Fixed Point Theory, 14(2013), No. 1, 141-150
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

THE FIXED POINT PROPERTY FOR SOME GENERALIZED NONEXPANSIVE MAPPINGS IN A NONREFLEXIVE BANACH SPACE

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Abstract. The space ℓ_1 can be equivalently renormed to enjoy the fixed point property for a large class of generalized nonexpansive mappings.
Key Words and Phrases: Fixed point, generalized nonexpansive mappings, nonreflexive Banach space.
2010 Mathematics Subject Classification: 47H10, 46B20, 47H09.

1. Introduction

One of the most important topics in Metric Fixed Point Theory is the study of nonexpansive mappings, i.e. those which have Lipschitz constant equal to 1. The natural framework of this subject are just the closed convex and bounded subsets of the Banach spaces. A Banach space X is said to have the fixed point property for nonexpansive mappings, (FPP) in short, provided that every nonexpansive self mapping of every closed convex bounded subset of X has a fixed point. It has been known from the outset of the theory of nonexpansive mappings that the (FPP) depends strongly on 'nice' geometrical properties of the space. Many classes of reflexive Banach spaces enjoy the (FPP), as for instance Hilbert or uniformly convex Banach spaces.

It has been a long time open question if any Banach space which has (FPP) is necessarily a reflexive space (see Question I in [5]). In 2008, P.K Lin [6] answered this question in the negative giving an equivalent renorming of ℓ_1, and thus a non-reflexive space, which has the (FPP). Hernández Linares and Japon extended in 2010 Lin's example and gave a larger class of nonreflexive spaces enjoying the (FPP), (see [4]).

On the other hand, considerable effort has been aimed to the study of the fixed point property for classes of nonlinear mappings which are more general than the nonexpansive ones. Very early, in the late sixties of the last century were introduced the so called Kannan type mappings, and a bit later some wider classes of generalized nonexpansive mappings. (See [1, 3, 10, 8] among others).

Even very recently, in a different approach T. Suzuki [9] defined the type (C) mappings, which properly contain the nonexpansive mappings. Type (C) mappings in turn have been generalized in two ways in [2].

Finally, the present authors defined in [7] the class which they named type (L) mappings, which contains both Suzuki's type (C) and several other generalized nonexpansive mappings.

A question naturally arises: Could a nonreflexive space have the FPP for such class of mappings? The aim of this paper is to answer this question in the affirmative. We will show that the space ℓ_1 can be renormed so that the resulting space enjoys the fixed point property for type (L) mappings.

2. Preliminaries

We will assume throughtout this paper that $(X, \|\cdot\|)$ is a Banach space, and C is a nonempty, closed, convex, bounded subset of X. A sequence (x_n) in C is called an *almost fixed point sequence for* T (a.f.p.s. for short) provided that $x_n - T(x_n) \to 0_X$. It is well known that every nonexpansive mapping $T : C \to C$ has a.f.p. sequences. The same holds if $T : C \to C$ satisfies Suzuki's condition (C) on C, (see [9], Lemma 6).

We now recall further concepts which will be useful in the forthcoming sections.

We begin with some classes of mappings. Definitions (1) and (2) are given in [2], and the first one is a generalization of condition (C) given by Suzuki in [9].

(1) For $\lambda \in (0, 1)$ we say that a mapping $T : C \to X$ satisfies *condition* (C_λ) on C if for all $x, y \in C$ with $\lambda\|x - Tx\| \le \|x - y\|$ one has that $\|Tx - Ty\| \le \|x - y\|$. Of course, the original Suzuki condition (C) is just $(C_{\frac{1}{2}})$.

(2) For $\mu \ge 1$ a mapping $T : C \to X$ is said to satisfy *condition* (E_μ) *on* C if for all $x, y \in C$,

$$\|x - Ty\| \le \mu\|x - Tx\| + \|x - y\|.$$

We say that T satisfies condition (E) on C if T satisfies (E_μ) on C for some $\mu \ge 1$. In [9] is shown that if a mapping satisfies Suzuki's condition $(C_{1/2})$ then it satisfies condition (E_3).

