AN EFFICIENT ALGORITHM FOR SOLVING HIGH ORDER STURM-LIOUVILLE PROBLEMS USING VARIATIONAL ITERATION METHOD

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Abstract. In this paper, a novel numerical algorithm based on generalized variational iteration method for the solution of every $2m$-order Sturm-Liouville problem for $m \geq 1$ is proposed. In this approach, a Lagrange multiplier is identified to establish suitable correction functional to construct an approximate solution which it is considered as the fixed point of the corresponding correction functional. It is proved that this algorithm converges to the corresponding exact solution. Error estimate for the algorithm is given. Numerical simulations show that this algorithm is easy to implement and produces accurate results. Numerical results are given.

Key Words and Phrases: Sturm-Liouville problems, Lagrange multiplier, eigenvalues, eigenfunctions, variational iteration method.

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1. Introduction

The Sturm-Liouville theory has many applications in applied mathematics, physics and engineering. Many physical phenomena, both in classical mechanics and in quantum mechanics are described mathematically by second-order Sturm-Liouville problems (see [1-3] for more details). While the problems arise in the stability of hydrodynamic and magnetohydrodynamic, are almost always of high order. It is either because they involve a coupled system of ordinary differential equations, or they have been reduced to a single differential equation of order $2m$, for an integer $m > 1$, (for example see [4-6]).

The numerical techniques referred to as high-order Sturm-Liouville problems have been less widely used compared with the more familiar second-order and fourth-order types. Although there are some available software codes like "SLEIGN" [7], "SLEIGN2" [8] and "SLEIGDGE" [9] for the solution of second-order Sturm-Liouville problems and "SLEUTH" [10] for solving fourth-order Sturm-Liouville problems, up to the knowledge of the authors there is no software and code for solving sixth-order Sturm-Liouville problems. In year 1998 Greenberg and Marletta [5] used the shooting method to approximate the eigenvalues of sixth-order Sturm-Liouville problems. In
Variational iteration method (VIM) has been successfully implemented to handle linear and non-linear differential equations (for example see [12-21]). The main property of VIM is its flexibility and ability to solve non-linear differential equations accurately and conveniently. The VIM was developed by J.H. He [12-20] during the years 1998-2010. This method has been extensively applied as a powerful tool for solving various kinds of problems, such as: autonomous ordinary differential equations and approximated solutions of some nonlinear problems. Recently, Goh, Noorani and Hashim [22] used the VIM for solving the chaotic Chen system. Safari, Ganji and Sadeghi [23] implemented the VIM for the solution of Benney-Lin equation while Wazwaz [24] used VIM for solving variational problems. Although, there have been a lot of papers on the ‘Variational Iteration Method’ for solving problems involving ordinary differential equations, with the knowledge of the authors there is no solution for high order Sturm-Liouville problems. This paper is an application of the fixed-point iteration method to higher order ODE eigenvalue problems. In this paper, we will extend the VIM for finding the eigenvalues of 2m-order non-singular Sturm-Liouville problem of the form

\[(−1)^m(p_m(x)y^{(m)})^{(m)} + (−1)^{m−1}(p_{m−1}(x)y^{(m−1)})^{(m−1)} + \cdots + (p_2(x)y''')'' - (p_1(x)y')' + p_0(x)y = Ew(x)y, \quad a < x < b, \] (1.1)

together with separated, self-adjoint boundary conditions imposed at \(x = a\) and \(x = b\).

We assume that all coefficient functions are real valued. The technical conditions for the problem to be non-singular are: the interval \((a, b)\) is finite; the coefficient functions \(p_k (0 \leq k \leq m−1)\), \(w(x)\) and \(1/p_m(x)\) are in \(L^1(a, b)\), and \(p_m(x)\) and weight function \(w(x)\) are both positive.

In the Section 2 we give some preliminary definitions for non-singular 2m-order Sturm-Liouville problems and basic idea of VIM method. In Section 3 we will propose a novel algorithm of VIM for the 2m-order Sturm-Liouville problems. In Section 4, while numerical results are discussed several work examples are solved to demonstrate high performance of the proposed method.

