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COMMON COUPLED FIXED POINT RESULTS FOR HYBRID NONLINEAR CONTRACTIONS IN METRIC SPACES

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Abstract. The concept of triangular function is introduced and two kind of hybrid nonlinear contractions involving a gauge function and a triangular function are considered. Several new common coupled fixed point theorems are established in complete metric spaces, and error estimates for iterations to approximate a fixed point are given. The presented results are general because the triangular function is abstract. As applications the existence and uniqueness of the common coupled solutions for a differential system and a integral system are proved respectively.
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1. Introduction

The Banach contraction principle is a very popular tool in solving unique existence problems in many branches of mathematical analysis and has many generalizations (see [5,6,19]). Boyd and Wong [4] extended the Banach contraction principle to the case of nonlinear contraction mappings involving a gauge function, and obtained some fixed point theorems by weakening the hypothesis on the gauge function. Coupled fixed points and their applications for binary mappings were considered by Bhaskar and Lakshmikantham [3]. Recently, some new results of coupled fixed points were presented in the case of ordered metric spaces (for example, see [1,2,7-9,12-18]).

Motivated by the work of [1-4,7-9,12-18], in this paper, we mainly focus on a binary mapping and an one-variable mapping satisfying some hybrid nonlinear contraction in usual metric spaces without order, and establish several now common coupled fixed point theorems. Also, we give some error estimates for iterations to approximate a fixed point, and present some applications to differential systems and integral systems.

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Throughout this paper, let $\mathbb{Z}^+$ be the set of all positive integers, $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and

$$\varphi^{-1}(\{0\}) = \{t \in \mathbb{R}^+ : \varphi(t) = 0\}.$$

If $\varphi^{-1}(\{0\}) = \{0\}$, then φ is called a gauge function. For $t \in \mathbb{R}^+$, by $\varphi^n(t)$ we denote the nth iteration of $\varphi(t)$. Recall that (cf. [10]) if X be a non-empty set, $T : X \times X \to X$ and $A : X \to X$ are two mappings, then an element $(u, v) \in X \times X$ is called a coupled coincidence point of T and A if $T(u, v) = Au$ and $T(v, u) = Av$; an element $u \in X$ is called a common coupled fixed point of T and A if $u = Au = T(u, u)$. A is said to be commutative with T if $AT(x, y) = T(Ax, Ay)$ for all $x, y \in X$.

Our main results given in Section 2. We introduce the concept of triangular function, and study two kind of hybrid nonlinear contractions involving a triangular function and a gauge function. In order to show the relevance and applicability of our results, several examples are given in Section 3.

2. Main results

Definition 2.1. A binary operation $\odot : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a triangular function if the following conditions are satisfied for any $a, b, c, d \in \mathbb{R}^+$:
(1) $a \odot b = b \odot a$;
(2) $a \odot b \le c \odot d$, whenever $a \le c$ and $b \le d$;
(3) $a \odot a \le a$.

A triangular function is said to be continuous if $a_n \odot b_n \to a \odot b$, whenever $a_n \to a$ and $b_n \to b$. A triangular function is said to be strictly increasing if $a \odot b < c \odot b$, whenever $a < c$.

Example 2.1. There are the four basic triangular functions ($a, b \in \mathbb{R}^+$):
(1) The minimum function $\odot_{\min}$ is defined by $a \odot_{\min} b = \min\{a, b\}$;
(2) The maximum function $\odot_{\max}$ is defined by $a \odot_{\max} b = \max\{a, b\}$;
(3) The average function $\odot_{\mathrm{av}}$ is defined by $a \odot_{\mathrm{av}} b = \frac{a+b}{2}$;
(4) The p-power mean function $\odot_{\mathrm{p}}$ is defined by $a \odot_{\mathrm{p}} b = \sqrt[p]{\frac{a^p+b^p}{2}}$, where $p > 1$.

It is clear that the above four triangular functions are all continuous, also $\odot_{\mathrm{av}}$ and $\odot_{\mathrm{p}}$ are all strictly increasing.

Lemma 2.1. Let X be a non-empty set. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings. If $T(X \times X) \subset A(X)$, then there exist two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$.

Proof. Let x_0, y_0 be two arbitrary points of X. Since $T(X \times X) \subset A(X)$, we can choose $x_1, y_1 \in X$ such that $Ax_1 = T(x_0, y_0)$ and $Ay_1 = T(y_0, x_0)$; Again, from $T(X \times X) \subset A(X)$ we can choose $x_2, y_2 \in X$ such that $Ax_2 = T(x_1, y_1)$ and $Ay_2 = T(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$, which completes the proof.

We first consider a hybrid nonlinear contraction with a gauge function φ, where $\varphi(t) < t$ for any $t > 0$.

Theorem 2.1. Let (X, d) be a complete metric space and $\odot$ a continuous triangular function. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\limsup_{r\to t} \varphi(r) < t$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that

$$d(T(x, y), T(z, w)) \leq \varphi[d(Ax, Az) \odot d(Ay, Aw)] \tag{2.1}$$

for all $x, y, z, w \in X$, where A is continuous and commutative with T and $T(X \times X) \subset A(X)$. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

Proof. Since $\varphi^{-1}(\{0\}) = \{0\}$ and $\varphi(t) < t$ for any $t > 0$, we have $\varphi(t) \leq t$ for any $t \in \mathbb{R}^+$. By Lemma 2.1, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$. From (2.1) it follows that

$$\begin{aligned} d(Ax_n, Ax_{n+1}) &= d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\leq \varphi[d(Ax_{n-1}, Ax_n) \odot d(Ay_{n-1}, Ay_n)]; \end{aligned} \tag{2.2}$$

$$\begin{aligned} d(Ay_n, Ay_{n+1}) &= d(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\ &\leq \varphi[d(Ay_{n-1}, Ay_n) \odot d(Ax_{n-1}, Ax_n)]. \end{aligned} \tag{2.3}$$

Write $d(Ax_n, Ax_{n+1}) = a_n$, $d(Ay_n, Ay_{n+1}) = b_n$, and $a_n \odot b_n = \alpha_n$. Then, operating by $\odot$ for (2.2) and (2.3) we get

$$\alpha_n \leq \varphi(\alpha_{n-1}) \odot \varphi(\alpha_{n-1}) \leq \varphi(\alpha_{n-1}) \leq \alpha_{n-1}. \tag{2.4}$$