(3) A mapping $T : C \to X$ is said to be a generalized nonexpansive mapping, (gne) for short, if there exists nonnegative constants a, b, c with $a+2b+2c \le 1$ such that for all $x, y \in C$

$$\|Tx - Ty\| \le a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|). \quad (2.1)$$

Although each one of the classes (C_λ), (E_μ), and (gne) mappings contains the class of nonexpansive mappings, in [2, 7] are given examples separating such classes.

3. Condition (L)

In a recent paper, [7] , the authors defined a class of nonlinear mappings as follows

Definition 3.1. A mapping $T : C \to C$ satisfies condition (L), (or it is an (L)-type mapping), on C provided that it fulfills the following two conditions.

(1) If a set $D \subset C$ is nonempty, closed, convex and T-invariant, (i.e. $T(D) \subset D$), then there exists an a.f.p.s. for T in D.
(2) For any a.f.p.s. (x_n) of T in C and each $x \in C$

$$\limsup_{n\to\infty} \|x_n - T(x)\| \le \limsup_{n\to\infty} \|x_n - x\|.$$

From now on, if not specified, when a mapping is said to satisfy condition (L) it will mean that it satisfies it on its domain.

Condition (1) and (2) in the previous definition are independent, as shown in the paper [7], where examples of mappings satisfying condition (1) and not condition (2) are given.

In the paper [7] it is proved that the class of (L)-type mappings contains strictly the following classes:

- nonexpansive mappings,
- mappings satisfying condition (C) of Suzuki,
- mappings satisfying condition (E) which in turn satisfy condition (1) in the definition of (L)-type mappings
- (gne) mappings, in many cases (for instance if $b < \frac{1}{2}$ or in the case $b = \frac{1}{2}$ in spaces with uniformly normal structure).

The following lemma refers to one of the most important features of (L)-type mappings. It is well known that nonexpansive mappings enjoy this property.

Lemma 3.2. *Let C be a nonempty bounded closed convex subset of a Banach space X and T a self-mapping satisfying condition (L) on C. Let (x_n) be an a.f.p.s. for T in C and let $\phi : C \to \mathbb{R}^+$ be the function defined by*

$$\phi(x) = \limsup_{n\to\infty} \|x - x_n\| .$$

Then ϕ is a lower semi-continuous function. Moreover, for any $d > \inf\{\phi(x) : x \in C\}$, the set $D = \{x \in C : \phi(x) \le d\}$ is a nonempty closed convex T-invariant subset of C.

Proof. We will prove that $D = \{x \in C : \limsup_{n\to\infty} \|x_n - x\| \le d\}$ is T-invariant. Let $x \in C$,

$$\phi(Tx) = \limsup_{n\to\infty} \|x_n - Tx\| \le \limsup_{n\to\infty} \|x_n - Tx\| = \phi(x) \le d$$

and so, for any $x \in D$, $Tx \in D$, this is, D is T-invariant. □

4. Fixed point theorem

In the paper [7] the authors gave fixed point results for (L)-type mappings in the setting of Banach spaces with the so called normal structure. On the other hand, P.K. Lin [6] and later C. Hernández and M.A. Japón [4] found nonreflexive Banach spaces with the (FPP) for nonexpansive mappings. To extend their results to (L)-type mappings we will be concerned with Banach spaces endowed with a a linear topology τ such that every bounded sequence has a τ-convergent subsequence. For instance, τ could be the weak star topology whenever the underlying Banach space is separable. For this kind of spaces we can recall the following lemma, which, in fact is a slight modification of Lemma 1 in [4].

