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A SPLITTING-RELAXED PROJECTION METHOD FOR SOLVING THE SPLIT FEASIBILITY PROBLEM

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Abstract. The split feasibility problem (SFP) is to find $x \in C$ so that $Ax \in Q$, where C is a nonempty closed convex subset of $\mathbb{R}^n$, Q is a nonempty closed convex subset of $\mathbb{R}^m$, and A is a matrix from $\mathbb{R}^n$ into $\mathbb{R}^m$. One of successful methods for solving the SFP is Byrne's CQ algorithm. However, to carry out the CQ algorithm, it is required that the closed convex subsets are simple and that the matrix norm is known in advance. Motivated by Tseng's splitting method and Yang's relaxed CQ algorithm, we propose in this paper a new method for solving the SFP, which overcomes the drawback of the CQ algorithm.
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1. Introduction

The problem under consideration in this article is formulated as finding a point $\hat{x}$ satisfying the property:

$$\hat{x} \in C \text{ and } A\hat{x} \in Q, \tag{1.1}$$

where C is a nonempty closed convex subsets of $\mathbb{R}^n$, Q is a nonempty closed convex subsets of $\mathbb{R}^m$, and A is a matrix from $\mathbb{R}^n$ into $\mathbb{R}^m$. Denote by S the solution set of problem (1.1) and assume that it is nonempty throughout the paper. Problem (1.1) is called by Censor and Elfving [5] the split feasibility problem (SFP) and has been proved very useful in dealing with a variety of signal processing problems [4]. Various algorithms have been invented to solve the SFP (see [2, 9, 10, 13, 14, 15, 17, 18] and reference therein). In particular, Byrne introduced his CQ algorithm:

$$x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n), \tag{1.2}$$

where the step τ is a real number in $(0, 2/\|A\|^2)$. Since it does not involve matrix inverses, the CQ algorithm is easily performed compared with the original algorithm in [5].

To implement the CQ algorithm, it is required that the closed convex subsets are so simple that the projections onto them are easily calculated. Now consider the nonempty closed convex subsets in (1.1) having the following form:

$$C = \{x \in \mathbb{R}^n : c(x) \le 0\},$$
$$Q = \{y \in \mathbb{R}^m : q(y) \le 0\},$$

where $c : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^m \to \mathbb{R}$ are both convex functions. In this situation, the efficiency of the CQ method is extremely affected because in general the computation of projections onto such subsets is very difficult. Motivated by Fukushima's relaxed projection method in [6], Yang [17] suggested calculating the projection onto a half-space containing the original subset instead of the latter set itself. More precisely, Yang introduced the following relaxed CQ algorithm:

$$x_{n+1} = P_{C_n}(x_n - \tau f_n(x_n)),$$

where $f_n = A^*(I - P_{Q_n})A$, C_n and Q_n are constructed as follows:

$$C_n = \{x \in \mathbb{R}^n : c(x_n) \le \langle \xi_n, x_n - x\rangle\}, \tag{1.3}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in \mathbb{R}^m : q(Ax_n) \le \langle \zeta_n, Ax_n - y\rangle\}, \tag{1.4}$$

where $\zeta_n \in \partial q(Ax_n)$. Obviously, for every $n \ge 0$, $C_n \supseteq C$ and $Q_n \supseteq Q$. More important, since the projections onto C_n and Q_n have the closed form, the relax CQ algorithm is easily implemented. Observe that in the relaxed version the step also relies on the matrix norm. So one has to estimate or calculate this value. This is however impossible in practice. To overcome this, Qu [9] introduced the following algorithm:

$$\begin{cases} y_n = P_{C_n}(x_n - \tau_n f_n(x_n)) \\ x_{n+1} = P_{C_n}(x_n - \tau_n f_n(y_n)), \end{cases} \tag{1.5}$$

where the step τ_n is chosen according to the Armijo-type rule. In such a way, the calculation for the matrix norm is avoided.

