Fixed Point Theory, 14(2013), No. 1, 185-192
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT THEOREMS FOR A CLASS OF NONLINEAR OPERATORS IN RIESZ SPACES

JINGXIAN SUN* AND YUJUN CUI**

*Department of Mathematics
Xuzhou Normal University
Xuzhou, 221116, China
E-mail: jxsun7083@sohu.com

**Department of Mathematics
Shandong University of Science and Technology
Qingdao 266590, China
E-mail: cyj720201@163.com

Abstract. In this paper, we study a class of nonlinear operator equations in Riesz spaces. By using the lattice structure and the partially ordered method, the existence and uniqueness of solutions for such equations are investigated without demanding the topological structure of the ordered vector space.
Key Words and Phrases: Riesz spaces, lattice structure, fixed point theorem.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

As a powerful mechanism for mathematical analysis, fixed point theory has many applications in areas such as mechanics, physics, transportation, control, economics, and optimization. Perhaps the most well known results in the theory of fixed points is Banach's contraction mapping principle:

Theorem A. *Let (X, d) be a complete metric space, $F : X \to X$ a contraction mapping. Then F has one and only one fixed point $u \in X$.*

As is well-known, Theorem A has been extensively used in differential and functional differential equations. There are also many variants of Theorem A[2,3]. In recent years, many papers investigated nonlinear operators in ordered Banach spaces (e.g., increasing operators, decreasing operators, etc.), obtained some fixed point theorems by using partial ordering inequalities, and gave some applications to the ordinary differential equations, partial differential equations, and integral equations [1, 4, 5, 7, 9].

However all of above results are obtained under the assumption that the space has topological structure. In this paper, we study a class of nonlinear operator equations in Riesz spaces. By using the lattice structure and the partially ordered method, the existence and uniqueness of solutions for such equations are investigated without demanding the topological structure of the ordered vector space.

For convenience, let us introduce some definitions and symbols which can be found in [8].

Let E be a nonempty set. The relation R in E is called a partial ordering of E whenever R is transitive, reflective and anti-symmetric, i.e., whenever

(i) it follows from xRy and yRz that xRz,

(ii) xRx holds for all $x \in E$,

(iii) it follows from xRy and yRx that $x = y$.

If R is a partial ordering in E, we will usually write $x \leq y$(or, equivalently, $y \geq x$) for xRy. If E is partial ordered, Y a nonempty subset of E, and $x_0 \in E$ satisfies $x_0 \geq y$ for all $y \in Y$, then x_0 is called an upper bound of Y. If x_0 is an upper bound of Y such that $x_0 \leq x_0'$ for any other upper bound x_0' of Y, then x_0 is called a least upper bound or supremum of Y. In this case, x_0 is uniquely determined and we denoted by $x_0 = \sup Y$. The notions of lower bound and greatest lower bound or infimum are defined similarly. Notations: $x_0 = \inf Y$ if x_0 is the infimum of Y.

Definition 1.1. The partially ordered set E is called a lattice if every subset consisting of two elements has a supremum and an infimum.

If E is a lattice in the partial ordering "$\leq$". For $x \in E$, let

$$x^+ = \sup\{x, \theta\}, \quad x^- = \sup\{-x, \theta\},$$

x^+ and x^- are called the positive and the negative part of x respectively.

Definition 1.2. The real linear space E, with elements $x, y, \ldots$, is called an ordered vector space if E is partial ordering in such a manner that the partial ordering is compatible with the algebraic structure of E, i.e.,

(i) $x \leq y$ implies $x + z \leq y + z$ for every $z \in E$,

(ii) $x \geq \theta$ implies $\alpha x \geq \theta$ for every real number $\alpha \geq 0$.

Given the ordered vector space E, the subset $P := \{x : x \in E,\ x \geq \theta\}$ is called the positive cone of E. Elements of P are called positive elements. Take $|x| = x^+ + x^-$, then $|x|$ is called the module of x.

The ordered vector space E is called a Riesz space if E is also a Lattice. The Riesz space E is called Archimedean if, given $x, y \in P$ such that $nx \leq y$ holds for $n = 1, 2, \ldots$, we have $x = \theta$.

Lemma 1.3. [7] *If E is a Riesz space, then the following holds:*

(i) $x^+,\ x^- \in P$, $x = x^+ - x^-$ *with* $\sup\{x^+, x^-\} = \theta$; $|x| \in P$,

(ii) $\theta \leq x^+ \leq |x|$, $\theta \leq x^- \leq |x|$, $-x^- \leq x \leq x^+$, $-|x| \leq x \leq |x|$,

(iii) if $x, y \in E$, *then* $|x + y| \leq |x| + |y|$.