Lemma 4.1. *Let $(X, \|\cdot\|)$ be a Banach space endowed with a linear topology τ such that every bounded sequence has a τ-convergent subsequence. Let C be a closed, convex, bounded subset of X and $T : C \to C$ be a mapping satisfying condition (L) without fixed points. Then there exists some $a > 0$ and a convex closed T-invariant subset D of C such that for each a.f.p.s. (x_n) in D with $x_n \xrightarrow{\tau} x$*

$$\limsup_{n\to\infty} \|x_n - x\| \geq a.$$

Proof. Otherwise, for any $a > 0$, and any nonempty closed convex and bounded set $D \subset C$

$$\exists (x_n) \text{ a.f.p.s. in } D, x_n \xrightarrow{\tau} x \in X, \text{and} \limsup_{n\to\infty} \|x_n - x\| < a. \tag{4.1}$$

Thus, taking $a = \frac{1}{2^2}$ and $D = C$, there exists an a.f.p.s. (x_n^1) in C with $x_n^1 \xrightarrow{\tau} x^1 \in X$ such that

$$\frac{1}{2^2} > \limsup_{n\to\infty} \|x_n^1 - x^1\|.$$

Since

$$\|x_n^1 - x_m^1\| \leq \|x_n^1 - x^1\| + \|x^1 - x_m^1\|,$$

then

$$\limsup_{n\to\infty} \|x_n^1 - x_m^1\| \leq \limsup_{n\to\infty} \|x_n^1 - x^1\| + \|x^1 - x_m^1\|$$

and

$$\begin{aligned}\limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n^1 - x_m^1\| &\leq \limsup_{n\to\infty} \|x_n^1 - x^1\| + \limsup_{m\to\infty} \|x^1 - x_m^1\| \\ &= 2\limsup_{n\to\infty} \|x_n^1 - x^1\|.\end{aligned}$$

Hence

$$\frac{1}{2^2} > \limsup_{n\to\infty} \|x_n^1 - x^1\| \geq \frac{1}{2} \limsup_{n\to\infty}\limsup_{m\to\infty} \|x_n^1 - x_m^1\|$$

Consequently for m large enough, $x_m^1 \in D_1$ where

$$D_1 = \{z \in C : \limsup_{n\to\infty} \|x_n^1 - z\| \leq \frac{1}{2}\}.$$

Therefore, given that T satisfies condition (L) on C, from Lemma 3.2, D_1 is a nonempty, convex, closed and T-invariant subset of C.

Taking now $a = \frac{1}{2^3}$ and $D = D_1$, from (4.1) we can assure the existence of an a.f.p.s. $(x_n^2) \in D_1$ with $x_n^2 \xrightarrow{\tau} x^2 \in X$ such that

$$\frac{1}{2^3} > \limsup_{n\to\infty} \|x_n^2 - x^2\| \geq \frac{1}{2} \limsup_{n\to\infty}\limsup_{m\to\infty} \|x_n^2 - x_m^2\|.$$

Then, the set

$$D_2 = \{z \in D_1 : \limsup_{n\to\infty} \|x_n^2 - z\| \leq \frac{1}{2^2}\}$$

is again a nonempty, convex, closed, T-invariant subset of D_1. We construct in this way a decreasing sequence (D_n) of convex closed bounded T-invariant subsets of C such that $diam(D_n) \leq \frac{1}{2^{n-1}}$ and so, by Cantor Theorem, $\cap_n D_n$ is a singleton, which has to be a fixed point, and we get a contradiction. $\square$

Let $(X, \|\cdot\|)$ be a Banach space endowed with a linear topology τ such that if (x_n) is a bounded sequence on X, then there is a subsequence (x_{n_j}) of (x_n) τ-converging to some $x \in X$.

Denote $R_1(x) = \|x\|$ and assume that there exists a family of seminorms $R_k : X \to [0, +\infty)$ $(k \geq 2)$, such that satisfy the following properties:

(1) For all $k \geq 1$, $R_k(x) \leq \|x\|$ for all $x \in X$.
(2) $\lim\limits_{k\to\infty} R_k(x) = 0$ for all $x \in X$.

Moreover, if (x_n) is norm-bounded and $x_n \xrightarrow{\tau} 0_X$, then for all $k \geq 1$, and any $x \in X$

(3) $\limsup\limits_{n\to\infty} R_k(x_n) = \limsup\limits_{n\to\infty} \|x_n\|$.
(4) $\limsup\limits_{n\to\infty} R_k(x_n + x) = \limsup\limits_{n\to\infty} R_k(x_n) + R_k(x)$.