2. Preliminaries

In this section we introduce some notation and definitions necessary for this work.

2.1. 2m-order Sturm-Liouville problems. Let us rewrite equation (1.1) in the following form

\[(-1)^m(p_m(x)y^{(m)})^{(m)} = F(y, y', \ldots, y^{2m-2}, E)\]
\[= (Ew(x) - p_0(x))y - \{(-1)^{m-1}(p_{m-1}(x)y^{(m-1)})^{(m-1)} + \cdots + (p_2(x)y''')''\}, \quad a < x < b, \] (2.1)
subject to some $2m$ point specified conditions at the boundary $x \in \{a, b\}$ on

$$
u_k = y^{(k-1)}, \quad 1 \leq k \leq m,$$

$$v_1 = p_1y' - (p_2y'')' + (p_3y''')' + \cdots + (-1)^{m-1}(p_m y^{(m)})^{(m-1)},$$

$$v_2 = p_2y'' - (p_3y''')' + (p_4y^{(4)})'' + \cdots + (-1)^{m-2}(p_m y^{(m)})^{(m-2)},$$

$$\vdots$$

$$v_k = p_k y^{(k)} - (p_{k+1} y^{(k+1)})' + (p_{k+2} y^{(k+2)})'' + \cdots + (-1)^{m-k}(p_m y^{(m)})^{(m-k)},$$

$$\vdots$$

$$v_m = p_m y^{(m)}.$$  \hfill (2.2)

The eigenvalues $E_k$, $k = 1, 2, 3, \ldots$ can be ordered as an increasing sequence, i.e.,

$$E_1 \leq E_2 \leq E_3 \leq \cdots,$$

where $\lim_{k \to \infty} E_k = \infty$ and each eigenvalue has multiplicity at most $m$. The restriction on the multiplicity arises from the fact that for each $E_k$, $k = 1, 2, 3, \ldots$ there are at most $m$ linear independent solutions of the differential equation satisfying either of the endpoint conditions, [6, 25].

Let $L^2_w(a, b)$, be the space of functions $f(x)$ on $(a, b)$ such that

$$\int_a^b |f(x)|^2 w(x)dx < \infty.$$  

$L^2_w(a, b)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}w(x)dx,$$

and norm $\|f\|^2 = \langle f, f \rangle$.

2.2. Basic idea of variational iteration method. Consider the general nonlinear differential equation given in the form

$$Ly(x) + Ny(x) = g(x),$$  \hfill (2.3)

where $g(x)$ is a given function, $L$ and $N$ are some linear and nonlinear operators respectively. By using the variational iteration method, a correction functional can be constructed as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(Ly_n(s) + Ny_n(s) - g(s)) \, ds, \quad n \geq 0,$$  \hfill (2.4)

where $\lambda$ is a general Lagrange multiplier [26], which can be identified optimally via the variational theory, the index $n$ means the $n$th order approximation for $y_n$, and $\tilde{y}_n$ is a restricted variation with the property $\delta \tilde{y}_n = 0$, (see for example, [12-20]).
From equations (3.6), the Lagrange multiplier can be derived as following form

\[ VIM \text{ for } 2m \]

VIM generalized algorithm.

3.1. Consequently, we obtain the following stationary conditions

\[ \lambda = \frac{(s - x)^{2m-1}}{(2m - 1)!}. \]
Now, by substituting equation (3.7) into (3.3) we get the following iteration formula
\[ y_{n+1}(x) = y_n(x) + \int_0^x (s - x)^{(2m-1)} (Ly_n(s) + Ny_n(s)) ds, \] (3.8)
where \( y_0(x) \) can be chosen to be the solution of the equation \( Ly(x) = y_0^{(2m)}(x) = 0 \).

Thus
\[ y_0(x) = \sum_{i=0}^{2m-1} c_i x^i, \] (3.9)
where \( c_0, c_1, \ldots, c_{2m-1} \) are some constants. From (3.8), we obtain the successive approximations of problem (1.1) and the exact solution can be derived in the following form
\[ y(x) = \lim_{n \to \infty} y_n(x). \] (3.10)

In fact, the solution of problem (3.1) is considered as the fixed point of the correction functional (3.8) under the suitable choice of the initial term that is given by (3.9).