The sequence $\{x_n\}$ in the ordered vector space E is called increasing if $x_1 \leq x_2 \leq \ldots$. This will be denoted by $x_n \uparrow$. If $x_n \uparrow$ and $x = \sup\{x_n\}$ exists in E, we will write $x_n \uparrow x$. Let u be a fixed nonzero element in the positive cone P of the Riesz space E. The sequence $\{x_n\}$ in E is said to converge $u-$uniformly to the element $x \in E$ whenever, for every $\varepsilon > 0$, there exists a natural number $n_0 = n_0(\varepsilon)$ such that $|x - x_n| \leq \varepsilon u$ holds for all $n \geq n_0$. If E is Archimedean, the $u-$uniform limit of a sequence, if existing, is unique and we denoted by $x_n \to x(ru)$. Furthermore, if

$x_n \uparrow$ and $x_n \to x(ru)$, then $x_n \uparrow x$. The sequence $\{x_n\}$ in E is called an $u-$uniform Cauchy sequence whenever, for every $\varepsilon > 0$, there exists a natural number $n_1 = n_1(\varepsilon)$ such that $|x_n - x_m| \leq \varepsilon u$ holds for all $n, m \geq n_1$.

Definition 1.4. The Riesz space E is called $u-$uniformly complete whenever every $u-$uniform Cauchy sequence has an $u-$uniform limit.

Definition 1.5. [7] Let u_0 be a fixed nonzero element in the positive cone P of the Riesz space E. The linear operator B is said to be increasing if $B(P) \subset P$. The linear operator B is said to be u_0-bounded if for every nonzero $x \in P$ a natural number n and two positive numbers α, β can be found such that $\alpha u_0 \leq B^n x \leq \beta u_0$.

$u^* \in E$ be called a positive characteristic vector of linear operator B if $u^* \in P\backslash\{\theta\}$ and there exists $\lambda > 0$ such that $Bu^* = \lambda u^*$.

Lemma 1.6. *Let u^* be a positive characteristic vector of u_0-bounded increasing operator B. Then B is a u^*-bounded operator.*

Proof. It follows from the definition 1.5 that for some α_0 and β_0 the inequalities

$$\alpha_0 u_0 \leq u^* \leq \beta_0 u_0 \tag{1.1}$$

are satisfied. Then from the inequalities

$$\alpha u_0 \leq B^n x \leq \beta u_0$$

it follows that

$$\frac{\alpha}{\beta_0} u^* \leq B^n x \leq \frac{\beta}{\alpha_0} u^*.$$

This completes the proof of the lemma.

2. Fixed points of nonlinear operators in Riesz spaces

In this section, we present the main results of this paper.

Theorem 2.1. *Let E be an u_0-uniformly complete Archimedean Riesz space, $A : E \to E$ a nonlinear operator. Suppose that there exists an u_0-bounded increasing operator $B : E \to E$ such that the following conditions are satisfied:*

(i) there exist $\lambda \in (0, 1)$ and a nonzero positive element u^* such that

$$Bu^* = \lambda u^*.$$

(ii) for $\forall\ x, y \in E$,

$$|Ax - Ay| \leq B(|x - y|).$$

Then A has an unique fixed point $x^* \in E$, and for any $x_0 \in E$, let $x_n = Ax_{n-1}(n = 1, 2, \dots)$, then $\{x_n\}$ converge u_0- uniformly to the element x^*.

Proof. First, by (1.1), we know that E is a u_0-uniformly complete Riesz space if and only if E is also a u^*-uniformly complete Riesz space.

Next we show that A has at most one fixed point. Suppose there exist two elements $x, y \in E$ with $x = Ax$ and $y = Ay$. By Lemma 1.6, there exist $n_1 \in N$ and $\beta > 0$ such that

$$B^{n_1}(|x - y|) \leq \beta u^*.$$

Then for all $n \in N$, we have

$$|x - y| = |A^{n_1 n} x - A^{n_1 n} y| \leq B^n \left(B^{n_1}(|x - y|)\right) \leq \beta B^n u^* \leq \beta \lambda^n u^*.$$

Since E is Archimedean, we get $x = y$. This implies that A has at most one fixed point.