Recently the SFP was investigated under a more general framework. In particular, Xu [16] considered the SFP in Hilbert spaces (see also [13]) and Schöpfer et al. [11] in uniformly smooth Banach spaces. In this paper, we will deal with the SFP in infinite-dimensional Hilbert spaces. In other words, we consider problem (1.1) with A a linear bounded operator from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$. We note that Qu's algorithm is in fact a combination of the relaxed projection method and Korpelevich's extragradient method [8]. Recently, Tseng [12] introduced a modified forward-backward splitting method for finding zeros of the sum of two monotone operators. Motivated by Tseng's splitting method and the relaxed projection method, we propose in this paper a new algorithm for solving the SFP. An iterative implementation of the proposed algorithm is easy and moreover one need not know the exact value of the matrix norm in advance.

2. Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces. $I : \mathcal{K} \to \mathcal{K}$ denotes the identity operator, $\omega_w(x_n)$ the set of cluster points in the weak topology, "$\to$" strong convergence and "$\rightharpoonup$" weak convergence.

Let T be an operator on $\mathcal{H}$. Then T is called

(i) κ-inverse strongly monotone (κ-ism) if there is $\kappa > 0$ so that

$$\langle Tx - Ty, x - y\rangle \geq \kappa\|Tx - Ty\|^2, \ x, y \in \mathcal{H};$$

(ii) κ-Lipschitz continuous if there is $\kappa > 0$ so that

$$\|Tx - Ty\| \leq \kappa\|x - y\|, \ x, y \in \mathcal{H};$$

(iii) firmly nonexpansive if it is 1-ism;

(iv) nonexpansive if it is 1-Lipschitz continuous.

Lemma 2.1 (Byrne [3]). *Let $A : \mathcal{H} \to \mathcal{K}$ be a linear bounded operator and let $\varrho = \|A\|^2$. Then the operator $A^*(I - P_Q)A$ is $(1/\varrho)$-ism and hence (ϱ)-Lipschitz continuous.*

One of important examples for firmly nonexpansive operators is the projection operator. Let C be a nonempty closed convex subset of $\mathbb{R}^n$. Denote by P_C the projection from $\mathbb{R}^n$ onto C; that is,

$$P_C x = \arg\min_{y \in C} \|x - y\|, \quad x \in \mathbb{R}^n.$$

It is well known that the projection is characterized by the following variational inequality:

$$\langle P_C x - x, P_C x - z\rangle \leq 0, \ \forall z \in C. \tag{2.1}$$

Firmly nonexpansive operators have the following properties.

Lemma 2.2 (Gobel-Kirk [7]). *Let T be an operator on $\mathcal{H}$. Then the following are equivalent.*

(i) *T is firmly nonexpansive,*

(ii) *$I - T$ is firmly nonexpansive,*

(iii) *$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \ x, y \in \mathcal{H}$.*

Recall that a function $f : \mathcal{H} \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $\lambda \in [0, 1]$ and for all $x, y \in \mathcal{H}$. A function $f : \mathcal{H} \to \mathbb{R}$ is called subdifferentiable at x if there exists at least one subgradient at x. The set of subgradients of f at the point x is called the subdifferential of f at x, and is denoted by $\partial f(x)$. A function f is called subdifferentiable if it is subdifferentiable at all $\mathcal{H}$. A convex function $f : \mathcal{H} \to \mathbb{R}$ is called weakly lower semicontinuious at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n);$$

weakly lower semicontinuious if it is weakly lower semicontinuious at all $\mathcal{H}$.

The following concept plays an important role in the sequential analysis. Assume that C is a closed convex nonempty subset and (x_n) is a sequence in $\mathcal{H}$. The sequence (x_n) is called Fejér monotone w.r.t. C, if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \ \forall z \in C.$$

Lemma 2.3. *Let C be a nonempty closed convex subset in $\mathcal{H}$. If the sequence (x_n) is Fejér monotone w.r.t. C, then the following hold.*

(i) *$x_n \rightharpoonup x^* \in C$ if and only if $\omega_w(x_n) \subseteq C$;*
(ii) *The sequence $\{P_C x_n\}$ converges strongly;*
(iii) *If $x_n \rightharpoonup x^* \in C$, then $x^* = \lim_{n\to\infty} P_C x_n$.*