Note that exactly \( m \) conditions are specified initially at \( x = a \), (these \( m \) conditions arise in different forms based on nature of the problem such as order of the highest derivative appearing in each condition must be less than \( 2m \)). Now, if these \( m \) conditions at \( x = a \) have the following form
\[ y_n(a, E) = y'_n(a, E) = \cdots = y^{(m-1)}_n(a, E) = 0, \]
then the approximate solution will be
\[ y_n(x, E) = \sum_{i=m}^{2m-1} c_i f_{n_i}(x, E), \quad n > 0. \] (3.11)

By using other conditions at endpoint \( b \), for example \( y_n(b, E) = y'_n(b, E) = \cdots = y^{(m-1)}_n(b, E) = 0 \), we get the following system
\[
\begin{align*}
\sum_{i=m}^{2m-1} c_i f_{n_i}(b, E) &= 0, \\
\sum_{i=m}^{2m-1} c_i f'_{n_i}(b, E) &= 0, \\
& \vdots \\
\sum_{i=m}^{2m-1} c_i f^{(m-1)}_{n_i}(b, E) &= 0,
\end{align*}
\] (3.12)
for \( c_m, c_{m+1}, \ldots, c_{2m-1} \). By Cramer’s rule, we will get a nontrivial solution for the system (3.12) if
\[
M_n(E) = \det \begin{pmatrix}
f_{n_m}(b, E) & f_{n_{m+1}}(b, E) & \cdots & f_{n_{2m-1}}(b, E) \\
f'_{n_m}(b, E) & f'_{n_{m+1}}(b, E) & \cdots & f'_{n_{2m-1}}(b, E) \\
\vdots & \vdots & \cdots & \vdots \\
f^{(m-1)}_{n_m}(b, E) & f^{(m-1)}_{n_{m+1}}(b, E) & \cdots & f^{(m-1)}_{n_{2m-1}}(b, E)
\end{pmatrix} = 0, \] (3.13)
which is a polynomial in \( E \). Therefore the eigenvalues of the problem (1.1) are the roots of \( M_n(E) \). In practice, stopping criterion for approximated eigenvalue \( E_k \) in the \( n \)-th iteration is
\[ |E_k - E_{k-1}| \leq \epsilon, \] (3.14)
where \( \epsilon \) may be made arbitrary small according to the accuracy required. Subsection 3.1 may be summarized in the following algorithm.

**Algorithm 3.1.**

Step 1: Use equation (3.9) and the initial conditions to construct \( y_0(x) \).

Step 2: Use iteration formula (3.8) to generate the sequence \( \{y_n\}_{n=1}^K \) for some positive integer \( K \).

Step 3: Construct the function \( M_n(E) \) as indicated in (3.13).

Step 4: Find roots of the \( M_n(E) \), by using (3.14).

**Note.** It is obvious that, roots in Step 4 are eigenvalues of the problem (1.1).

3.2. **Convergence analysis.** In this subsection, we discuss the convergence of generalized VIM presented in the previous subsection. From (3.8) we may define the operator \( L \) in the following form

\[
L[y] = \int_0^x \frac{(s-x)(2m-1)}{(2m-1)!} (Ly(s) + Ny(s)) \, ds. \quad (3.15)
\]

By using (3.15), we can construct the following components

\[
f_0 = y_0, \quad f_{n+1} = L[\sum_{i=0}^n f_i]. \quad (3.16)
\]

We conclude that, the exact solution is in the following form

\[
y(x) = \lim_{n \to \infty} y_n(x) = \sum_{n=0}^{\infty} f_n(x). \quad (3.17)
\]

Now, by using initial approximation \( f_0 = y_0 \) (see (3.9)), the approximation solution can be considered by taking \( n \)-terms of the series (3.17), that is \( y_n(x) = \sum_{i=0}^n f_i(x) \). The variational iteration method makes a sequence \( \{y_n\} \), here, we show that the sequence \( \{y_n\} \) converges to the solution of problem (1.1). To do this, we state and prove the following theorems.

