ALGEBRAS. A FIXED POINT APPROACH

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Abstract. Using fixed point methods, we investigate the Hyers–Ulam–Rassias stability and superstability of involutions on Banach algebras. Moreover, we show that under some conditions on an approximate involution, the Banach algebra has a C^*-algebra structure.
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1. Introduction

The *stability problem* of functional equations started with the question concerning stability of group homomorphisms proposed by S.M. Ulam [34] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison. In 1941, Hyers [18] gave a first affirmative answer to the question of Ulam for Banach spaces as follows:

If E and $E^{'}$ are Banach spaces and $f : E \longrightarrow E^{'}$ is a mapping for which there is $\varepsilon > 0$ such that
$\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all $x, y \in E$, then there is a unique additive mapping $L : E \longrightarrow E^{'}$ such that $\|f(x) - L(x)\| \le \varepsilon$ for all $x \in E$.

Hyers' theorem was generalized by Rassias [30] for linear mappings by considering an unbounded Cauchy difference.

The paper of Rassias [31] has provided a lot of influence in the development of what we now call the *generalized Hyers–Ulam stability* or as *Hyers–Ulam–Rassias stability* of functional equations. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems, the reader refer to [3]–[23], [27, 28] and [32] .

In 1991 J.A.Baker [1] used the Banach fixed point theorem to give Hyers–Ulam stability results for a nonlinear functional equation. In 2003, V.Radu [29] applied the fixed point alternative theorem for Hyers– Ulam–Rassias stability. D. Mihet [25] applied the Luxemburg–Jung fixed point theorem in generalized metric spaces to study the Hyers– Ulam stability for two functional equations in a single variable and L. Gavruta [15] used the Matkowski's fixed point theorem to obtain a new general result concerning the Hyers–Ulam stability of a functional equation in a single variable. In 2003 Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [5]. They could present a short and a simple proof (different of the "*direct method*", initiated by Hyers in 1941) for the generalized Hyers–Ulam stability of Jensen functional equation [5], for Cauchy functional equation [2].

In this paper, by using fixed point methods, we prove that if there is an approximately involution $f : A \to A$ on Banach algebra A, then there exists an involution $I : A \to A$ which is near to f. Moreover, under some conditions on f, the algebra A has a C^*-algebra structure with involution I.

Throughout this paper assume that $n_0 \in \mathbb{N}$ is a positive integer. Suppose that $\mathbb{T}^1 := \{z \in \mathbb{C} : |z| = 1\}$ and that $\mathbb{T}^1_{\frac{1}{n_o}} := \{e^{i\theta};\ 0 \le \theta \le \frac{2\pi}{n_o}\}$. It is easy to see that $\mathbb{T}^1 = \mathbb{T}^1_{\frac{1}{1}}$. Moreover, we suppose that A is a Banach algebra. For a given mapping $f : A \to A$, we define

$$D_\mu f(x, y) = 2\bar{\mu} f(\frac{x + y}{2}) + 2\bar{\mu} f(\frac{x - y}{2}) - 2f(\mu x)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y \in A$.

We refer the reader to [26] for more information on C^*-algebras.

2. Main results

Before proceeding to the main results, we recall the following theorem by Margolis and Diaz.

Theorem 2.1. *(The alternative of fixed point* [24, 33]*). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$d(T^m x, T^{m+1}x) = \infty$ *for all* $m \ge 0$,

or other exists a natural number m_0 such that

⋆ $d(T^m x, T^{m+1}x) < \infty$ *for all* $m \ge m_0$;

⋆ *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*

⋆ *y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;*

⋆ $d(y, y^*) \le \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

Theorem 2.2. *Let $f : A \to A$ be a mapping, for which there exists a function $\phi : A^2 \to [0, \infty)$ such that*

$$\|D_\mu f(x, y)\| \le \phi(x, y), \tag{2.1}$$

$$\|f(xy) - f(y)f(x)\| \le \phi(x, y), \tag{2.2}$$

$$\lim_m 2^{-m} f(2^m \lim_n 2^{-n} f(2^n x)) = x \tag{2.3}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ *and all* $x, y \in A$. *If there exists an* $L < 1$ *such that* $\phi(x, y) \leq 2L\phi(\frac{x}{2}, \frac{y}{2})$ *for all* $x, y \in A$, *then there exists a unique involution* $I : A \to A$ *such that*

$$\|f(x) - I(x)\| \leq \frac{L}{2-L}\phi(x, 0) \tag{2.4}$$

for all $x \in A$. *Moreover, if*

$$|\|xf(x)\| - \|x\|^2| \leq \phi(x, x) \tag{2.5}$$

for all $x \in A$, *then* A *is a* C^*-*algebra with involution* $x^* = I(x)$ *for all* $x \in A$.