Let X be a Banach space and $(R_k(\cdot))$ satisfying the conditions stated above. Let (γ_k) be any nondecreasing sequence of real numbers in $(0,1)$ such that $\lim_k \gamma_k = 1$. Then, it is obvious that the expression

$$|||x||| = \sup_{k\geq 1} \gamma_k R_k(x)$$

defines a norm on X which is an equivalent norm on $(X, \|\cdot\|)$. In fact, for all $x \in X$

$$\gamma_1 \|x\| \leq |||x||| \leq \|x\|.$$

In order to simplify the statement of the following results, we will introduce the following definition.

Definition 4.2. Given a sequence of seminorms (R_k) on X satisfying the above conditions 1-4, and a nondecreasing sequence (γ_k) in $(0,1)$ with $\lim_k \gamma_k = 1$, we will refer to the corresponding norm $|||\cdot|||$ as an *HJ*-norm on X.

Remark 4.3. In [4] are given some examples of Banach spaces endowed with an HJ-norm, which hence fall into the scope of our main theorem. The first example is the sequence space ℓ_1, considering the seminorms given by $R_k(x) = \left\|\sum_{n=k}^{+\infty} x_n\right\|_1$, the sequence $\gamma_k = \frac{8^k}{1+8^k}$ and τ the weak star topology. They recover in this way the result of Lin in [6].

We will need some lemmas in order to prove our main theorem.

Lemma 4.4. *(Hernández and Japón, Lemma 2 in* [4]*).*

Let X be a Banach space and $|||\cdot|||$ an HJ-norm on X. Then, for two bounded sequences (x_n), (y_n) in X, the following statements hold.

(1) *If $x_n \xrightarrow{\tau} 0$, then*
$$\limsup_{n\to\infty} |||x_n||| = \limsup_{n\to\infty} \|x_n\|.$$

(2) *If* $x_n \xrightarrow{\tau} x$ *and* $y_n \xrightarrow{\tau} y$ *then*

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n - y_m\| \geq \limsup_{n\to\infty} \|x_n - x\| + \limsup_{m\to\infty} \|y_m - y\| .$$

Remark 4.5. If $x_n \xrightarrow{\tau} x$ and $y_n \xrightarrow{\tau} x$ then, since

$$\|x_n - y_m\| \leq \|x_n - x\| + \|x - y_m\| ,$$

it follows that

$$\limsup_{n\to\infty} \|x_n - y_m\| \leq \limsup_{n\to\infty} \|x_n - x\| + \|x - y_m\|$$

and therefore

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n - y_m\| \leq \limsup_{n\to\infty} \|x_n - x\| + \limsup_{m\to\infty} \|x - y_m\| .$$

Taking into account the statement 2 of the above lemma,

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n - y_m\| = \limsup_{n\to\infty} \|x_n - x\| + \limsup_{m\to\infty} \|x - y_m\| .$$

This fact will be useful later on, just in the proof of the main theorem.

The proofs of the next lemma and the latter theorem are closely modeled on the corresponding proofs of Lemma 3 and Theorem 1 of [4].

Lemma 4.6. *Consider the Banach space X endowed with an HJ-norm $\|\cdot\|$, and let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \to C$ be a fixed point free mapping satisfying condition (L). Let K be any closed convex T-invariant subset of C and denote*

$$\rho = \inf\{\limsup_{n\to\infty} \|x_n - x\| : (x_n) \text{ is an a.f.p.s. in } K \text{ and } x_n \xrightarrow{\tau} x\}.$$