**Theorem 3.2.** Let \( L : L^2_w(a,b) \to L^2_w(a,b) \) be an operator in a Hilbert space satisfy in (3.15). Then the series solution \( y(x) \) defined by (3.17) for problem (1.1) converges if there exist \( 0 < \alpha < 1 \) such that

\[
\|f_{n+1}\|_{L^2_w} \leq \alpha \|f_n\|_{L^2_w}. \quad (3.18)
\]

**Proof.** Define the sequence of the partial sums \( s_n \) such that \( s_0 = f_0, \quad s_n = \sum_{i=0}^n f_i \), we see that the sequence \( \{s_n\} \) is well-defined. Let us first prove that \( \{s_n\} \) is a Cauchy sequence in the \( L^2_w(a,b) \) space. For this purpose, we see that

\[
\|s_{n+1} - s_n\|_{L^2_w} \leq \alpha \|s_n - s_{n-1}\|_{L^2_w} \leq \cdots \leq \alpha^n \|s_1 - s_0\|_{L^2_w}. \quad (3.19)
\]

Then for any \( m \geq n \), we have

\[
\|s_m - s_n\|_{L^2_w} \leq \|s_{n+1} - s_n\|_{L^2_w} + \|s_{n+2} - s_{n+1}\|_{L^2_w} + \cdots + \|s_m - s_{m-1}\|_{L^2_w} \\
\leq \alpha^n [1 + \alpha + \cdots + \alpha^{m-n-1}] \|s_1 - s_0\|_{L^2_w} \\
\leq \frac{\alpha^n}{1 - \alpha} \|s_1 - s_0\|_{L^2_w}. \quad (3.20)
\]
Since \( \alpha \in (0, 1) \), then \( \| s_m - s_n \|_{L^2_w} \to 0 \) as \( m, n \to \infty \). Thus \( \{s_n\} \) is a Cauchy sequence in the \( L^2_w(a, b) \) space, therefore the series solution converges and the proof is complete.

**Theorem 3.3.** If the series solution (3.17) generated by using iteration formula (3.8) converges, then it converges to an exact solution of the problem (1.1).

**Proof.** If the series (3.17) converges, we can write \( y(x) = \lim_{n \to \infty} y_n(x) = \sum_{n=0}^{\infty} f_n(x) \), then

\[
\lim_{n \to \infty} f_n = 0. \tag{3.21}
\]

We can write,

\[
\sum_{n=0}^{m} [f_{n+1} - f_n] = (f_1 - f_0) + \cdots + (f_{m+1} - f_m) = f_{m+1} - f_0. \tag{3.22}
\]

Hence,

\[
\sum_{n=0}^{\infty} [f_{n+1} - f_n] = \lim_{n \to \infty} f_n - f_0 = -f_0. \tag{3.23}
\]

Now, by applying linear operator \( L = \frac{d^{2m}}{dx^{2m}} \), to both sides of (3.23) and since \( f_0 = y_0 = \sum_{i=0}^{2m-1} c_i x^i \), see (3.9), we get

\[
\sum_{n=0}^{\infty} L[f_{n+1} - f_n] = -L f_0 = 0. \tag{3.24}
\]

Now, from (3.16), we have

\[
L[f_{n+1} - f_n] = L[\mathcal{L}\sum_{i=0}^{n} f_i] - L[\sum_{i=0}^{n-1} f_i], \tag{3.25}
\]

and by using (3.15), we obtain

\[
L[f_{n+1} - f_n] = L[\int_0^x \frac{(x-s)^{(2m-1)}}{(2m-1)!} (L[\sum_{i=0}^{n} f_i] + N[\sum_{i=0}^{n} f_i]) ds] - L[\sum_{i=0}^{n-1} f_i] - N[\sum_{i=0}^{n-1} f_i], \quad n \geq 1. \tag{3.26}
\]

Now, since the linear differential operator \( L = \frac{d^{2m}}{dx^{2m}} \) is the left inverse to \( 2m \)-fold integral operator, then (3.26) becomes

\[
L[f_{n+1} - f_n] = L[f_n] + N[f_n]. \tag{3.27}
\]