Therefore, $\{\alpha_n\}$ is a nonnegative and non-increasing sequence; and hence, $\lim_{n\to\infty} \alpha_n = \alpha \geq 0$. If we assume that $\alpha > 0$, then, from (2.4) and the hypothesis concerning φ it follows that

$$\alpha = \lim_{n\to\infty} \alpha_n \leq \liminf_{n\to\infty} \varphi(\alpha_{n-1}) \leq \limsup_{r\to\alpha} \varphi(r) < \alpha,$$

which is a contradiction. Hence, $\alpha = 0$. By (2.4), (2.2) and (2.3) we have

$$\lim_{n\to\infty} \varphi(\alpha_n) = 0, \ \lim_{n\to\infty} a_n = 0 \text{ and } \lim_{n\to\infty} b_n = 0. \tag{2.5}$$

Setting $c_n = d(Ax_n, Ay_n)$, we claim that

$$\lim_{n\to\infty} c_n = 0. \tag{2.6}$$

In fact, (2.1) yields

$$\begin{aligned} c_n &= d(T(x_{n-1}, y_{n-1}), T(y_{n-1}, x_{n-1})) \leq \varphi[d(Ax_{n-1}, Ay_{n-1}) \odot d(Ay_{n-1}, Ax_{n-1})] \\ &= \varphi[c_{n-1} \odot c_{n-1}] \leq c_{n-1} \odot c_{n-1} \leq c_{n-1}. \end{aligned} \tag{2.7}$$

This shows that $\{c_n\}$ is a nonnegative and non-increasing sequence; and hence, $\lim_{n\to\infty} c_n = c \geq 0$. By the continuity of $\odot$ we have $\lim_{n\to\infty} c_n \odot c_n = c \odot c$. If we assume that $c > 0$, then, from (2.7) it follows that

$$c = \lim_{n\to\infty} c_n \leq \liminf_{n\to\infty} \varphi[c_{n-1} \odot c_{n-1}] \leq \limsup_{r\to c\odot c} \varphi(r) < c \odot c \leq c,$$

which is a contradiction. Hence (2.6) holds. The next step we check that $\{Ax_n\}$ is a Cauchy sequence. If it is not, then there exist a number $\varepsilon_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ such that $m_i > n_i \geq i$ and

$$d(Ax_{m_i}, Ax_{n_i}) \geq \varepsilon_0, \text{ for all } i \in \mathbb{Z}^+. \tag{2.8}$$

Without loss of generality, we can assume that m_i is the smallest number exceeding n_i for which (2.8) holds. Then, by (2.8), we have

$$\varepsilon_0 \leq d(Ax_{m_i}, Ax_{n_i}) \leq d(Ax_{m_i}, Ax_{m_i-1}) + d(Ax_{m_i-1}, Ax_{n_i}) \leq a_{m_i-1} + \varepsilon_0.$$

It follows from (2.5) that $d(Ax_{m_i}, Ax_{n_i}) \to \varepsilon_0$ as $i \to \infty$. Observe that

$$\begin{aligned} d(Ay_{m_i}, Ay_{n_i}) &\leq c_{m_i} + d(Ax_{m_i}, Ax_{n_i}) + c_{n_i} \text{ and} \\ d(Ax_{m_i}, Ax_{n_i}) &\leq c_{m_i} + d(Ay_{m_i}, Ay_{n_i}) + c_{n_i}. \end{aligned}$$

According to (2.6), this means that $d(Ay_{m_i}, Ay_{n_i}) \to \varepsilon_0$ as $i \to \infty$. Setting $d(Ax_{m_i}, Ax_{n_i}) \odot d(Ay_{m_i}, Ay_{n_i}) = \beta_i$, from the continuity of $\odot$ we see that $\beta_i \to \varepsilon_0 \odot \varepsilon_0$. In view of (2.1),

$$\begin{aligned} d(Ax_{m_i}, Ax_{n_i}) &\leq d(Ax_{m_i}, Ax_{m_i+1}) + d(Ax_{m_i+1}, Ax_{n_i+1}) + d(Ax_{n_i+1}, Ax_{n_i}) \\ &\leq a_{m_i} + \varphi(\beta_i) + a_{n_i}. \end{aligned} \tag{2.9}$$

From (2.9) and (2.5) it follows that

$$\varepsilon_0 = \lim_{i\to\infty} d(Ax_{m_i}, Ax_{n_i}) \leq \liminf_{i\to\infty} \varphi(\beta_i) \leq \limsup_{r\to\varepsilon_0\odot\varepsilon_0} \varphi(r) < \varepsilon_0 \odot \varepsilon_0 \leq \varepsilon_0,$$

which is a contradiction. Hence, $\{Ax_n\}$ is a Cauchy sequence. For $m, n \in \mathbb{Z}^+$, we have

$$d(Ax_m, Ax_n) - c_m - c_n \leq d(Ay_m, Ay_n) \leq d(Ax_m, Ax_n) + c_m + c_n.$$

This shows that $\{Ax_n\}$ is also a Cauchy sequence. Since X is complete, there exist $u, v \in X$ such that $\lim_{n\to\infty} Ax_n = u$ and $\lim_{n\to\infty} Ay_n = v$. The continuity of A implies that

$$\lim_{n\to\infty} AAx_n = Au \text{ and } \lim_{n\to\infty} AAy_n = Av.$$

Taking into account the commutativity of A with T, from (2.1) we have

$$\begin{aligned} d(AAx_n, T(u,v)) &= d(AT(x_{n-1}, y_{n-1}), T(u,v)) = d(T(Ax_{n-1}, Ay_{n-1}), T(u,v)) \\ &\leq \varphi[d(AAx_{n-1}, Au) \odot d(AAy_{n-1}, Av)]. \end{aligned} \tag{2.10}$$

The continuity of $\odot$ implies that $\lim_{n\to\infty} d(AAx_{n-1}, Au) \odot d(AAy_{n-1}, Av) = 0$. Since $\varphi(t) < t$ for any $t > 0$, we have $\lim_{t\to 0^+} \varphi(t) = 0$. Thus, letting $n \to \infty$ in (2.10), we get $\lim_{n\to\infty} AAx_n = T(u, v)$. Hence, $T(u, v) = Au$. Similarly, we can show that $T(v, u) = Av$. To prove $Au = v$ and $Av = u$, by using (2.1) we note that

$$d(Au, Ay_n) = d(T(u, v), T(y_{n-1}, x_{n-1})) \leq \varphi[d(Au, Ay_{n-1}) \odot d(Av, Ax_{n-1})]; \tag{2.11}$$

$$d(Av, Ax_n) = d(T(v, u), T(x_{n-1}, y_{n-1})) \leq \varphi[d(Av, Ax_{n-1}) \odot d(Au, Ay_{n-1})]. \tag{2.12}$$

