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SOME RELATIONSHIPS BETWEEN SUFFICIENT CONDITIONS FOR THE FIXED POINT PROPERTY

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Abstract. It is shown that E-convex Banach spaces satisfy the so called Prus-Sczepanik condition which in turn is sufficient for the fixed point property of nonexpansive mappings in Banach spaces. Moreover, we study the independence between several of these sufficient conditions.
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1. Introduction

Let C be a subset of a Banach space $(X, \|\cdot\|)$. A mapping $T\colon C \to X$ is called *nonexpansive* whenever $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$.

The space $(X, \|\cdot\|)$ has the *fixed point property* (FPP) if every nonexpansive self-mapping of each nonempty bounded closed convex subset C of X has a fixed point. If the same property holds for every weakly compact convex subset of X we say that $(X, \|.\|)$ has the *weak fixed point property* (WFPP for short).

It has been known from the outset of the study of this property (around the early sixties of the last century) that it depends strongly on 'nice' geometrical properties of the space. For instance, a celebrated result due to W.A. Kirk ([23], 1965) establishes that the Banach spaces with *normal structure* (NS) have the (WFPP). In particular, uniformly convex Banach spaces, and hence Hilbert spaces, have normal structure. (See definitions below).

The problem of whether every superreflexive Banach spaces enjoys (FPP) is a classical open question in Fixed Point Theory.

Since 1965 until now a considerable amount of papers dealing with sufficient conditions for (FPP) have been published. It turns out that to check that a given Banach

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space enjoys one of these conditions is not an easy task. Moreover, some links between these geometrical conditions remained hidden for years. For instance, it was longtime open the question whether the uniformly non square Banach spaces have (FPP), but in 2003 it was discovered (see [25, 15]) that all these spaces indeed satisfy the so called Domínguez Benavides' condition $M(X) > 1$ (see [3]) (which in reflexive spaces implies (FPP)) and that it was known nine years before the publication of [15].

The aim of this note is twofold. First, to show one of these relationships, which seems to be new. Namely, that E-convex Banach spaces (which have (FPP) according to [32]) in fact satisfy another of such sufficient conditions for (FPP) due Prus-Sckepanik [29]. Second, to clarify the independence between several of the remaining most important sufficient conditions for (FPP).

2. Notations and basic definitions

Throughout this paper we will use the standard notation in Banach space geometry, as it appears, for instance in [16]. In particular, given a Banach space $(X, \|.\|)$, we will denote the closed balls and the spheres as follows: $B_X := \{x \in X : \|x\| \leq 1\}$, $S_X := \{x \in X : \|x\| = 1\}$ and for $r > 0, x \in X$, $B[x, r] = x + rB_X$. If X^* is the dual space of X, then for $x \in X$,

$$\nabla(x) := \{f \in S_{X^*} : f(x) = \|x\|\}.$$

The weak convergence of a sequence (x_n) in X to $x_0 \in X$ will be denoted as $x_n \rightharpoonup x_0$. Given the Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $p \in [1, \infty)$ the ℓ_p direct sum of X and Y will be denoted as $X \oplus_p Y$. In the same way, $X \oplus_\infty Y$ will be the space $X \times Y$ endowed with the norm $\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$.

For a bounded sequence (x_n) in X we often will use the notation

$$D(x_n) := \limsup_n \left(\limsup_m \|x_n - x_m\| \right).$$

Recall that the modulus of convexity of $(X, \|\cdot\|)$ is the function $\delta_X : [0, 2] \to [0, 1]$ given by

$$\delta_X(\varepsilon) \quad := \inf \left\{ 1 - \left\| \tfrac{1}{2}(x + y) \right\| : x, y \in B_X, \|x - y\| \geq \varepsilon \right\}.$$

The *characteristic of convexity of* $(X, \|\cdot\|)$, is the real number

$$\varepsilon_0(X) := \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}.$$

The space $(X, \|\cdot\|)$ is *uniformly convex* whenever $\varepsilon_0(X) = 0$.

Several sufficient conditions for (FPP) have been stated in terms of some coefficients. For instance, J. García-Falset in [11] defined, for a given Banach space $(X, \|\cdot\|)$ the coefficient

$$R(X) := \sup\{\liminf_{n\to\infty} \|x_n + x\| : \ x, x_n \in B_X \ (n = 1, 2, \ldots), \ \ x_n \rightharpoonup 0 \ \}.$$

Moreover he proved (see [12]) that if $R(X) < 2$ then $(X, \|\cdot\|)$ has (WFPP). Later on, T. Domíguez Benavides in [3] defined for $a \geq 0$,

$$R(a, X) := \sup\{\liminf \|x + x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences (x_n) in the unit ball of X such that $D(x_n) \leq 1$. Moreover, he also defined

$$M(X) := \sup \left\{ \frac{1+a}{R(a,X)} : a \geq 0 \right\}.$$

The main result in [3] that if $M(X) > 1$ then X has the (WFPP). It turns out that $R(X) < 2 \Rightarrow M(X) > 1$.