Then, $\rho > 0$ and for any a.f.p.s. (x_n) in K and for all $z \in K$ we have

$$\limsup_{n\to\infty} \|x_n - z\| \geq 2\rho.$$

Proof. From Lemma 4.1 it follows immediately that $\rho > 0$. Now suppose, for a contradiction, that there exists an a.f.p.s. (x_n) in K and $z \in K$ such that

$$r = \limsup_{n\to\infty} \|x_n - z\| < 2\rho$$

We define

$$K' = \{w \in K : \limsup_{n\to\infty} \|x_n - w\| \leq r\}.$$

Since $z \in K'$ then K' is nonempty, and, of course, bounded, closed and convex. Bearing in mind that T satisfies condition (L) on C, from Lemma 3.2 it follows that the set K' is T-invariant. Again because T satisfies condition (L) on C, we can take an a.f.p.s. (y_n) in K'. Without loss of generality we may suppose that $y_n \xrightarrow{\tau} y$ and $x_n \xrightarrow{\tau} x$. Using the previous lemma,

$$\begin{aligned} r &\geq \limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n - y_m\| \\ &\geq \limsup_{n\to\infty} \|x_n - x\| + \limsup_{m\to\infty} \|y_m - y\| \geq \rho + \rho = 2\rho \end{aligned}$$

which is a contradiction. □

Theorem 4.7. *Let $|||\cdot|||$ be an HJ-norm on X. Then, $(X, |||\cdot|||)$ has the FPP for mappings satisfying condition (L).*

Proof. Assume for a contradiction, that T is fixed point free. Let D be as in the conclusion of Lemma (4.1). Define

$$c = \inf\{\limsup_{n\to\infty} |||x_n - x||| : (x_n) \text{ is an a.f.p.s. in } D \text{ and } x_n \xrightarrow{\tau} x\}$$

which is greater than zero by Lemma 4.6.

Without loss of generality we can assume that $c = 1$. Take $0 < \varepsilon_1 < \frac{1}{2}$ and an a.f.p.s. (x_n) in D such that $x_n \xrightarrow{\tau} x$ and $\limsup_{n\to\infty} |||x_n - x||| < 1 + \varepsilon_1$. Again, by translation, we can assume that $x = 0$.

Let us consider now

$$K = \{z \in D : \limsup_{n\to\infty} |||x_n - z||| \le 2 + 2\varepsilon_1\},$$

which we claim it is nonempty. From Remark 4.5, since $x_n \xrightarrow{\tau} 0$,

$$\limsup_{m\to\infty} \limsup_{n\to\infty} |||x_n - x_m||| = \limsup_{n\to\infty} |||x_n||| + \limsup_{m\to\infty} |||x_m||| < 2 + 2\varepsilon_1.$$

Then we can find an increasing sequence of positive integers (m_k) such that $\limsup_{n\to\infty} |||x_n - x_{m_k}||| < 2 + 2\varepsilon_1$, that is, $x_{m_k} \in K$. Hence from Lemma 3.2 the set K is closed, convex, T-invariant and nonempty. Therefore, since T is a type (L) mapping, we can consider

$$\rho := \inf\{\limsup_{n\to\infty} |||y_n - y||| : (y_n) \text{ is an a.f.p.s. in } K \text{ and } y_n \xrightarrow{\tau} y\}.$$

It is clear that

$$1 = c \le \rho \le \limsup_{k\to\infty} |||x_{m_k}||| \le \limsup_{n\to\infty} |||x_n||| < 1 + \varepsilon_1.$$

We are going to find an a.f.p.s. (y_n) in K and $z \in K$ such that

$$\limsup_{n\to\infty} |||y_n - z||| < 2\rho$$

and then we obtain a contradiction according to Lemma 4.6.

Notice the following fact: If (y_n) is an a.f.p.s. for T in K and $y_n \xrightarrow{\tau} y$, then for any $k \in \mathbb{N}$, bearing in mind Property 4 of the sequence of seminorms R_k,

$$\begin{aligned}
2 + 2\varepsilon_1 &\ge \limsup_{m\to\infty} \limsup_{n\to\infty} |||x_n - y_m||| \\
&= \limsup_{m\to\infty} \limsup_{n\to\infty} |||x_n - (y_m - y) - y||| \\
&\ge \gamma_k \limsup_{m\to\infty} \limsup_{n\to\infty} R_k(x_n - (y_m - y) - y) \\
&= \gamma_k \limsup_{m\to\infty} \left[\limsup_{n\to\infty} R_k(x_n) + R_k((y_m - y) - y)\right] \\
&= \gamma_k \left[\limsup_{n\to\infty} R_k(x_n) + \limsup_{m\to\infty} R_k(y_m - y) + R_k(y)\right].
\end{aligned}$$