Writing $d(Au, Ay_{n-1}) \odot d(Av, Ax_{n-1}) = \rho_n$ and operating by $\odot$ for (2.11) and (2.12), we get

$$\rho_{n+1} \leq \varphi(\rho_n) \odot \varphi(\rho_n) \leq \varphi(\rho_n) \leq \rho_n. \tag{2.13}$$

Therefore, $\{\rho_n\}$ is a nonnegative non-increasing sequence; and hence, $\lim_{n\to\infty} \rho_n = \rho \geq 0$. If we assume that $\rho > 0$, then from (2.13) it follows that

$$\rho = \lim_{n\to\infty} \rho_{n+1} \leq \liminf_{n\to\infty} \varphi(\rho_n) \leq \limsup_{r\to\rho} \varphi(r) < \rho,$$

which is a contradiction. Hence, $\rho = 0$. From (2.13) we infer that $\lim_{n\to\infty} \varphi(\rho_n) = 0$. Thus, by (2.12) and (2.11) we have

$$\lim_{n\to\infty} Ax_n = Av \text{ and } \lim_{n\to\infty} Ay_n = Au.$$

Hence, $Au = v$ and $Av = u$. We claim that $u = v$. In fact, if $u \neq v$, then $d(u, v) > 0$. From (2.1) and the hypothesis on φ it follows that

$$\begin{aligned} d(u, v) &= d(Av, Au) = d(T(v, u), T(u, v)) \leq \varphi[d(Av, Au) \odot d(Au, Av)] \\ &< d(Av, Au) \odot d(Au, Av) \leq d(Av, Au) = d(u, v), \end{aligned}$$

a contradiction. Hence, $u = v$. Also, the uniqueness of u follows from (2.1). This makes end to the proof.

Lemma 2.2. (cf.[10,11]) Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\varphi^{-1}(\{0\}) = \{0\}$.
(1) If $\varphi(t) < t$ and $\limsup_{r\to t} \varphi(r) < t$ for all $t > 0$, then $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.
(2) If φ is nondecreasing and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, then $\varphi(t) < t$ for all $t > 0$.

Using Lemma 2.2, from Theorem 2.1 we have the following consequence.

Theorem 2.2. Let (X, d) be a complete metric space and $\odot$ a continuous triangular function. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function which satisfies that $\varphi^{-1}(\{0\}) = \{0\}$, φ is nondecreasing and right-continuous, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that

$$d(T(x, y), T(z, w)) \leq \varphi[d(Ax, Az) \odot d(Ay, Aw)]$$

for all $x, y, z, w \in X$, where A is continuous and commutative with T, and $T(X \times X) \subset A(X)$. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

Proof. Since φ is nondecreasing and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$, by Lemma 2.2(2), we have $\varphi(t) < t$ for any $t > 0$. Since φ is right-continuous, we have $\limsup_{r\to t} \varphi(r) = \lim_{r\to t^+} \varphi(r) = \varphi(t) < t$ for any $t > 0$. Thus, the hypotheses of Theorem 2.1 are satisfied. So, the conclusion of Theorem 2.2 follows from Theorem 2.1 immediately.

If we remove the hypothesis of right-continuity of φ in Theorem 2.2, and replace the hypothesis "$\lim_{n\to\infty} \varphi^n(t) = 0$" with the hypothesis "$\sum_{n=0}^{\infty} \varphi^n(t) < +\infty$", we can obtain the following existence, uniqueness and error estimates.

Theorem 2.3. Let (X,d) be a complete metric space and $\odot$ a continuous triangular function. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a function which satisfies that $\varphi^{-1}(\{0\})=\{0\}$, φ is nondecreasing and $s(t)=\sum\limits_{n=0}^{\infty}\varphi^n(t)<+\infty$ for any $t>0$. Let $T:X\times X\to X$ and $A:X\to X$ be two mappings such that

$$d(T(x,y),T(z,w))\le\varphi[d(Ax,Az)\odot d(Ay,Aw)] \tag{2.14}$$

for all $x,y,z,w\in X$, where A is continuous and commutative with T, and $T(X\times X)\subset A(X)$. Then

(1) there exists a unique $u\in X$ such that $u=Au=T(u,u)$.

(2) $d(Ax_n,u)\le s(\varphi^n(\alpha_0))$ and $d(Ay_n,u)\le s(\varphi^n(\alpha_0))$, where $\{x_n\}$ and $\{y_n\}$ are given by Lemma 2.1, and $\alpha_0=d(Ax_0,Ax_1)\odot d(Ay_0,Ay_1)$.

Proof. By Lemma 2.1, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X such that $Ax_{n+1}=T(x_n,y_n)$ and $Ay_{n+1}=T(y_n,x_n)$. Write $d(Ax_n,Ax_{n+1})=a_n$, $d(Ay_n,Ay_{n+1})=b_n$, and $a_n\odot b_n=\alpha_n$. By duplicating the proofs of (2.2)-(2.4), from (2.14) we get

$$a_n\le\varphi(\alpha_{n-1}),\ b_n\le\varphi(\alpha_{n-1});\ \alpha_n\le\varphi(\alpha_{n-1})\odot\varphi(\alpha_{n-1})\le\varphi(\alpha_{n-1}). \tag{2.15}$$

Since φ is nondecreasing, by (2.15), we have $\alpha_n\le\varphi(\alpha_{n-1})\le\varphi^2(\alpha_{n-2})\le\cdots\le\varphi^n(\alpha_0)$. Thus, from (2.15) it follows that

$$d(Ax_n,Ax_{n+1})\le\varphi^n(\alpha_0)\text{ and }d(Ay_n,Ay_{n+1})\le\varphi^n(\alpha_0). \tag{2.16}$$

Let $m,n\in\mathbb{Z}^+$ and $m>n$. By (2.16), we have

$$\begin{aligned}d(Ax_n,Ax_m)&\le d(Ax_n,Ax_{n+1})+d(Ax_{n+1},Ax_{n+2})+\cdots+d(Ax_{m-1},Ax_m)\\&\le\varphi^n(\alpha_0)+\varphi^{n+1}(\alpha_0)+\cdots+\varphi^{m-1}(\alpha_0)=\sum_{i=n}^{m-1}\varphi^i(\alpha_0).\end{aligned} \tag{2.17}$$