3. E-convexity implies Prus-Szczepanik condition

3.1. **Prus-Szczepanik condition (PSz).** It was introduced by S. Prus and M. Szczepanik in 2005 [29]. Given a Banach space X, for $x \in X$ and $\varepsilon > 0$ put

$$d(\varepsilon, x) = \inf_{(y_m)\in \mathfrak{N}_X} \limsup_{m\to\infty} \|x + \varepsilon y_m\| - \|x\|,$$

and

$$b_1(\varepsilon, x) = \sup_{(y_m)\in \mathfrak{M}_X} \liminf_{m\to\infty} \|x + \varepsilon y_m\| - \|x\|.$$

where

$$\mathfrak{N}_X := \{(x_n) : x_n \in S_X,\, n = 1, 2, \ldots, x_n \rightharpoonup 0_X\}$$

and

$$\mathfrak{M}_X := \{(x_n) : D(x_n) \leq 1, x_n \in B_X,\, n = 1, 2, \ldots, x_n \rightharpoonup 0_X\}.$$

Definition 3.1. *Let $(X, \|\cdot\|)$ be a non-Schur Banach space. If there exists $\varepsilon \in (0,1)$ such that for every $x \in S_X$ it is the case that $b_1(1,x) < 1-\varepsilon$ or $d(1,x) > \varepsilon$ we say that $(X, \|\cdot\|)$ satisfies the Prus-Szczepanik condition.*

The main result in [29] is the following.

Theorem 3.2. *Let X be a Banach space without the Schur property. If X satisfies the Prus-Szczepanik condition then X has the (WFPP).*

Properties which are stronger than (PSz) condition are, among others, the following.

(1) Uniform noncreasyness (introduced by S. Prus in 1997) (see [28]) and its generalizations. (See [14, 8, 7]).
(2) Property $M(X) > 1$. (See [3]). In particular this last condition covers all the uniformly nonsquare Banach spaces. (See [25, 15]). Other reflexive Banach spaces X with $M(X) > 1$ are those which satisfy $R(X) < 2$. (See [12]).

3.2. **Conditions depending on the dual space.** Very recently, in 2008, P.N. Dowling, B. Randrianantoanina and B. Turett [6], solved a question posed by S. Saejung in [31], showing that (superreflexive) spaces with O-convex dual have the (FPP). They presented this result also as a generalization of the above referred Eva Mazcuñán's result of 2003 (see [25], [15]).

That is, we have that:

$$\begin{array}{l} \varepsilon_0(X) < 2 \Rightarrow M(X) > 1 \Rightarrow (PSz) \Rightarrow (FPP) \\ \Downarrow \\ X^*O-\text{convex} \\ \Downarrow \\ (FPP) \end{array}$$

We will recall a bit more details about this last result.

For $\eta \in (0,2)$, a subset A of X is said to be symmetrically η-separated if the distance between any two distinct points of $A \cup (-A)$ is at least η and a Banach space X is O-convex if the unit ball B_X contains no symmetrically $(2-\varepsilon)$-separated subset of cardinality n for some $\varepsilon > 0$ and some $n \in \mathbb{N}$. In other words,

Definition 3.3. *A Banach space X is O-convex if there exists $\varepsilon \in (0,1)$ and a positive integer $n \geq 2$ such that for every $x_1, x_2, \dots x_n \in S_X$, there exist $i, j \in \{1, \dots .n\}$ with $i < j$ such that*

$$\min\{\|x_i - x_j\|, \|x_i + x_j\|\} \leq 2 - \varepsilon.$$

Since $\varepsilon_0(X) < 2$ if and only if $\varepsilon_0(X^*) < 2$, uniformly nonsquare Banach spaces have O-convex dual.

Naidu and Sastry in [26] also characterized the dual property of O-convexity. For $\varepsilon > 0$, a convex subset A of B_X is an ε-flat if $A \cap (1-\varepsilon)B_X = \emptyset$. A collection $\mathcal{D}$ of ε-flats is jointly complemented (jcc in short) if, for each distinct ε-flats A and B in $\mathcal{D}$, the sets $A \cap B$ and $A \cap (-B)$ are nonempty. For a positive integer n define

$$E(n, X) = \inf\{\varepsilon > 0 : B_X \text{ contains a jcc of } \varepsilon - \text{flats of cardinality } n\}.$$

Definition 3.4. *A Banach space X is said to be E-convex if $E(n, X) > 0$ for some $n \in \mathbb{N}$.*

It turns out that a Banach space X is O-convex if and only if its dual space X^* is E-convex, and that E-convex Banach spaces are superreflexive. The main result in [6] is the following.

Theorem 3.5. *Every E-convex Banach space enjoys (FPP).*

The main goal of this section is to show that E-convex Banach spaces enjoy the Prus-Sczepanik condition.