For any given $x_0 \in E$, Let $x_n = Ax_{n-1}(n = 1, 2, \dots,)$. We first show that $\{x_n\}$ is a u^*-uniform Cauchy sequence. By Lemma 1.2, there exist $n_1 \in N$ and $\beta > 0$ such that

$$B^{n_1}(|x_1 - x_0|) \leq \beta u^*.$$

Notice for $p \in N$ that

$$\begin{aligned} |x_{n_1+p+1} - x_{n_1+p}| &= |Ax_{n_1+p} - Ax_{n_1+p-1}| \leq B(|x_{n_1+p} - x_{n_1+p-1}|) \\ &\leq \cdots \leq B^{n_1+p}(|x_1 - x_0|) \leq \beta\lambda^p u^*. \end{aligned}$$

Thus for $n > n_1$ and $m \in N$

$$\begin{aligned} |x_{n+m+1} - x_n| &= |x_{n+m+1} - x_{n+m} + \cdots + x_{n+1} - x_n| \\ &\leq |x_{n+m+1} - x_{n+m}| + \cdots + |x_{n+1} - x_n| \\ &\leq \beta\left[\lambda^{n+m-n_1} + \cdots + \lambda^{n-n_1}\right] u^* \\ &= \beta\frac{\lambda^{n-n_1}(1 - \lambda^{m+1})}{1 - \lambda} u^*. \end{aligned}$$

This shows that $\{x_n\}$ is a u^*-uniform Cauchy sequence and since that E is u^*-uniformly complete there exists $x \in E$ such that x_n is u^*- uniformly convergent to the element x^*. So, for every $\varepsilon > 0$, there exists a natural number $n_0 = n_0(\varepsilon)$ such that $|x^* - x_n| \leq \varepsilon u$ holds for all $n \geq n_0$. Thus, for every $n \geq n_0$, we have

$$\begin{aligned} |x^* - Ax^*| &\leq |x^* - x_n| + |Ax_{n-1} - Ax^*| \\ &\leq |x^* - x_n| + B|x_{n-1} - x^*| \leq \varepsilon u^* + \varepsilon\lambda u^* \leq 2\varepsilon u^*, \end{aligned}$$

Therefore x^* is the unique fixed point of A. This completes the proof.

Theorem 2.2. *Let u^* be a nonzero positive element, E an u^*-uniformly complete Archimedean Riesz space, $A : E \to E$ a nonlinear operator. Suppose that there exists an u_0-bounded increasing operator $B : E \to E$ such that the hypotheses (ii) of Theorem 2.1 is satisfied and there exists $\lambda \in (0, 1)$ such that*

$$Bu^* \leq \lambda u^*.$$

Then A has an unique fixed point $x^* \in E$, and for any $x_0 \in E$, let $x_n = Ax_{n-1}$(n=1,2,...); then $\{x_n\}$ is u^*- uniformly convergent to the element x^*.

The proof of this theorem is based on the following lemma and the method used in Theorem 2.1.

Lemma 2.3. *If B is an u_0-bounded increasing operator, and there exists $\lambda \in (0, 1)$ such that $Bu^* \leq \lambda u^*$. Then for every nonzero $x \in P$, there exist a natural number n and a positive numbers β_1 such that $B^n x \leq \beta_1 u^*$.*

Proof. It follows from the definition 1.5 that for some n, n_1, α and β the inequalities

$$\alpha u_0 \leq B^{n_1} u^*, \quad B^n x \leq \beta u_0$$

are satisfied. Then from the inequalities

$$Bu^* \leq \lambda u^*$$

it follows that

$$B^n x \le \frac{\beta\lambda^{n_1}}{\alpha} u^*.$$

This completes the proof of the lemma.

Theorem 2.4. *Let E be an u_0-uniformly complete Archimedean Riesz space, $A : E \to E$ a nonlinear operator. Suppose that there exists an u_0-bounded increasing operator $B : E \to E$ such that the hypotheses (i) of Theorem 2.1 is satisfied and*

$$-B(x-y) \le Ax - Ay \le B(x-y),\ \forall\ x, y \in E,\ x \ge y, \tag{2.1}$$

Then A has an unique fixed point $x^* \in E$, and for any $x_0 \in E$, let $x_n = Ax_{n-1}(n = 1, 2, \dots)$, then $\{x_n\}$ converge u_0- uniformly to the element x^*.