Since $s(\alpha_0)=\sum\limits_{n=0}^{\infty}\varphi^n(\alpha_0)<+\infty$, from (2.17) we see that $\{Ax_n\}$ is a Cauchy sequence. In the same manner, we can show that $\{Ay_n\}$ is also a Cauchy sequence. In view of the completeness of X, there exist $u,v\in X$ such that $\lim\limits_{n\to\infty}Ax_n=u$ and $\lim\limits_{n\to\infty}Ay_n=v$. The continuity of A implies that

$$\lim_{n\to\infty}AAx_n=Au\text{ and }\lim_{n\to\infty}AAy_n=Av.$$

From (2.14) and the commutativity of A with T, we have

$$\begin{aligned}d(AAx_n,T(u,v))&=d(AT(x_{n-1},y_{n-1}),T(u,v))=d(T(Ax_{n-1},Ay_{n-1}),T(u,v))\\&\le\varphi[d(AAx_{n-1},Au)\odot d(AAy_{n-1},Av)].\end{aligned} \tag{2.18}$$

The continuity of $\odot$ implies that $\lim\limits_{n\to\infty}d(AAx_{n-1},Au)\odot d(AAy_{n-1},Av)=0$. For any $t>0$, since $s(t)<+\infty$, we have $\lim\limits_{n\to\infty}\varphi^n(t)=0$. It follows from Lemma 2.2(2) that

$$\varphi(t)<t,\ \text{for all }t>0. \tag{2.19}$$

This implies that $\lim_{t\to 0^+} \varphi(t) = 0$. Hence, letting $n \to \infty$ in (2.18), we have $\lim_{n\to\infty} AAx_n = T(u,v)$; and so $T(u,v) = Au$. Similarly, we can show that $T(v,u) = Av$. The next step we show that $Au = v$ and $Av = u$. Write $d(Au, Ay_{n-1}) \odot d(Av, Ax_{n-1}) = \rho_n$. By duplicating the proofs of (2.11)-(2.13), from (2.14) we get

$$d(Au, Ay_n) \le \varphi(\rho_n),\ d(Av, Ax_n) \le \varphi(\rho_n);\ \rho_{n+1} \le \varphi(\rho_n) \odot \varphi(\rho_n) \le \varphi(\rho_n). \quad (2.20)$$

Since φ is nondecreasing, we have

$$\rho_{n+1} \le \varphi(\rho_n) \le \varphi^2(\rho_{n-1}) \le \cdots \le \varphi^n(\rho_1) \to 0 \text{ as } n \to \infty.$$

From (2.20) it follows that $\lim_{n\to\infty} Ax_n = Av$ and $\lim_{n\to\infty} Ay_n = Au$. Hence, $Au = v$ and $Av = u$. By duplicating the remainder of the proof of Theorem 2.1, from (2.14) and (2.19) we have $u = v$. The uniqueness of u follows from (2.14). So, the assertion (1) holds.

Letting $n \to \infty$ in (2.17), we have

$$d(Ax_n, u) \le \sum_{i=n}^{\infty} \varphi^i(\alpha_0) = \sum_{j=0}^{\infty} \varphi^j[\varphi^n(\alpha_0)] = s(\varphi^n(\alpha_0)).$$

Similarly, we have $d(Ay_n, v) \le s(\varphi^n(\alpha_0))$. So, the inequalities in (2) hold, which completes the proof.

Remark 2.1. If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function, then φ is invertible. Let $\psi = \varphi^{-1}$ be the inverse function of φ, then from (2.1) we get the following equivalent contraction

$$\psi[d(T(x,y), T(z,w))] \le d(Ax, Az) \odot d(Ay, Aw), \text{ for } x, y, z, w \in X. \quad (2.21)$$

In general, the gauge function φ in Theorems 2.1-2.3 are not necessarily strictly increasing, and so not necessarily invertible. Hence, the inequality (2.21) is a new contraction which is not equivalent to the one in Theorems 2.1-2.3.

Next, we consider a hybrid nonlinear contraction with a gauge function ψ, where $\psi(t) > t$ for any $t > 0$.

Theorem 2.4. Let (X, d) be a complete metric space. Let $\odot$ be a continuous and strictly increasing triangular function. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\psi^{-1}(\{0\}) = \{0\}$, $\psi(t) > t$ and $\liminf_{r\to t} \psi(r) > t$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that

$$\psi[d(T(x,y), T(z,w))] \le d(Ax, Az) \odot d(Ay, Aw) \quad (2.22)$$

for all $x, y, z, w \in X$, where A is continuous and commutative with T, and $T(X\times X) \subset A(X)$. Then there exists a unique $u \in X$ such that $u = Au = T(u,u)$.

Proof. Since $\psi^{-1}(\{0\}) = \{0\}$ and $\psi(t) > t$ for any $t > 0$, we have $\psi(t) \ge t$ for any $t \in \mathbb{R}^+$. By Lemma 2.1, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X

such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$. Write $d(Ax_n, Ax_{n+1}) = a_n$, $d(Ay_n, Ay_{n+1}) = b_n$, and $a_n \odot b_n = \alpha_n$. In view of (2.22),

$$\begin{aligned} a_n &\le \psi(a_n) = \psi[d(T(x_{n-1}, y_{n-1}), T(x_n, y_n))] \\ &\le d(Ax_{n-1}, Ax_n) \odot d(Ay_{n-1}, Ay_n) = \alpha_{n-1}; \end{aligned} \tag{2.23}$$

$$\begin{aligned} b_n &\le \psi(b_n) = \psi[d(T(y_{n-1}, x_{n-1}), T(y_n, x_n))] \\ &\le d(Ay_{n-1}, Ay_n) \odot d(Ax_{n-1}, Ax_n) = \alpha_{n-1}. \end{aligned} \tag{2.24}$$

Then, operating by $\odot$ for (2.23) and (2.24) we get

$$\alpha_n \le \psi(a_n) \odot \psi(b_n) \le \alpha_{n-1} \odot \alpha_{n-1} \le \alpha_{n-1}.$$