Thus

\[
\sum_{n=0}^{m} L[f_{n+1} - f_n] = L \sum_{n=0}^{m} f_n + N \sum_{n=0}^{m} f_n, \tag{3.28}
\]

and so,

\[
\sum_{n=0}^{\infty} L[f_{n+1} - f_n] = L \sum_{n=0}^{\infty} f_n + N \sum_{n=0}^{\infty} f_n. \tag{3.29}
\]

Therefore, from (3.24) and (3.29), we see that \( y(x) = \sum_{n=0}^{\infty} f_n(x) \) must be an exact solution.
Theorem 3.4. For the approximation solution \( y_n = \sum_{i=0}^{n} f_i \), if the series solution (3.17) is convergent to an exact solution \( y(x) \), then the error estimate is

\[
e_n = \frac{\alpha^n}{1 - \alpha} \| f_1 \|_{L^2_w}.
\]  (3.30)

Proof. From Theorem 3.1, letting \( m \to \infty \) in (3.20), we get

\[
\|y(x) - \sum_{i=0}^{n} f_i \|_{L^2_w} \leq \frac{\alpha^n}{1 - \alpha} \| f_1 \|_{L^2_w},
\]  (3.31)

and this completes the proof.

Note that, Theorem 3.2 is a special case of Banach fixed point theorem [27].

4. Work examples

In this section the efficiency of the generalized variational iteration method (proposed in Section 3) is illustrated. Three classes of work examples of second-order, fourth-order and sixth-order Sturm-Liouville problems are discussed. In order to compare our results with others, each problem represent a specific example of eigenvalue problems that are frequently studied in the context of Sturm-Liouville operators. In Examples 4.1-4.3, \( \epsilon \) is chosen to be \( 10^{-10} \).

Example 4.1. Consider the following second-order eigenvalue problem

\[
y^{(2)}(x) + Ey(x) = 0, \quad x \in (0, 1),
\]  (4.1)

subject to boundary conditions

\[
y'(0) = 0, \quad y(1) = 0.
\]  (4.2)

Thus, from (3.7) the Lagrange multiplier becomes

\[
\lambda = s - x.
\]

and corresponding to (3.8), the correction functional for equation (4.1) is give by

\[
y_{n+1}(x) = y_n(x) + \int_{0}^{x} (s - x) \left( \frac{d^2}{ds^2} y_n(s) + Ey_n(s) \right) ds.
\]  (4.3)

Now, choose \( y_0 \) so that \( L(y_0) = 0 \) and \( y'(0) = 0 \). Simple calculations implies that \( y_0(x) = c_0 \), where \( c_0 \) is a constant. By applying iteration formula (4.3), we get the following approximations,

\[
y_1(x) = c_0(1 - \frac{E x^2}{2}),
\]

\[
y_2(x) = c_0(1 - \frac{E x^2}{2} + E^2 \frac{x^4}{24}),
\]

\[
\vdots
\]

It is easy to see that

\[
y_n(x, E) = c_0 f_n(x, E), \quad n > 0.
\]  (4.4)

Using the boundary condition at 1, we get an equation of the form

\[
y_n(1, E) = c_0 f_n(1, E) = 0.
\]  (4.5)
Table 1. Convergence behavior of the first three eigenvalues for Example 4.1

<table>
<thead>
<tr>
<th>n</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2.4674011003</td>
<td>22.2316528426</td>
<td>46.2659547508</td>
</tr>
<tr>
<td>8</td>
<td>2.4674011003</td>
<td>22.2047858586</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>2.4674011003</td>
<td>22.2067176578</td>
<td>58.8534769990</td>
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<tr>
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<td>22.206046809</td>
<td>62.385801420</td>
</tr>
<tr>
<td>11</td>
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<td>22.206101110</td>
<td>61.6158764684</td>
</tr>
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</tr>
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</table>

Since $c_0 \neq 0$, this shows that the last equation has nonzero solution if
\[ f_{n_0}(1, E) = 0, \] (4.6)
which is a polynomial in $E$. The eigenvalues of problem (4.1)-(4.2) are the roots of (4.6). The first three eigenvalues of problem (4.1)-(4.2) are calculated and given in the Table 1. This Table shows that after at most 20 iterations these 3 eigenvalues converge to correct solution up to 10 decimal points.