Since $x_n \xrightarrow{\tau} 0$ and $y_n - y \xrightarrow{\tau} 0$, by Property 3 in the definition of the seminorms R_k, and the first statement in Lemma 4.4, then

$$\begin{aligned} 2+2\varepsilon_1 &\geq \gamma_k \left[\limsup_{n\to\infty} R_k(x_n) + \limsup_{m\to\infty} R_k(y_m - y) + R_k(y)\right] \\ &= \gamma_k \left[\limsup_{n\to\infty} \|x_n\| + \limsup_{m\to\infty} \|y_m - y\| + R_k(y)\right] \\ &= \gamma_k \left[\limsup_{n\to\infty} |||x_n||| + \limsup_{m\to\infty} |||y_m - y||| + R_k(y)\right] \geq \gamma_k[2 + R_k(y)], \end{aligned}$$

which yields

$$R_k(y) \leq 2\left(\frac{1+\varepsilon_1}{\gamma_k} - 1\right).$$

Put $p := 1 + \varepsilon_1 + 2\left(\frac{1+\varepsilon_1}{\gamma_1} - 1\right)$. Of course $p > 1 + \varepsilon_1 > \rho$. Choose $\delta \in (\varepsilon_1, \frac{1}{2})$. Since $\rho \geq 1$ and $-2\delta > -1$ then $\rho - 2\delta > 0$ and we can also choose $\varepsilon_2 \in (0, \rho - 2\delta)$.

Since, by Lemma 4.4, $\limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} |||x_n||| < 1 + \varepsilon_1$, we can find $x \in K$ such that $\|x\| < 1 + \varepsilon_1$. Also from Property 2 in the definition of the seminorms R_k, there exists $m \in \mathbb{N}$ such that if $k \geq m$

$$R_k(x) < \varepsilon_2,$$

and since $\lim_k \gamma_k = 1$,

$$\frac{1+\varepsilon_1}{1+\delta} < \gamma_k.$$

We may take $\lambda \in (0,1)$ such that

$$\lambda < \frac{\rho(1-\gamma_m)}{\gamma_m(p-\rho)}.$$

Since

$$(2-\lambda)\rho + \lambda(\varepsilon_2 + 2\delta) = 2\rho - \lambda(\rho - (2\delta + \varepsilon_2)) < 2\rho$$

and

$$\gamma_m[(2-\lambda)\rho + \lambda p] = 2\gamma_m\rho + \gamma_m\lambda(p-\rho) < 2\gamma_m\rho + \rho(1-\gamma_m) = \rho(1+\gamma_m) < 2\rho,$$

we can find $\varepsilon_3 > 0$ small enough such that

$$(2-\lambda)(\rho + \varepsilon_3) + \lambda(\varepsilon_2 + 2\delta) < 2\rho$$

and

$$\gamma_m[(2-\lambda)(\rho+\varepsilon_3) + \lambda p] < 2\rho.$$

Take an a.f.p.s. (y_n) in K such that $y_n \xrightarrow{\tau} y$ and $\limsup_{n\to\infty} |||y_n - y||| < \rho + \varepsilon_3$. From Lemma 4.4

$$\limsup_{n\to\infty} \|y_n - y\| = \limsup_{n\to\infty} |||y_n - y||| < \rho + \varepsilon_3.$$

There exists $s \in \mathbb{N}$ such that $\|y_N - y\| < \rho + \varepsilon_3$ for all $N \geq s$ and define

$$z = (1-\lambda)y_s + \lambda x$$

which belongs to K because K is convex.