Therefore, $\{\alpha_n\}$ is a nonnegative non-increasing sequence; and hence, $\lim_{n\to\infty} \alpha_n = \alpha \ge 0$. From (2.23) and (2.24) we see that $\{a_n\}$ contains a convergent subsequence $\{a_{n_k}\}$, and $\{b_{n_k}\}$ contains a convergent subsequence, for simplicity we still denote it by $\{b_{n_k}\}$. Suppose that $\lim_{k\to\infty} a_{n_k} = a$ and $\lim_{k\to\infty} b_{n_k} = b$. From the continuity of $\odot$ and $a_{n_k} \odot b_{n_k} = \alpha_{n_k}$ it follows that $a \odot b = \alpha$. If we assume that $\alpha > 0$, then, it follows from $0 \odot 0 = 0$ that $a > 0$ or $b > 0$. Without loss of generality, we can assume that $a > 0$. Then, by (2.23) and (2.24) we have

$$a < \liminf_{r\to a} \psi(r) \le \liminf_{k\to\infty} \psi(a_{n_k}) \le \lim_{k\to\infty} \alpha_{n_k} = \alpha \text{ and } b \le \liminf_{k\to\infty} \psi(b_{n_k}) \le \alpha.$$

From the strict monotonicity of $\odot$ it follows that $\alpha = a \odot b < \alpha \odot \alpha \le \alpha$, which is a contradiction. Hence, $\alpha = 0$, and so by (2.23) and (2.24) we infer that

$$\lim_{n\to\infty} a_n = 0 \text{ and } \lim_{n\to\infty} b_n = 0. \tag{2.25}$$

Now we set $c_n = d(Ax_n, Ay_n)$ and prove that

$$\lim_{n\to\infty} c_n = 0. \tag{2.26}$$

In fact, by (2.22), we have

$$\begin{aligned} c_n &\le \psi(c_n) = \psi[d(T(x_{n-1}, y_{n-1}), T(y_{n-1}, x_{n-1}))] \\ &\le d(Ax_{n-1}, Ay_{n-1}) \odot d(Ay_{n-1}, Ax_{n-1}) = c_{n-1} \odot c_{n-1} \le c_{n-1}. \end{aligned} \tag{2.27}$$

This implies that $\{c_n\}$ is a nonnegative and non-increasing sequence; and hence, $\lim_{n\to\infty} c_n = c \ge 0$. If we assume that $c > 0$, then, from (2.27) it follows that

$$c < \liminf_{r\to c} \psi(r) \le \liminf_{n\to\infty} \psi(c_n) \le \lim_{n\to\infty} c_{n-1} = c,$$

which is a contradiction. Hence, (2.26) holds. Now we show that $\{Ax_n\}$ is a Cauchy sequence. If it is not, then there exist a number $\varepsilon_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ such that $m_i > n_i \ge i$ and

$$d(Ax_{m_i}, Ax_{n_i}) \ge \varepsilon_0, \text{ for all } i \in \mathbb{Z}^+. \tag{2.28}$$

Further, we can choose m_i corresponding to n_i in such a way that it is the smallest integer with $m_i > n_i$ and satisfying (2.28). Then, by (2.28), we have

$$\varepsilon_0 \le d(Ax_{m_i}, Ax_{n_i}) \le d(Ax_{m_i}, Ax_{m_i-1}) + d(Ax_{m_i-1}, Ax_{n_i}) \le a_{m_i-1} + \varepsilon_0.$$

It follows from (2.25) that $d(Ax_{m_i}, Ax_{n_i}) \to \varepsilon_0$ as $i \to \infty$. Observe that

$$d(Ay_{m_i}, Ay_{n_i}) \le c_{m_i} + d(Ax_{m_i}, Ax_{n_i}) + c_{n_i} \text{ and}$$
$$d(Ax_{m_i}, Ax_{n_i}) \le c_{m_i} + d(Ay_{m_i}, Ay_{n_i}) + c_{n_i}.$$

Taking into account (2.26), this implies that $d(Ay_{m_i}, Ay_{n_i}) \to \varepsilon_0$ as $i \to \infty$. By (2.22), we have

$$\begin{aligned} & \psi[d(Ax_{m_i}, Ax_{n_i})] = \psi[d(T(x_{m_i-1}, y_{m_i-1}), T(x_{n_i-1}, y_{n_i-1}))] \\ \le\ & d(Ax_{m_i-1}, Ax_{n_i-1}) \odot d(Ay_{m_i-1}, Ay_{n_i-1}) \qquad (2.29) \\ \le\ & [a_{m_i-1} + d(Ax_{m_i}, Ax_{n_i}) + a_{n_i-1}] \odot [b_{m_i-1} + d(Ay_{m_i}, Ay_{n_i}) + b_{n_i-1}]. \end{aligned}$$

From (2.29) and (2.25) it follows that

$$\begin{aligned} \varepsilon_0 &< \liminf_{r \to \varepsilon_0} \psi(r) \le \liminf_{i \to \infty} \psi[d(Ax_{m_i}, Ax_{n_i})] \\ &\le \lim_{i \to \infty} d(Ax_{m_i}, Ax_{n_i}) \odot \lim_{i \to \infty} d(Ay_{m_i}, Ay_{n_i}) = \varepsilon_0 \odot \varepsilon_0 \le \varepsilon_0, \end{aligned}$$

a contradiction. Hence, $\{Ax_n\}$ is a Cauchy sequence. For $m, n \in \mathbb{Z}^+$, we have

$$d(Ax_m, Ax_n) - c_m - c_n \le d(Ay_m, Ay_n) \le d(Ax_m, Ax_n) + c_m + c_n.$$

This shows that $\{Ax_n\}$ is also a Cauchy sequence. Since X is complete, there exist $u, v \in X$ such that $\lim_{n\to\infty} Ax_n = u$ and $\lim_{n\to\infty} Ay_n = v$. In view of the continuity of A,

$$\lim_{n\to\infty} AAx_n = Au \text{ and } \lim_{n\to\infty} AAy_n = Av.$$

From (2.22) and the commutativity of A with T, we have

$$\begin{aligned} d(AAx_n, T(u,v)) &\le \psi[d(AAx_n, T(u,v))] = \psi[d(AT(x_{n-1}, y_{n-1}), T(u,v))] \\ &= \psi[d(T(Ax_{n-1}, Ay_{n-1}), T(u,v))] \\ &\le d(AAx_{n-1}, Au) \odot d(AAy_{n-1}, Av). \qquad (2.30) \end{aligned}$$