In Fig. 1, the first three normalized eigenfunctions correspond to eigenvalues $E_1$, $E_2$, and $E_3$ are plotted, where the normalized eigenfunction $\overline{y}_k$ is given by
\[ \overline{y}_k(x, E_k) = \frac{y_k(x, E_k)}{\int_0^1 |y_k(x, E_k)| \, dx}, \quad k = 1, 2, 3. \] (4.7)

Example 4.2. Consider the following fourth-order Sturm-Liouville problem
\[ y^{(4)}(x) = Ey(x), \quad x \in (0, 1), \] (4.8)
subject to
\[ y(0) = y'(0) = 0, \quad y(1) = y''(1) = 0. \] (4.9)
In elasticity, the fourth-order Sturm-Liouville equations are associated to the steady state Euler-Bernoulli equation for the deflection $y$ of a vibrating beam, with the other quantities having physical meaning, for example $p > 0$ is the flexural rigidity of the beam, $Ey''$ is the bending moment and $Ew - q$ is the frequency of vibration, (for example see [6, 28, 29]). Let us choose $y_0(x)$ so that $Ly_0(x) = 0$, $y(0) = y'(0) = 0$. A simple calculation implies that $y_0(x) = c_2 x^2 + c_3 x^3$, where $c_2$, $c_3$ are some constants. By using (3.7) and (3.8), the iteration formula for (4.8), can be constructed as,
\[ y_{n+1}(x) = y_n(x) + \int_0^x (s - x)^3 6 (y_n^{(4)}(s) - Ey_n(s)) \, ds. \] (4.10)
Figure 1. The first three normalized eigenfunctions for Example 4.1, where △ and - stand for approximated and exact solutions.

By applying iteration formula (4.10) we get the following approximations,

\[ y_1(x) = c_2 \left( x^2 + E \frac{x^6}{360} \right) + c_3 \left( x^3 + E \frac{x^7}{540} \right), \]

\[ y_2(x) = c_2 \left( x^2 + E \frac{x^6}{360} + E^2 \frac{x^{10}}{16140} \right) + c_3 \left( x^3 + E \frac{x^7}{540} + E^2 \frac{x^{11}}{604800} \right), \]

\[ \vdots \]
We see that
\[ y_n(x, E) = c_2 f_{n_2}(x, E) + c_3 f_{n_3}(x, E). \]

Now by using the boundary conditions at 1, we get the following system
\[ y_n(1, E) = c_2 f_{n_2}(1, E) + c_3 f_{n_3}(1, E) = 0, \]
\[ y_n(1, E) = c_2 f'''_{n_2}(1, E) + c_3 f'''_{n_3}(1, E) = 0. \]

Hence, we have nonzero solution for \( c_2 \) and \( c_3 \), if
\[ M(E) = \det \begin{pmatrix} f_{n_2}(1, E) & f_{n_3}(1, E) \\ f'''_{n_2}(1, E) & f'''_{n_3}(1, E) \end{pmatrix} = 0. \quad (4.11) \]

By computing roots of (4.11), we can obtain the eigenvalues of problem (4.8)-(4.9). In this example, the first eigenvalue \( E_1 \) is obtained after sixth iteration and the second eigenvalue \( E_2 \) is obtained in the ninth iteration. Following the same approach, the remaining eigenvalues \( E_k \), \( k = 3, 4, 5, 6 \) are obtained, and these are listed in Table 2. The first three normalized eigenfunctions are plotted in Fig. 2. Our numerical results are close to those values obtained by ADM, (see Table 1 in Ref. [29]). In Table 2 it is shown that at most 20 iterations are needed that all of 6 eigenvalues converge to the correct solution up to 10 decimal points.