Let us prove that $\limsup_{n\to\infty} \|\!|y_n - z\|\!| < 2\rho$. In order to do this, we will prove that there exists $M > 0$ such that for all k and $N \geq s$ we have

$$\gamma_k R_k(y_N - z) < M < 2\rho.$$

We split the proof into two cases:

Case 1: $k \geq m$

$$\begin{aligned}
\gamma_k R_k(y_N - z) &= \gamma_k R_k(y_N - (1-\lambda)y_s - \lambda x) \\
&\leq R_k(y_N - y - (1-\lambda)(y_s - y) - \lambda(x - y)) \\
&\leq R_k(y_N - y) + (1-\lambda)R_k(y_s - y) + \lambda R_k(x - y) \\
&\leq \|y_N - y\| + (1-\lambda)\|y_s - y\| + \lambda R_k(x - y) \\
&\leq (\rho + \varepsilon_3) + (1-\lambda)(\rho + \varepsilon_3) + \lambda(R_k(x) + R_k(y)) \\
&\leq (2-\lambda)(\rho + \varepsilon_3) + \lambda(\varepsilon_2 + R_k(y)) \\
&\leq (2-\lambda)(\rho + \varepsilon_3) + \lambda\left(\varepsilon_2 + 2\left(\tfrac{1+\varepsilon_1}{\gamma_k} - 1\right)\right) \\
&< (2-\lambda)(\rho + \varepsilon_3) + \lambda(\varepsilon_2 + 2\delta) < 2\rho.
\end{aligned}$$

Case 2: $k < m$

$$\begin{aligned}
\gamma_k R_k(y_N - z) &\leq \gamma_m R_k(y_N - (1-\lambda)y_s - \lambda x) \\
&= \gamma_m \left[R_k(y_N - y - (1-\lambda)(y_s - y) - \lambda(x - y))\right] \\
&\leq \gamma_m \left[R_k(y_N - y) + (1-\lambda)R_k(y_s - y) + \lambda R_k(x - y)\right] \\
&\leq \gamma_m \left[(\rho + \varepsilon_3) + (1-\lambda)(\rho + \varepsilon_3) + \lambda(R_k(x) + R_k(y))\right] \\
&\leq \gamma_m \left[(2-\lambda)(\rho + \varepsilon_3) + \lambda(1 + \varepsilon_1 + R_k(y))\right] \\
&\leq \gamma_m \left[(2-\lambda)(\rho + \varepsilon_3) + \lambda\left(1 + \varepsilon_1 + 2\left(\tfrac{1+\varepsilon_1}{\gamma_1} - 1\right)\right)\right] \\
&= \gamma_m \left[(2-\lambda)(\rho + \varepsilon_3) + \lambda p\right] < 2\rho.
\end{aligned}$$

Take

$$M = \max\{(2-\lambda)(\rho + \varepsilon_3) + \lambda(\varepsilon_2 + 2\delta), \gamma_m \left[(2-\lambda)(\rho + \varepsilon_3) + \lambda p\right]\}.$$

Then, for all $N \geq s$, $\|\!|y_N - z\|\!| < M < 2\rho$. Hence $\limsup_{n\to\infty} \|\!|y_n - z\|\!| < 2\rho$ and this contradiction finishes the proof. □

Remark 4.8. From the inclusions of classes of mappings discussed in the preliminaries and Theorem 4.4. of [7], which states that continuous mappings with condition (C_λ) either satisfy condition (L) or have a fixed point, we can conclude from the above theorem that Banach spaces endowed with an HJ-norm $(X, \|\!|\cdot\|\!|)$ have the FPP for mappings belonging to anyone of the following classes.

(1) Nonexpansive mappings.
(2) Generalized nonexpansive mappings (with $b \neq \frac{1}{2}$)
(3) Mappings which satisfy condition (C).
(4) Continuous mappings satisfying condition (C_λ) (for some $\lambda \in (0,1)$).

On the other hand, since condition (L) depends on the norm, it can be found mappings which fail to satisfy condition (L) in some Banach space $(X, \|.\|)$ but satisfying such condition with respect to the corresponding HJ-norm.