Proof. For any $x, y \in E$, from Lemma 1.3, we have

$$x \ge \frac{1}{2}(x + y - |x-y|), \quad y \ge \frac{1}{2}(x + y - |x-y|).$$

By (2.1), we know

$$Ax - A\left(\frac{1}{2}(x+y-|x-y|)\right) \le B\left(x - \frac{1}{2}(x+y-|x-y|)\right) = B\left(\frac{x-y+|x-y|}{2}\right),$$

$$Ax - A\left(\frac{1}{2}(x+y-|x-y|)\right) \ge -B\left(x - \frac{1}{2}(x+y-|x-y|)\right) = -B\left(\frac{x-y+|x-y|}{2}\right),$$

i.e.,

$$-B\left(\frac{x-y+|x-y|}{2}\right) \le Ax - A\left(\frac{1}{2}(x+y-|x-y|)\right) \le B\left(\frac{x-y+|x-y|}{2}\right). \tag{2.2}$$

By using the same method, we get

$$-B\left(\frac{y-x+|x-y|}{2}\right) \le Ay - A\left(\frac{1}{2}(x+y-|x-y|)\right) \le B\left(\frac{y-x+|x-y|}{2}\right). \tag{2.3}$$

Subtracting (2.3) from (2.2), we obtain

$$-B(|x-y|) \le Ax - Ay \le B(|x-y|),\ \forall\ x, y \in E.$$

This implies that

$$|Ax - Ay| \le B(|x-y|),\ \forall\ x, y \in E.$$

Hence, our conclusion follows from Theorem 2.1. This completes the proof.

Corollary 2.5. *Let E be an u_0-uniformly complete Archimedean Riesz space, $A : E \to E$ a nonlinear operator. Suppose that there exist an u_0-bounded increasing operator $B : E \to E$ and two natural numbers n_0, n_1 such that the hypotheses (i) of Theorem 2.1 is satisfied, and*

$$-B^{n_0}(x-y) \le A^{n_1}x - A^{n_1}y \le B^{n_0}(x-y),\ \forall\ x, y \in E,\ x \ge y.$$

Then A has an unique fixed point $x^* \in E$.

Proof. Let $A_1 = A^{n_1}$, $B_1 = B^{n_0}$. It is easy to see that A_1, B_1 satisfy all conditions of Theorem 2.4. Thus, A_1 has a unique fixed point x^* in E, and so A has a unique fixed point x^* in E. This completes the proof.

Theorem 2.6 *Let u^* be a nonzero positive element, E an u^*-uniformly complete Archimedean Riesz space, $A : E \to E$ a nonlinear operator, $x_0 \in E$ with $x_0 \leq Ax_0$. Suppose that there exists a linear increasing operator $B : E \to E$ which satisfies the following conditions:*

(i) for any $x \in P$, there exists a positive number β which are dependent on x such that

$$Bx \leq \beta u^*, \tag{2.4}$$

(ii) there exists $\lambda \in (0, 1)$ such that

$$Bu^* \leq \lambda u^*.$$

(iii) for any $x, y \in D = \{x \in E \mid x \geq x_0\}$, $x \geq y$, we have

$$\theta \leq Ax - Ay \leq B(x - y).$$

Then A has an unique fixed point $x \in D$.

Proof. Since A is increasing on D and $x_0 \leq Ax_0$, so we obtain $A(D) \subset D$. Let $x_n = Ax_{n-1}(n = 1, 2, \ldots)$, then we have

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots .$$

By (2.4), there exist $\beta > 0$ such that

$$B(x_1 - x_0) \leq \beta u^*.$$

Then for $\forall\ n \in N$, we have

$$\begin{aligned} \theta &\leq x_{n+1} - x_n = Ax_n - Ax_{n-1} \leq B(x_n - x_{n-1}) \\ &= B(Ax_{n-1} - Ax_{n-2}) \leq \cdots \leq B^n(x_1 - x_0) \leq \beta\lambda^{n-1}u^*. \end{aligned}$$

Thus for $n, m \in N$

$$\begin{aligned} |x_{n+m} - x_n| &= |x_{n+m} - x_{n+m-1} + \cdots + x_{n+1} - x_n| \\ &\leq |x_{n+m} - x_{n+m-1}| + \cdots + |x_{n+1} - x_n| \\ &\leq \beta\left[\lambda^{n+m-2} + \cdots + \lambda^{n-1}\right] u^* \\ &= \beta\frac{\lambda^{n-1}(1 - \lambda^m)}{1 - \lambda} u^*. \end{aligned}$$

This shows that $\{x_n\}$ is a u^*-uniform Cauchy sequence and since that E is uniformly complete there exists x such that x_n is u^*- uniformly convergent to the element x. So we can get $x_n \uparrow x^*$ and $x^* \in D$. Moreover, for every $n \in N$, we have

$$|x^* - Ax^*| \leq |x^* - x_n| + |Ax_{n-1} - Ax^*| \leq |x^* - x_n| + B|x_{n-1} - x^*|.$$

Therefore x^* is a fixed point of A.