**Example 4.3.** Consider the following sixth-order boundary value problem
\[ -y^{(6)}(x) = E y(x), \quad x \in (0, \pi), \quad (4.12) \]
subject to homogeneous boundary value conditions
\[ y(0) = y''(0) = y^{(4)}(0) = 0, \]
\[ y(\pi) = y''(\pi) = y^{(4)}(\pi) = 0. \quad (4.13) \]

Let \( L(y) = -y'' \) and \( N(y) = -Ey(x) \). As an initial approximating solution, let us choose \( y_0 \) so that \( L(y_0) = 0 \) and \( y(0) = y''(0) = y^{(4)}(0) = 0 \), this implies that
Figure 2. The first three normalized eigenfunctions for Example 4.2, where $\circ$ and $-$ stand for approximated and exact solutions

$y_0(x) = c_1x + c_3x^3 + c_5x^5$, where $c_1$, $c_3$, and $c_5$ are some constants. From equation (3.7), the Lagrange multiplier can be derived as

$$\lambda = \frac{(s - x)^5}{5!},$$
and corresponding to equation (3.8), the iteration formula for equation (4.12) is given by,
\[ y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^5}{5!} \left( -\gamma_n^{(6)}(s) - Ey_n(s) \right) ds. \] (4.14)

By applying iteration formula (4.14) we get the following approximations,
\[ y_1(x) = c_1(x + E \frac{x^7}{5040}) + c_3(x^3 + E \frac{x^9}{60480}) + c_5(x^5 + E \frac{x^{11}}{332640}), \]
\[ y_2(x) = c_1 \left( x + \frac{E_2}{5!} x^7 + E \frac{E_3}{3!} x^9 + \frac{E_5}{2!} \right) + c_3 \left( x^3 + \frac{E_3}{3!} x^9 + \frac{E_5}{2!} \right) + c_5 \left( x^5 + \frac{E_5}{3!} x^{11} + \frac{E_7}{4!} \right), \]
\[ \vdots \]

It is easy to see that
\[ y_n(x, E) = c_1 f_{n_1}(x, E) + c_3 f_{n_3}(x, E) + c_5 f_{n_5}(x, E), \quad n > 0. \] (4.15)

Using the boundary conditions at \( x = \pi \), namely, \( y(\pi) = y''(\pi) = y^{(4)}(\pi) = 0 \), we get three equations of the form
\[ c_1 f_{n_1}(\pi, E) + c_3 f_{n_3}(\pi, E) + c_5 f_{n_5}(\pi, E) = 0, \]
\[ c_1 f''_{n_1}(\pi, E) + c_3 f''_{n_3}(\pi, E) + c_5 f''_{n_5}(\pi, E) = 0, \]
\[ c_1 f^{(4)}_{n_1}(\pi, E) + c_3 f^{(4)}_{n_3}(\pi, E) + c_5 f^{(4)}_{n_5}(\pi, E) = 0. \]

We see that the last system have nonzero values for \( c_1, c_3 \) and \( c_5 \), if
\[ M_n(E) = \det \begin{pmatrix} f_{n_1}(\pi, E) & f_1''_{n_1}(\pi, E) & f^{(4)}_{n_1}(\pi, E) \\ f_{n_3}(\pi, E) & f_1''_{n_3}(\pi, E) & f^{(4)}_{n_3}(\pi, E) \\ f_{n_5}(\pi, E) & f_1''_{n_5}(\pi, E) & f^{(4)}_{n_5}(\pi, E) \end{pmatrix} = 0. \] (4.16)

**Table 3.** Convergence behavior of the first ten eigenvalues for Example 4.3

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The first ten eigenvalues of problem (4.12)-(4.13) are given in Table 3. The first normalized eigenfunction $\tilde{y}_1$ is given by

$$\tilde{y}_1(x, E_1) = \frac{y_1(x, E_1)}{\int_0^\pi |y_1(x, E_1)| \, dx} \quad (4.17)$$

and it is plotted in Fig. 3. Here the first estimated eigenvalue $E_1$ is obtained in the fourth iteration and the second eigenvalue $E_2$ is obtained in the sixth iteration. The other eigenvalues are found in the same way (see Table 3). These results are more accurate than the results obtained by using the ADM (see Table 2 in Ref. [11]) and results obtained by using shooting method (see Table 6.1 in Ref. [5]). It is well known that the exact eigenvalues are given by $E_k = k^6$.