Proof. The (i) and (ii) are taken from [1, Theorem 2.16]. To show (iii), let $\hat{x}$ be the limit point of the sequence $\{P_C x_n\}$. It follows from the characterizing inequality (2.1) that

$$\langle x_n - P_C x_n, x^* - P_C x_n \rangle \leq 0.$$

Letting $n \to \infty$ yields

$$\langle x^* - \hat{x}, x^* - \hat{x} \rangle \leq 0,$$

that is, $x^* = \hat{x}$ and thus the proof is complete. □

3. Relaxed projection method

In this section we consider the nonempty closed convex subsets in the SFP with the following form:

$$C = \{x \in \mathcal{H} : c(x) \leq 0\}, \ Q = \{y \in \mathcal{K} : q(y) \leq 0\},$$

where $c : \mathcal{H} \to \mathbb{R}$ and $q : \mathcal{K} \to \mathbb{R}$ are both convex functions. Assume that both ∂c and ∂q are nonempty and uniformly bounded on bounded sets. It is worth noting that in finite setting, every convex function is subdifferentiable everywhere and its subdifferentials are uniformly bounded on bounded sets (see [1, Corollary 7.9]). So our assumptions are automatically satisfied in finite setting. Define C_n and Q_n as follows:

$$C_n = \{x \in \mathcal{H} : c(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \tag{3.1}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in \mathcal{K} : q(Ax_n) \leq \langle \zeta_n, Ax_n - y \rangle\}, \tag{3.2}$$

where $\zeta_n \in \partial q(Ax_n)$. Obviously, for every $n \geq 0$, $C_n \supseteq C$ and $Q_n \supseteq Q$.

For every n, set $f_n = A^*(I - P_{Q_n})A$. Now let us present our algorithm for finding a solution of the SFP.

Algorithm 3.1. Take an initial guess $x_0 \in \mathcal{H}$.

Step 1. Given x_n, compute the next iteration:

$$\left[\begin{array}{l} y_n = P_{C_n}(x_n - \tau_n f_n(x_n)) \\ x_{n+1} = P_{C_n}(y_n - \tau_n(f_n(y_n) - f_n(x_n))), \end{array}\right. \tag{3.3}$$

where $\tau_n = \sigma\beta^{m(n)}$ with $m(n)$ the smallest nonnegative integer m so that

$$\sigma\beta^m \|f_n(x_n) - f_n(y_n)\| \le \sqrt{1-\theta}\|x_n - y_n\|, \tag{3.4}$$

with $\sigma > 0, \beta > 0, \theta \in (0,1)$.

Step 2. If $x_{n+1} = x_n$, stop; otherwise go to step 1.

Remark 3.2. Since f_n is Lipschitz continuous, the step at each iteration is well defined. By the definition of τ_n, $\tau_n = \sigma\beta^{m(n)}$. If $m(n) = 0$, then $\tau_n = \sigma$. Otherwise, we have

$$\sigma\beta^{m(n)-1}\|f_n(x_n) - f_n(y_n)\| > \sqrt{1-\theta}\|x_n - y_n\|$$
$$\Leftrightarrow \|f_n(x_n) - f_n(y_n)\| > \frac{\sqrt{1-\theta}}{\sigma\beta^{m(n)-1}}\|x_n - y_n\|.$$

Since f_n is ϱ-Lipschitz continuous, it follows that

$$\frac{\sqrt{1-\theta}}{\sigma\beta^{m(n)-1}} < \varrho \Leftrightarrow \sigma\beta^{m(n)-1} > \frac{\sqrt{1-\theta}}{\varrho}.$$