In the following we will show that x^* is the unique fixed point of A. Suppose there exists a elements $x \in D$ with $x = Ax$. By the condition, there exists $\beta_1 > 0$ such that

$$B(x - x_0) \leq \beta_1 u^*.$$

and for any $n \in N$, we have

$$x \geq x^* \geq x_n.$$

Then for all $n \in N$, we have

$$\begin{aligned} |x - x^*| &\leq |x - x_n| + |x^* - x_n| \leq |A^n x - A^n x_0| + |A^n x^* - A^n x_0| \\ &\leq B^n(x - x_0) + B^n(x^* - x_0) \leq 2\beta_1 B^n u^* \leq 2\beta_1 \lambda^{n-1} u^*. \end{aligned}$$

Since E is Archimedean, we get $x = x^*$. This completes the proof.

Remark 2.7. In this paper, we do not suppose that the space has topological structure, which is usually done in [1, 2, 3, 4, 5, 6, 9]. For example, the normality of cone and the norm of operator, used in the proof of Theorem 3.1.13 and Theorem 3.1.14 in [6], are based on the topological structure of space. The existence of a unique solution is proved in the case where the ordered vector space has only the lattice structure and the operators A or A^{n_1} satisfy an ordering relation. Moreover, we give the iterative forms.

3. Application

Throughout the remainder of this paper we apply the above result to the following Hammerstein integral equation:

$$x(t) = \int_0^1 k(t,s) f(s, x(s)) ds \tag{3.1}$$

where $f : [0,1] \times R \to R$ is continuous; R denotes the real numbers; and $k(t,s)$ is given by

$$k(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $I = [0,1]$ and E be a real linear space of all continuous functions from I into R. We also let $P = \{x \in E \mid x(t) \geq 0,\ t \in I\}$, then it is easy to verify that E is a Riesz space which is ordered by the positive cone P. Take $u^0(t) = t(1-t)$, then E is an u_0-uniformly complete Archimedean Riesz space.

Defined the operators on E:

$$(Ax)(t) = \int_0^1 k(t,s) f(s, x(s)) ds,$$

and

$$(B_1 x)(t) = \int_0^1 k(t,s) x(s) ds.$$

It is well known that the operator B_1 is an u_0-bounded increasing operator and $\pi^2 B_1 u^* = u^*$ with $u^*(t) = sin\pi t$.

Applying Theorem 2.1 with $B = \alpha\pi^2 B_1$, we can get the following result.

Proposition 3.1. *Suppose that there exists $\alpha \in [0,1)$ such that*

$$|f(t,x) - f(t,y)| \leq \alpha\pi^2 |x - y|, \quad \forall\ t \in [0,1],\ x, y \in R. \tag{3.2}$$

Then Hammerstein integral equation (3.1) has a unique solution x^* in E, and for any $x_0 \in E$, the iterative sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$ converges to x^*.

Remark 3.2. Endowed with supremum norm $\|x\| = \sup_{t \in I} |x(t)|$, E is a Banach space. Then for $\forall\ x, y \in E$, we have

$$\|Ax - Ay\| \leq \frac{\alpha\pi^2}{8} \|x - y\|.$$

This implies that A is a Lipschitzian map. But for $\alpha \in [\frac{8}{\pi^2}, 1)$, A is not a contraction map. Hence our results in essence improve and generalize the relevant results in [2, 6, 9].

Acknowledgments. The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original paper. The Project Supported by the National Science Foundation of China (10971179,11126094, 11071141) and Research Award Fund for Outstanding Young Scientists of Shandong Province (BS2010SF023).

References

[1] H. Amann, *Fixed point equation and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., **18**(1976), 620-709.

[2] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Application*, Cambridge University Press, Cambridge, 2004.

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[4] D. Guo, *Semi-ordered Methods in Nonlinear Functional analysis*, Shandong Sci. Tech. Press, Jinan, 2000 (in Chinese).

[5] D. Guo, V. Lakshmikantham, X. Liu,*Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996.

[6] D. Guo, Y.J. Cho, J. Zu, *Partial Ordering Methods in Nonlinear Problems*, Nova Science Publ., 2004.

[7] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, The Netherlands, 1964.

[8] W.A.J. Luxemburg, A.C. Zaanen, *Riesz Spaces*, North-Holland Publishing Company, Amsterdam London, 1971.

[9] X. Zhang, *Fixed Point Theorems for a Class of Nonlinear Operators in Banach Spaces and Applications*, Nonlinear Anal., **69**(2008),536–543.

Received: April 15, 2011; Accepted: December 28, 2011.