**Example 4.4.** We wanted to test our algorithm on a problem whose differential equation exhibits stiffness in at least part of its range: we chose the following problems

![Figure 3. The first normalized eigenfunction $\tilde{y}_1(x, E_1)$ for Example 4.3, where * and - stand for approximated and exact solutions](image-url)
(i) Consider the following second-order Sturm-Liouville problem

\[-y''(x) + \alpha x^2 y(x) = Ey(x), \quad x \in (0, 5),\]

subject to

\[y(0) = y(5) = 0.\]

Let \(L(y) = y''\) and \(N(y) = (\alpha x^2 y - Ey).\) Choose \(y_0\) so that \(L(y_0) = 0\) and \(y(0) = 0.\) Simple calculation implies \(y_0(x) = c_0.\) The iteration formula (3.8) becomes

\[y_{n+1}(x) = y_n(x) + \int_0^x (s - x) \left( \frac{d^2}{ds^2} y_n(s) + \alpha x^2 y_n(s) - Ey_n(s) \right) \, ds.\]

For \(\alpha = 0.01,\) the first two eigenvalues are: \(E_1 = 0.4637357700,\) \(E_2 = 1.6597620115.\)

(ii) Consider the following fourth-order Sturm-Liouville problem

\[y^{(4)}(x) - 0.02x^2 y'' - 0.04xy' + (0.0001x^3 - 0.02)y = Ey(x), \quad x \in (0, 5),\]

subject to

\[y(0) = y''(0) = 0, \quad y(5) = y''(5) = 0.\]

By using Algorithm 3.1, the first fifth eigenvalues are: \(E_1 = 0.2150508644,\) \(E_2 = 2.7548099347,\) \(E_3 = 13.2153515406,\) \(E_4 = 40.9508197591\) and \(E_5 = 99.0534781381.\) This result show that, the eigenvalues of this problem are the squares of eigenvalues of problem (4.18)-(4.19).

(iii) Consider the following sixth-order Sturm-Liouville problem

\[-y^{(6)}(x) + (3\alpha^2 x^2 y'')'' + ((8\alpha - 3\alpha^2 x^4)y')' + (\alpha^3 x^6 - 14\alpha^2 x^2)y = Ey(x), \quad x \in (0, 5),\]

subject to homogeneous boundary value conditions

\[y(0) = y''(0) = y^{(4)}(0) = 0,\]
\[y(5) = y''(5) = y^{(4)}(5) = 0.\]

By using Algorithm 3.1, the first fifth eigenvalues for \(\alpha = 0.01\) are: \(E_1 = 0.0997267728,\) \(E_2 = 4.57232104092,\) \(E_3 = 48.0416683775,\) \(E_4 = 262.0590748452\) and \(E_5 = 985.8390701194.\) We see that, the eigenvalues of this problem are the cubes of the eigenvalues of the second-order problem (4.18)-(4.19).

5. Conclusion

This paper suggests an effective numerical algorithm for the high order Sturm-Liouville problem and the results are of high accuracy. We have proposed a numerical technique based on fixed point variational iteration method for computing eigenvalues of the general \(2m\)-order Sturm-Liouville problems for \(m \geq 1.\) First, a generalization VIM algorithm for \(m = 1\) is explained for fourth and sixth order Sturm-Liouville problems numerically. Second, theoretical, convergence and numerical aspects of the generalization of VIM for \(2m\)-order Sturm-Liouville problems is discussed for general case \(m \geq 1.\) In this process a general formula for the Lagrange multiplier \(\lambda\) is given. Three different cases for (i) \(m = 1,\) (ii) \(m = 2\) and (iii) \(m = 3\) for the solution of second, fourth and sixth order Sturm-Liouville problems are discussed. Numerical results (obtained from proposed method in this paper) are compared with results that obtained by exact solution, Adomian decomposition method and shooting method.
Numerical results show that the variational iteration method is an efficient tool to compute eigenvalues of high-order Sturm-Liouville problems.

References


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