Proof. Putting $\mu = 1, y = 0$ in (2.1), we get

$$\|4f(\frac{x}{2}) - 2f(x)\| \leq \phi(x, 0) \tag{2.6}$$

for all $x \in A$. Hence,

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{4}\phi(2x, 0) \leq \frac{L}{2}\phi(x, 0) \tag{2.7}$$

for all $x \in A$. Consider the set $X := \{g \mid g : A \to B\}$ and introduce the generalized metric on X:

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0) \text{ for all } x \in A\}.$$

It is easy to show that (X, d) is complete. Now we define mapping $J : X \to X$ by

$$J(h)(x) = \frac{1}{2}h(2x)$$

for all $x \in A$. By definition of d and inequality $\phi(x, y) \leq 2L\phi(\frac{x}{2}, \frac{y}{2})$, one can show that

$$d(J(g), J(h)) \leq \frac{L}{2}d(g, h)$$

for all $g, h \in X$. It follows from (2.7) that

$$d(f, J(f)) \leq \frac{L}{2}.$$

By Theorem 2.1, J has a unique fixed point in the set $X_1 := \{I \in X : d(f, I) < \infty\}$. Let I be the fixed point of J. Also, we have $\lim_n d(J^n(f), I) = 0$. It follows that

$$\lim_n \frac{1}{2^n}f(2^n x) = I(x) \tag{2.8}$$

for all $x \in A$. It follows from $d(f, I) \leq \frac{1}{1-\frac{L}{2}}d(f, J(f))$ that

$$d(f, I) \leq \frac{L}{2-L}.$$

This implies the inequality (2.4). By inequality $\phi(x, y) \leq 2L\phi(\frac{x}{2}, \frac{y}{2})$, we have

$$\lim_j 2^{-j}\phi(2^j x, 2^j y) = 0 \tag{2.9}$$

for all $x, y \in A$. It follows from (2.1), (2.8) and (2.9) that

$$\begin{aligned}&\left\|2I(\frac{x+y}{2})+2I(\frac{x-y}{2})-2I(x)\right\|\\&=\lim_n \frac{1}{2^n}\|I(r^{n-1}(x+y))+I(2^{n-1}(x-y))-I(2^n x)\|\\&\le \lim_n \frac{1}{2^n}\phi(2^n x, 2^n y)=0\end{aligned}$$

for all $x, y \in A$. So

$$2I(\frac{x+y}{2})+2I(\frac{x-y}{2})=2I(x)$$

for all $x, y \in A$. Hence, I is Cauchy additive. By putting $y = x$ in (2.1), we have

$$\|2\bar{\mu}f(\frac{2x}{2})-2f(\mu x)\| \le \phi(x,x)$$

for all $x \in A$. This implies that

$$\|I(2\mu x)-2\bar{\mu}I(x)\|=\lim_n \frac{1}{2^n}\|f(2\mu 2^n x)-2\bar{\mu}f(2^n x)\| \le \lim_n \frac{1}{2^n}\phi(2^n x, 2^n x)=0$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$, $x \in A$. It follows by the last equation and additivity of I that $I(\mu x) = \bar{\mu} I(x)$ for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x \in A$. Now, we show that I is conjugate linear. We have to show that $I(\lambda x) = \bar{\lambda} I(x)$ for all $\lambda \in \mathbb{C},\ x \in X$. To this end, let $\lambda \in \mathbb{C}$. If λ belongs to $\mathbb{T}^1$, then there exists $\theta \in [0, 2\pi]$ such that $\lambda = e^{i\theta}$. We set $\lambda_1 = e^{\frac{i\theta}{n_o}}$, thus λ_1 belongs to $\mathbb{T}^1_{\frac{1}{n_o}}$ and $I(\lambda x) = I(\lambda_1^{n_o} x) = \bar{\lambda}_1^{n_o} I(x) = \bar{\lambda} I(x)$ for all $x \in X$.
If λ belongs to $n\mathbb{T}^1 = \{nz\ ;\ z \in \mathbb{T}^1\}$ for some $n \in \mathbb{N}$, then by additivity of I, $I(\lambda x) = \bar{\lambda} I(x)$ for all $x \in X$.
Let $t \in (0, \infty)$ then by archimedean property of $\mathbb{C}$, there exists a positive real number n such that the point $(t, 0)$ lies in the interior of circle with center at origin and radius n. Putting $t_1 := t + \sqrt{n^2 - t^2}\ i$, $t_2 := t - \sqrt{n^2 - t^2}\ i$. Then we have $t = \frac{t_1 + t_2}{2}$ and $t_1, t_2 \in n\mathbb{T}^1$. It follows that