In this case, we have $\tau_n = \sigma\beta^{m(n)} > \sqrt{1-\theta}\beta/\varrho$. Altogether

$$\tau_n \ge \min\{\sigma, \sqrt{1-\theta}\beta/\varrho\} := \tau. \tag{3.5}$$

Remark 3.3. If $x_n = x_{n+1}$ for some $n \ge 0$, then x_n is a solution of the SFP. In fact, pick any point $z \in S$. Clearly, $x_n \in C_n$. Substituting $x = x_n$ in (3.1) yields $c(x_n) \le 0$, i.e., $x_n \in C$. Note that $x_n = P_{C_n}(x_n - \tau_n f_n(x_n))$. By inequality (2.1), $\langle f_n(x_n), x_n - z\rangle \le 0$. Since $I - P_{Q_n}$ is firmly nonexpansive,

$$\begin{aligned}\|(I - P_{Q_n})Ax_n\|^2 &= \|(I - P_{Q_n})Ax_n - (I - P_{Q_n})Az\|^2 \\ &\le \langle (I - P_{Q_n})Ax_n, Ax_n - Az\rangle \\ &= \langle f_n(x_n), x_n - z\rangle \le 0.\end{aligned}$$

Thus $Ax_n \in Q_n$. By the definition of Q_n, we have $q(Ax_n) \le 0$, i.e., $Ax_n \in Q$ and therefore x_n must be a solution of the SFP.

We assume without loss of generality that Algorithm 3.1 generates an infinite iterative sequence in what follows.

Theorem 3.4. *Let (x_n) be the sequence generated by Algorithm 3.1. Then (x_n) converges weakly to a solution $x^*(:= \lim_n P_S x_n)$ of the SFP.*

Proof. By Lemma 2.3, it suffices to show that (i) (x_n) is Fejér-monotone w.r.t. S; (ii) every weak cluster point of (x_n) is in the solution set S.

To see this (i), let $z \in S$ be fixed. Clearly, $z \in C_n$ and thus $f_n(z) = 0$ for all $n \in \mathbb{N}$. Note that $x_{n+1} = P_{C_n}(y_n - \tau_n(f_n(y_n) - f_n(x_n)))$. Since P_{C_n} is nonexpansive, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|P_{C_n}(y_n - \tau_n(f_n(y_n) - f_n(x_n))) - z\|^2 \\ &\le \|y_n - z - \tau_n(f_n(y_n) - f_n(x_n))\|^2 \\ &= \|y_n - z\|^2 + \tau_n^2\|f_n(y_n) - f_n(x_n)\|^2 \\ &\quad - 2\tau_n\langle y_n - z, f_n(y_n) - f_n(x_n)\rangle,\end{aligned}$$

which together with (3.4) yields

$$\begin{aligned}\|x_{n+1}-z\|^2 \le \|y_n-z\|^2 + (1-\theta)\|y_n-x_n\|^2 \\ - 2\tau_n\langle y_n - z, f_n(y_n) - f_n(x_n)\rangle.\end{aligned} \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}\|y_n-z\|^2 &= \|x_n-z\|^2 + \|y_n-x_n\|^2 + 2\langle x_n - z, y_n - x_n\rangle \\ &= \|x_n-z\|^2 - \|y_n-x_n\|^2 + 2\langle y_n - z, y_n - x_n\rangle.\end{aligned}$$

We now estimate the last term in the above. By inequality (2.1), it follows that

$$\langle y_n - z, y_n - (x_n - \tau_n f_n(x_n))\rangle \le 0,$$

or equivalently,

$$\langle y_n - z, y_n - x_n\rangle \le -\tau_n\langle f_n(x_n), y_n - z\rangle, \tag{3.7}$$

so that

$$\|y_n-z\|^2 \le \|x_n-z\|^2 - \|y_n-x_n\|^2 - 2\tau_n\langle f_n(x_n), y_n - z\rangle. \tag{3.8}$$

Combining (3.6) and (3.8) yields

$$\|x_{n+1}-z\|^2 \le \|x_n-z\|^2 - \theta\|y_n-x_n\|^2 - 2\tau_n\langle f_n(y_n), y_n - z\rangle;$$

on the other hand, since f_n is monotone, it follows that

$$\langle f_n(y_n), y_n - z\rangle \ge \langle f_n(z), y_n - z\rangle = 0,$$

so that

$$\|x_{n+1}-z\|^2 \le \|x_n-z\|^2 - \theta\|y_n-x_n\|^2. \tag{3.9}$$

This shows that (i) holds and hence (x_n) is bounded. Moreover $\|y_n - x_n\| \to 0$ as $n \to \infty$.