The continuity of $\odot$ implies that $\lim_{n\to\infty} d(AAx_{n-1}, Au) \odot d(AAy_{n-1}, Av) = 0$. Thus, letting $n \to \infty$ in (2.30), we have $\lim_{n\to\infty} AAx_n = T(u,v)$. Hence, $T(u,v) = Au$. Similarly, we can show that $T(v,u) = Av$. The next step we show that $Au = v$ and $Av = u$. According to (2.22),

$$\begin{aligned} d(Au, Ay_n) &\le \psi[d(Au, Ay_n)] = \psi[d(T(u,v), T(y_{n-1}, x_{n-1}))] \\ &\le d(Au, Ay_{n-1}) \odot d(Av, Ax_{n-1})]; \qquad (2.31) \\ d(Av, Ax_n) &\le \psi[d(Av, Ax_n)] = \psi[d(T(v,u), T(x_{n-1}, y_{n-1}))] \\ &\le d(Av, Ax_{n-1}) \odot d(Au, Ay_{n-1}). \qquad (2.32) \end{aligned}$$

Write $d(Au, Ay_n) \odot d(Av, Ax_n) = \rho_n$. Then, operating by $\odot$ for (2.31) and (2.32) we get

$$\rho_n \le \psi(d(Au, Ay_n)) \odot \psi(d(Av, Ax_n)) \le \rho_{n-1} \odot \rho_{n-1} \le \rho_{n-1}.$$

Therefore, $\{\rho_n\}$ is a nonnegative and non-increasing sequence; and hence, $\lim_{n\to\infty} \rho_n = \rho \ge 0$. By duplicating the proof of $\alpha = 0$, we can show that $\rho = 0$. Thus, from (2.31) and (2.32) we have

$$\lim_{n\to\infty} Ax_n = Av \text{ and } \lim_{n\to\infty} Ay_n = Au.$$

Hence, $Au = v$ and $Av = u$. Finally, we prove that $u = v$. If $u \neq v$, then $d(u, v) > 0$. It follows from (2.22) and the hypothesis on ψ that

$$\begin{aligned} d(u,v) &< \psi[d(u,v)] = \psi[d(Av, Au)] = \psi[d(T(v,u), T(u,v))] \\ &\leq d(Av, Au) \odot d(Au, Av) \leq d(Av, Au) = d(u, v), \end{aligned}$$

a contradiction. Hence, we have $u = v$. The uniqueness of u also follows from (2.22). So, the proof of Theorem 2.4 is finished.

Lemma 2.3. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\psi^{-1}(\{0\}) = \{0\}$.
(1) If $\psi(t) > t$ and $\liminf_{r\to t} \psi(r) > t$ for all $t > 0$, then $\lim_{n\to\infty} \psi^n(t) = +\infty$ for all $t > 0$.
(2) If ψ is nondecreasing and $\lim_{n\to\infty} \psi^n(t) = +\infty$ for all $t > 0$, then $\psi(t) > t$ for all $t > 0$.

Proof. (1) Suppose that $t > 0$. Since $\psi(t) > t$, we have $\psi^n(t) > \psi^{n-1}(t) > \cdots > t$ for all $n \in \mathbb{Z}^+$. Thus, there exists $t < q_t \leq +\infty$ such that $\lim_{n\to\infty} \psi^n(t) = q_t$. If $q_t < +\infty$, then, from $\liminf_{r\to q_t} \psi(r) > q_t$ it follows that

$$q_t = \lim_{n\to\infty} \psi[\psi^{n-1}(t)] \geq \liminf_{r\to q_t} \psi(r) > q_t,$$

a contradiction. Hence, $q_t = +\infty$.

(2) Assume that there exists $t_0 \in \mathbb{R}^+$ such that $t_0 > 0$ and $\psi(t_0) \leq t_0$. Then, the monotonicity of ψ implies that $t_0 \geq \psi(t_0) \geq \psi^2(t_0) \geq \cdots \geq \psi^n(t_0)$ for all $n \in \mathbb{Z}^+$. From $\lim_{n\to\infty} \psi^n(t_0) = +\infty$ it follows that $t_0 = +\infty$, a contradiction. Hence, we have $\psi(t) > t$ for all $t > 0$, which is the desired conclusion.

Using Lemma 2.2, from Theorem 2.4 we have the following consequence. Since ψ is nondecreasing, the hypothesis of which the triangular function $\odot$ is strictly increasing in Theorem 2.4 can be removed.

Theorem 2.5. Let (X, d) be a complete metric space and $\odot$ a continuous triangular function. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function which satisfies that $\psi^{-1}(\{0\}) = \{0\}$, ψ is nondecreasing and left-continuous, and $\lim_{n\to\infty} \psi^n(t) = +\infty$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that

$$\psi[d(T(x,y), T(z,w))] \leq d(Ax, Az) \odot d(Ay, Aw)$$

for all $x, y, z, w \in X$, where A is continuous and commutative with T, and $T(X \times X) \subset A(X)$. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

Proof. Since ψ is nondecreasing and $\lim_{n\to\infty} \psi^n(t) = +\infty$ for any $t > 0$, by Lemma 2.3(2), we have $\psi(t) > t$ for any $t > 0$. Since ψ is left-continuous, we have $\liminf_{r\to t} \psi(r) = \lim_{r\to t^-} \psi(r) = \psi(t) > t$ for any $t > 0$. Thus, except the strict monotonicity of $\odot$, the hypotheses of Theorem 2.4 are satisfied. We show that Theorem 2.5 is still true without the strict monotonicity condition of $\odot$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences given in Lemma 2.1. Write $d(Ax_n, Ax_{n+1}) = a_n$, $d(Ay_n, Ay_{n+1}) = b_n$, and $a_n \odot b_n = \alpha_n$. From (2.22) we have (2.23) and (2.24). In view of (2.23) and (2.24),

$$\psi(\max\{a_n, b_n\}) = \max\{\psi(a_n), \psi(b_n)\} \le \alpha_{n-1} \text{ and} \tag{2.33}$$

$$\alpha_n \le \psi(a_n) \odot \psi(b_n) \le \alpha_{n-1} \odot \alpha_{n-1} \le \alpha_{n-1}. \tag{2.34}$$

Hence, $\{\alpha_n\}$ is a nonnegative and non-increasing sequence due to (2.34), and so $\lim_{n\to\infty} \alpha_n = \alpha \ge 0$. Since ψ is nondecreasing and $\alpha_n \ge \alpha$ for all $n \in \mathbb{Z}^+$, by (2.33) we have

$$\psi(\alpha_n) = \psi(a_n \odot b_n) \le \psi(\max\{a_n, b_n\} \odot \max\{a_n, b_n\}) \le \psi(\max\{a_n, b_n\}) \le \alpha_{n-1};$$

$$\psi^n(\alpha) \le \psi^n(\alpha_n) = \psi^{n-1}(\alpha_{n-1}) \le \cdots \le \psi(\alpha_1) \le \alpha_0.$$

If $\alpha > 0$, then it follows from (2.35) that $\alpha_0 \ge \lim_{n\to\infty} \psi^n(\alpha) = +\infty$, a contradiction. Hence, $\alpha = 0$. By duplicating the remainder of the proof of Theorem 2.4, the proof of Theorem 2.5 is completed.