$$I(tx)=I(\frac{t_1+t_2}{2}x)=\frac{\bar{t_1}}{2}I(x)+\frac{\bar{t_2}}{2}I(x)=\bar{t}I(x)=tI(x)$$

for all $x \in X$.
On the other hand, there exists $\theta \in [0, 2\pi]$ such that $\lambda = |\lambda| e^{i\theta}$. It follows that

$$I(\lambda x)=I(|\lambda|e^{i\theta}x)=|\lambda|I(e^{i\theta}x)=|\lambda|e^{-i\theta}I(x)=\bar{\lambda}I(x)$$

for all $x \in X$. Hence $I : A \to A$ is conjugate $\mathbb{C}-$linear. It follows from (2.2) that

$$\begin{aligned}&\|I(xy)-I(y)I(x)\|\\&=\lim_n \left\|\frac{1}{2^n}I((2^n x)y)-(I(y)I(x))\right\|\\&\le \lim_n \frac{1}{2^{2n}}\phi(2^n x, 2^n x) \le \lim_n \frac{1}{2^n}\phi(2^n x, 2^n x)\\&=0\end{aligned}$$

for all $x, y \in A$. This means that

$$I(xy) = I(y)I(x)$$

for all $x, y \in A$. On the other hand by (2.3)

$$I(I(x)) = \lim_m 2^{-m} f(2^m \lim_n 2^{-n} f(2^n x)) = x$$

for all $x \in A$. Hence $I : A \to A$ is an involution satisfying (2.4). To prove the uniqueness property of I, let $I' : A \to A$ be an involution satisfies (2.4). according to (2.4),

$$\|I(x) - I'(x)\| = \lim_n \| \frac{1}{2^n} f(2^n x) - \frac{1}{2^{2n}} I'(2^n x)\| \leq \lim_n \frac{1}{2^n} (\frac{L}{1-L}) \phi(2^n x, 0) = 0$$

for all $x, y \in A$. This means that $I = I'$.
Now, suppose I satisfies (2.5). Then we have

$$\begin{aligned}
&|\|xI(x)\| - \|x\|^2| \\
&= \lim_n | \left\| \frac{1}{2^{2n}} (2^n x) \frac{1}{2^{2n}} f(2^n x) \right\| - \frac{1}{2^{4n}} \|2^n x\|^2 | \\
&\leq \lim_n \frac{1}{2^{4n}} \phi(2^n x, 2^n x) \leq \lim_n \frac{1}{2^n} \phi(2^n x, 2^n x) \\
&= 0
\end{aligned}$$

for all $x \in A$. Hence A is a C^*-algebra with involution $x^* = I(x)$ for all $x \in A$. □

We prove the following Hyers–Ulam–Rassias stability problem for involutions on Banach algebras.

Corollary 2.3. *Let $p \in (0, 1)$ and $\theta \in [0, \infty)$ be real numbers. Suppose $f : A \to A$ with $f(1) = 1$, satisfies*

$$\|D_\mu f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p),$$
$$\|f(xy) - f(y)f(x)\| \leq \theta(\|x\|^p + \|y\|^p),$$
$$\lim_m 2^{-m} f(2^m \lim_n 2^{-n} f(2^n x)) = x$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y \in A$. Then there exists a unique involution $I : A \to A$ such that

$$\|f(x) - I(x)\| \leq \frac{2^{p-1}\theta}{2 - 2^{p-1}} \|x\|^p$$

for all $x \in A$. Moreover, if

$$|\|xf(x)\| - \|x\|^2| \leq 2\theta \|x\|^p$$

for all $x \in A$, then A is a C^*-algebra with involution I.

Proof. It follows from Theorem 2.2 by putting $\phi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in A$, and $L = 2^{p-1}$. □

As a consequence of Theorem 2.2, we obtain the superstability of involutions on Banach algebras as follow.

Corollary 2.4. *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Suppose $f : A \to A$ satisfies*

$$\|D_\mu f(x,y)\| \le \theta(\|x\|^p\|y\|^p),$$
$$\|f(xy) - f(y)f(x)\| \le \theta(\|x\|^p\|y\|^p),$$
$$\lim_m 2^{-m} f(2^m \lim_n 2^{-n} f(2^n x)) = x$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y \in A$. Then f is an involution on A. Moreover, if

$$|\|xf(x)\| - \|x\|^2| \le \theta\|x\|^{2p}$$

for all $x \in A$, then A is a C^*-algebra with involution f.

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