We next show (ii) holds. Take a subsequence (x_{n_k}) of (x_n) so that $x_{n_k} \rightharpoonup x'$. Note that $y_n \in C_n$. This implies that

$$c(x_n) \le \langle \xi_n, x_n - y_n\rangle \le \xi\|y_n - x_n\| \to 0,$$

where ξ satisfies $\|\xi_n\| \le \xi$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ yields $\limsup_n c(x_n) \le 0$. By the weak lower semicontinuity of c,

$$c(x') \le \liminf_{k\to\infty} c(x_{n_k}) \le \limsup_{k\to\infty} c(x_{n_k}) \le 0,$$

which implies $x' \in C$.

It remains to show $Ax' \in Q$. Take $M > 0$ so that $M \ge (\varrho\|x_n - z\| + (1/\tau)\|y_n - z\|)$ for all $n \in \mathbb{N}$, where τ is defined as in (3.5). On the one hand, it follows from (3.7) that

$$\begin{aligned}\langle f_n(x_n), y_n - z\rangle &= \frac{1}{\tau_n}\langle \tau_n f_n(x_n), y_n - z\rangle \\ &\le \frac{1}{\tau_n}\langle x_n - y_n, y_n - z\rangle \\ &\le \frac{1}{\tau}\|x_n - y_n\|\|y_n - z\|;\end{aligned}$$

on the other hand, since f_n is ϱ-Lipschitz continuous,

$$\begin{aligned}\langle f_n(x_n), x_n - y_n\rangle &\le \|f_n(x_n) - f_n(z)\|\|x_n - y_n\| \\ &\le \varrho\|x_n - z\|\|x_n - y_n\|.\end{aligned}$$

Combing the last two inequalities yields

$$\langle f_n(x_n), x_n - z\rangle \le M\|x_n - y_n\|. \tag{3.10}$$

Recalling that f_n is $(1/\varrho)$-ism, we get

$$\begin{aligned}\langle f_n(x_n), x_n - z\rangle &= \langle f_n(x_n) - f_n(z), x_n - z\rangle \\ &\ge \frac{1}{\varrho}\|f_n(x_n) - f_n(z)\|^2 = \frac{1}{\varrho}\|f_n(x_n)\|^2,\end{aligned}$$

which together with (3.10) yields

$$\|f_n(x_n)\|^2 \le \varrho M\|x_n - y_n\| \to 0.$$

Hence $f_n(x_n) \to 0$ as $n \to \infty$. Moreover, since $I - P_{Q_n}$ is firmly nonexpansive, it follows that

$$\begin{aligned}\|(I - P_{Q_n})Ax_n\|^2 &= \|(I - P_{Q_n})Ax_n - (I - P_{Q_n})Az\|^2 \\ &\le \langle (I - P_{Q_n})Ax_n, Ax_n - Az\rangle \\ &= \langle A^*(I - P_{Q_n})Ax_n, x_n - z\rangle \\ &\le \|f_n(x_n)\|\|x_n - z\| \to 0.\end{aligned}$$

Since $P_{Q_n}(Ax_n) \in Q_n$, it follows that

$$\begin{aligned}q(Ax_n) &\le \langle \zeta_n, Ax_n - P_{Q_n}(Ax_n)\rangle \\ &\le \zeta\|(I - P_{Q_n})Ax_n\| \to 0,\end{aligned}$$

where ζ satisfies $\|\zeta_n\| \le \zeta$ for all $n \in \mathbb{N}$. However, the weak continuity of A yields that $Ax_{n_k} \rightharpoonup Ax'$, which together with the weak lower semicontinuity of q now yields

$$q(Ax') \le \liminf_{k\to\infty} q(Ax_{n_k}) \le 0,$$

that is, $Ax' \in Q$. Consequently, $\omega_w(x_n) \subseteq S$ and the proof is finished. □

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