Remark 2.2. In [1-3,7-9,12-18], the existence problems of coupled fixed point for contractions are studied in partial order spaces. Differing from these results, Theorems 2.1-2.5 do not relate to partial order. To the best of our knowledge, there is no work reported on the existence and uniqueness of coupled fixed point for hybrid nonlinear contractions in usual metric spaces without order.

3. Applications

Each common fixed point result for the mappings T and A in Section 2 implies a corresponding fixed point result for T, if we take the mapping A as the identity mapping I. For example, from Theorem 2.1 we obtain the following consequence.

Corollary 3.1. Let (X, d) be a complete metric space and $\odot$ a continuous triangular function. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\limsup_{r\to t} \varphi(r) < t$ for any $t > 0$. Let $T : X \times X \to X$ be a mapping such that

$$d(T(x, y), T(z, w)) \le \varphi[d(x, z) \odot d(y, w)]$$

for all $x, y, z, w \in X$. Then there exists a unique $u \in X$ such that $u = T(u, u)$.

Since each hybrid contraction with a gauge function includes the case of linear contraction as a special case, each fixed point result in Section 2 implies the same fixed point result for linear contraction. Hence, we have the following consequence.

Corollary 3.2. Let (X, d) be a complete metric space. Let $\odot$ be a continuous triangular function, and $\alpha \in (0, 1)$ a real number. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that

$$d(T(x, y), T(z, w)) \le \alpha[d(Ax, Az) \odot d(Ay, Aw)]$$

for all $x, y, z, w \in X$, where A is continuous and commutative with T, and $T(X \times X) \subset A(X)$. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

In all results in Section 2, the triangular function $\odot$ is abstract. Hence, Theorems 2.1-2.5 are generalizations and unifications of many results. For example, taking $a \odot b = \ln\left(\frac{e^a+e^b}{2}\right)$ (clearly, it is a continuous and strictly increasing triangular function) in Theorem 2.4, we get the following consequence.

Corollary 3.3. Let (X, d) be a complete metric space. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\psi^{-1}(\{0\}) = \{0\}$, $\psi(t) > t$ and $\liminf_{r\to t} \psi(r) > t$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that

$$\psi[d(T(x, y), T(z, w))] \leq \ln\left[\frac{e^{d(Ax,Az)} + e^{d(Ay,Aw)}}{2}\right]$$

for all $x, y, z, w \in X$, where A is continuous and commutative with T, and $T(X\times X) \subset A(X)$. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

Taking $\odot = \odot_{\text{av}}$ and taking A as identity mapping I in Corollary 3.2, we get the following consequence.

Corollary 3.4. Let (X, d) be a complete metric space and $\alpha \in (0, 1)$ a real number. Let $T : X \times X \to X$ be a mapping such that

$$d(T(x, y), T(z, w)) \leq \frac{\alpha}{2}[d(x, z) + d(y, w)]$$

for all $x, y, z, w \in X$. Then there exists a unique $u \in X$ such that $u = T(u, u)$.

Example 3.1. Let $X = [-1, 1] \subset \mathbb{R}$, $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Let $A = I$, $\odot = \odot_{\text{av}}$ and $\psi(t) = \begin{cases} 2t, & t \in [0, 7/8]; \\ 7/4, & t \in (7/8, 1]. \end{cases}$ Then ψ is nondecreasing, continuous and $\lim_{n\to\infty} \psi^n(t) = +\infty$ for any $t > 0$. For $x, y \in X$, define $T : X \times X \to X$ as follows: $T(x, y) = 1 - x^2/8 - |y|/4$. Then, for each $x, y, z, w \in X$, we have

$$|T(x, y) - T(z, w)| = \left|\frac{z^2 - x^2}{8} + \frac{|w| - |y|}{4}\right| \leq \frac{1}{4}(|x - z| + |y - w|).$$

This means that $\psi(d(T(x, y), T(z, w))) \leq d(x, z) \odot_{\text{av}} d(y, w)$. Thus, all hypotheses of Theorem 2.5 are satisfied. Therefore, T has a unique fixed point in X. Indeed, $u = \sqrt{33} - 5$ is a unique fixed point of T.

Example 3.2. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. For $x, y \in X$, define $T : X\times X \to X$ and $A : X \to X$ as follows: $T(x, y) = x+y/3+2$, $Ax = 2x + 6$. Clearly, A is continuous and commutative with T. Let $\odot = \odot_{\max}$ and

$$\varphi(t) = \begin{cases} \frac{4t}{4+t}, & t \in [0, 1]; \\ \frac{2t}{3}, & t \in (1, +\infty). \end{cases}$$

Then, for each $x, y, z, w \in X$, we have

$$|T(x, y) - T(z, w)| = \left|(x - z) + \frac{y - w}{3}\right| \leq \frac{4}{3} \max\{|x - z|, |y - w|\} \text{ and}$$

$$|Ax - Az| \odot_{\max} |Ay - Aw| = 2 \max\{|x - z|, |y - w|\}.$$

Setting $t = \max\{|x-z|, |y-w|\}$, we have $\frac{4t}{3} \le \varphi(2t)$, i.e.,

$$|T(x,y) - T(z,w)| \le \varphi(|Ax - Az| \odot_{\max} |Ay - Aw|).$$

Thus, all hypotheses of Theorem 2.1 are satisfied. Therefore, T and A have a unique common coupled fixed point in X.