Example 4.9. Consider the Banach space $(\ell_1, \|\cdot\|_1)$ and the equivalent HJ-norm $|||\cdot|||$. Put $b = \gamma_1^{-1}$. Let $K := \{x \in \ell_1 : |||x||| \leq 1/b\}$. It is clear that for every $x \in \ell_1$, $|||x||| \leq \|x\|_1 \leq \gamma_1^{-1} |||x||| = b\, |||x|||$.

We define $U : K \to K$ by

$$U(x) := \frac{\|x\|_1 + b\, |||x|||}{2} e_1$$

For $x \in K$,

$$b\, |||Ux||| = b\frac{1}{2}(\|x\|_1 + b\, |||x|||)\frac{1}{b} \leq b\, |||x||| \leq 1,$$

then, indeed $Ux \in K$.

It is clear that the mapping U is $|||\cdot|||$-nonexpansive, since for $x, y \in K$,

$$\begin{aligned} |||Ux - Uy||| &= \left| \tfrac{\|x\|_1 - \|y\|_1}{2} + \tfrac{b(|||x||| - |||y|||)}{2} \right| |||e_1||| \\ &\leq \tfrac{1}{2}[\|x - y\|_1 + b\, |||x - y|||]\, |||e_1||| \\ &\leq b\, |||x - y|||\, |||e_1||| \\ &= |||x - y||| . \end{aligned}$$

Thus, the mapping U satisfies condition (L) on K w.r.t. the norm $|||\cdot|||$. We will see that the mapping U fails to satisfy condition (L) on K w.r.t. the standard norm $\|\cdot\|_1$. To see this, take $x = \gamma_1\gamma_2^{-1}e_2$, and $(x_n) \equiv (0_{\ell_1})$. Of course (x_n) is an a.f.p.s. for U because 0_{ℓ_1} is a fixed point of U. Since $|||e_2||| = \max\{\gamma_1, \gamma_2\} = \gamma_2$ and $|||x||| = \gamma_1 = 1/b$, one has that

$$\limsup_{n\to\infty} \|x_n - Ux\|_1 = \frac{\|x\|_1 + b\, |||x|||}{2}\|e_1\|_1 = \frac{\gamma_1\gamma_2^{-1} + 1}{2} > \|x\|_1 = \limsup_{n\to\infty} \|x_n - x\|_1.$$

References

[1] J. Bogin, *A generalization of a fixed point theorem of Goebel, Kirk and Shimi,* Canad. Math. Bull., **19**(1976), No. 1, 7-12.

[2] J. García-Falset, E. Llorens- Fuster, T. Suzuki, *Some generalized nonexpansive mappings,* J. Math. Anal. Appl., 2010, doi:10.1016/jmaa.2010.08.069.

[3] K. Goebel, W.A. Kirk, T.N. Shimi, *A fixed point theorem in uniformly convex spaces*, Boll. Un. Mat. Ital., **7**(4)(1973), 67–75.

[4] C.A. Hernandez-Linares, M.A. Japon, *A renorming in some Banach spaces with applications to fixed point theory,* J. Funct. Anal., **258**(2010), 3452–3468.

[5] W.A. Kirk, *Some questions in metric fixed point theory*, Recent Advances in Metric Fixed Point Theory, Seville 1995, 73–97, Ciencias, 48, Univ. Seville, 1996.

[6] P.K. Lin, *There is an equivalent norm on ℓ_1 that has the fixed point property,* Nonlinear Anal., **68**(2008), 2303–2308.

[7] E. Llorens-Fuster, E. Moreno-Galvez, *The fixed point theory for some generalized nonexpansive mappings*, Abstr. Appl. Anal. 2011, Art. ID 435686, 15 pp.

[8] M.A. Smyth, *Aspects of the Fixed Point Theory for some Metrically Defined Maps,* Ph. D. Dissertation, University of Newcastle, 1994.

[9] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings,* J. Math. Anal. Appl., **340**(2008), 1088-1095.

[10] C.S. Wong, *On Kannan maps*, Proc. Amer. Math. Soc., **47**(1975), 105-111.

Received: March 30, 2011; Accepted: December 20, 2011.