Example 3.3. Let $X = C[t_0 - \mu, t_0 + \mu]$ be the space of continuous functions defined on $[t_0 - \mu, t_0 + \mu]$, where $t_0, \mu \in \mathbb{R}$ and $\mu > 0$. Obviously, this space with the metric given by

$$d(x,y) = \sup\{|x(t) - y(t)| : t \in [t_0 - \mu, t_0 + \mu]\}.$$

Let $f, g : [t_0 - \mu, t_0 + \mu] \times \mathbb{R} \to \mathbb{R}$ be two continuous functions such that there exist $L_1, L_2 > 0$ with $\mu(L_1 + L_2) \le 1$ and

$$|f(t,x) - f(t,y)| \le L_1\sqrt{\ln[1 + (x-y)^2]} \text{ and } |g(t,x) - g(t,y)| \le L_2\sqrt{\ln[1 + (x-y)^2]}$$

for all $(t,x), (t,y) \in [t_0 - \mu, t_0 + \mu] \times \mathbb{R}$. Consider the following initial value problem for coupled differential system:

$$\begin{cases} x'(t) = f(t, x(t)) + g(t, y(t)); \\ y'(t) = f(t, y(t)) + g(t, x(t)); \\ x(t_0) = y(t_0) = \eta, \ \eta \in \mathbb{R}. \end{cases} \tag{3.1}$$

It is easy to see that the problem (3.1) is equivalent to the following system of integral equations:

$$\begin{cases} x(t) = \eta + \int_{t_0}^{t} [f(\tau, x(\tau)) + g(\tau, y(\tau))] d\tau; \\ y(t) = \eta + \int_{t_0}^{t} [f(\tau, y(\tau)) + g(\tau, x(\tau))] d\tau. \end{cases} \tag{3.2}$$

We define the mapping $T : X \times X \to X$ by

$$T(x,y)(t) = \eta + \int_{t_0}^{t} [f(\tau, x(\tau)) + g(\tau, y(\tau))] d\tau, \text{ for } t \in [t_0 - \mu, t_0 + \mu].$$

Then, for each $x, y, z, w \in X$, we have

$$\begin{aligned} d(T(x,y), T(z,w)) &\le \sup_{|t - t_0| \le \mu} \left| \int_{t_0}^{t} [|f(\tau, x(\tau)) - f(\tau, z(\tau))| + |g(\tau, y(\tau)) - g(\tau, w(\tau))|] d\tau \right| \\ &\le \mu \left(L_1 \sqrt{\ln[1 + (d(x,z))^2]} + L_2 \sqrt{\ln[1 + (d(y,w))^2]} \right) \\ &\le \mu(L_1 + L_2) \max\left\{ \sqrt{\ln[1 + (d(x,z))^2]}, \sqrt{\ln[1 + (d(y,w))^2]} \right\} \\ &= \mu(L_1 + L_2) \sqrt{\ln[1 + (d(x,z) \odot_{\max} d(y,w))^2]}. \end{aligned} \tag{3.3}$$

Let $A = I$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\varphi(t) = \mu(L_1 + L_2)\sqrt{\ln[1 + t^2]}$. Then φ is continuous, $\varphi^{-1}(\{0\}) = \{0\}$ and $\varphi(t) < t$ due to $\mu(L_1 + L_2) \le 1$. Thus, from (3.3) we see that all hypotheses of Theorem 2.1 are satisfied. Therefore, T have a unique common coupled fixed point u in X, i.e., the integral system (3.2) has a unique common coupled solution u in X. Therefore, the problem (3.1) has a unique common coupled solution $u = u(t)$.

Example 3.4. Let $L^2[a,b], L^2([a,b]\times[a,b])$ be Lebesgue spaces, where $a,b\in\mathbb{R}$ and the metric in $L^2[a,b]$ is given by

$$d(x,y)=\left(\int_a^b |x(t)-y(t)|^2dt\right)^{1/2}.$$

Let $\eta\in L^2[a,b]$, $K\in L^2([a,b]\times[a,b])$ and $M=\int_a^b\int_a^b|K(t,s)|^2dsdt$. Let $f,g:[a,b]\times\mathbb{R}\to\mathbb{R}$ be the functions such that $t\mapsto f(t,x)$ and $t\mapsto g(t,x)$ are Lebesgue measurable, and there exist $L_1,L_2>0$ with $\sqrt{M}(L_1+L_2)<1$ and

$$|f(t,x)-f(t,y)|\le L_1|x-y| \text{ and } |g(t,x)-g(t,y)|\le L_2|x-y|$$

for all $(t,x),(t,y)\in[a,b]\times\mathbb{R}$. Consider the following coupled integral system:

$$\begin{cases} x(t)=\eta(t)+\int_a^b K(t,s)[f(s,x(s))+g(s,y(s))]ds; \\ y(t)=\eta(t)+\int_a^b K(t,s)[f(s,y(s))+g(s,x(s))]ds. \end{cases} \tag{3.4}$$

We define the mapping $T:L^2[a,b]\times L^2[a,b]\to L^2[a,b]$ by

$$T(x,y)(t)=\eta(t)+\int_a^b K(t,s)[f(s,x(s))+g(s,y(s))]ds,\ \text{for } t\in[a,b].$$

Then, for each $x,y,z,w\in X$, by the Minkowski inequality and the Hölder inequality we have

$$\begin{aligned} & d(T(x,y),T(z,w)) \\ = & \left[\int_a^b dt\left(\int_a^b K(t,s)[f(s,x(s))-f(s,z(s))+g(s,y(s))-g(s,w(s))]ds\right)^2\right]^{1/2} \\ \le & L_1\left[\int_a^b dt\left(\int_a^b |K(t,s)||x(s)-z(s)|ds\right)^2\right]^{1/2} \\ & +L_2\left[\int_a^b dt\left(\int_a^b |K(t,s)||y(s)-w(s)|ds\right)^2\right]^{1/2} \\ \le & \sqrt{M}[L_1d(x,z)+L_2d(y,w)]\le\sqrt{M}(L_1+L_2)\ d(x,z)\odot_{\max} d(y,w). \end{aligned} \tag{3.5}$$

Let $A=I$ and $\alpha=\sqrt{M}(L_1+L_2)$. From (3.5) we see that all hypotheses of Corollary 3.2 are satisfied. Therefore, T have a unique common coupled fixed point u in $L^2[a,b]$. Therefore, the integral system (3.4) has a unique common coupled solution $u=u(t)$ in $L^2[a,b]$.

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