Lemma 3.6. *Let X be a Banach space. Suppose that for every $\varepsilon \in (0,1)$ there exists $x_\varepsilon \in S_X$ such that $b_1(1, x_\varepsilon) \geq 1 - \varepsilon$. Then, given $\varepsilon \in (0,1)$, there exists a point $x_\varepsilon \in S_X$ and a weakly null sequence (z_n) in B_X such that for every pair (m,n) of positive integers,*

$$\|z_n - z_m\| \leq 1 + 2\varepsilon \text{ and } \|x_\varepsilon + z_n\| > 2 - 2\varepsilon.$$

Proof. Fix $\varepsilon \in (0,1)$. Since there exists $x_\varepsilon \in S_X$ such that $b_1(1, x_\varepsilon) \geq 1 - \varepsilon$ then there exists $(z_n) \in \mathfrak{M}_X$ such that

$$\sup_n \{\inf_{k \geq n} \|x_\varepsilon + z_k\|\} = \liminf_{n \to \infty} \|x_\varepsilon + z_n\| > 2 - 2\varepsilon.$$

Therefore, there exists a positive integer L_0 such that,

$$\inf_{k\geq L_0} \|x_\varepsilon + z_k\| > 2 - 2\varepsilon$$

and then, for every $k \geq L_0$

$$\|x_\varepsilon + z_k\| > 2 - 2\varepsilon. \tag{3.1}$$

The weakly null sequence (z_n) satisfies $D(z_n) \leq 1$, or, in other words,

$$\inf_n \sup_{k\geq n} \left(\limsup_m \|z_k - z_m\| \right) \leq 1.$$

Hence there exists a positive integer N_0 such that

$$\sup_{k\geq N_0} \left(\limsup_m \|z_k - z_m\| \right) \leq 1 + \varepsilon$$

and consequently, for every $k \geq N_0$,

$$\inf_m \sup_{l\geq m} \|z_k - z_l\| = \limsup_m \|z_k - z_m\| \leq 1 + \varepsilon\,.$$

Thus, for every $k \geq N_0$ there exists a positive integer $M = M(k)$ such that for every $l \geq M$,

$$\|z_k - z_l\| \leq 1 + 2\varepsilon. \tag{3.2}$$

In particular for $k = N_0$, according with (3.2) there exists a positive integer M_0 such that, for every $l \geq \max\{M_0, N_0\}$, one has

$$\|z_{N_0} - z_l\| \leq 1 + 2\varepsilon.$$

Choose a positive integer $N_1 > \max\{M_0, N_0\}$. Then, for every $l \geq N_1$,

$$\|z_{N_0} - z_l\| \leq 1 + 2\varepsilon \tag{3.3}$$

and hence $\|z_{N_0} - z_{N_1}\| \leq 1 + 2\varepsilon$.

Taking $k = N_1$, again according with (3.2) there exists a positive integer M_1 such that, for every $l \geq \max\{M_1, N_1\}$, one has

$$\|z_{N_1} - z_l\| \leq 1 + 2\varepsilon.$$

Choose a positive integer $N_2 > \max\{M_1, N_1\}$. Then, for every $l \geq N_2$,

$$\|z_{N_1} - z_l\| \leq 1 + 2\varepsilon, \tag{3.4}$$

and hence $\|z_{N_1} - z_{N_2}\| \leq 1 + 2\varepsilon$. From (3.3) $\|z_{N_0} - z_{N_2}\| \leq 1 + 2\varepsilon$ also holds.

In this way we inductively can get a subsequence (z_{N_i}) of (z_n) such that for every pair (i, j) of positive integers

$$\|z_{N_i} - z_{N_j}\| \leq 1 + 2\varepsilon.$$

Finally we can find a positive integer i_0 such that $N_{i_0} \geq L_0$. Then, according with (3.1), for every $i \geq i_0$,

$$\|x_\varepsilon + z_{N_i}\| > 2 - 2\varepsilon.$$

Thus, the subsequence $(z_{N_i})_{i\geq i_0}$ of (z_n) satisfies the required conditions.

□

Theorem 3.7. *If X is an E-convex Banach space then there exists $\varepsilon \in (0,1)$ such that for every $x \in S_X$, $b_1(1,x) < 1-\varepsilon$.*

Proof. Let X be an E-convex Banach space. We shall argue by contradiction. Suppose that for all $\varepsilon \in (0,1)$ there exists $x_\varepsilon \in S_X$ such that $b_1(1,x_\varepsilon) \geq 1-\varepsilon$.

Then, if we fix $\varepsilon \in \left(0, \frac{1}{3}\right)$, from the above lemma we know that there exist $x_\varepsilon \in S_X$ and a sequence (z_n) in S_X such that for every pair (m,n) of positive integers,

$$\|z_n - z_m\| \leq 1 + 2\varepsilon \text{ and } \|x_\varepsilon + z_n\| > 2 - 2\varepsilon.$$

Take $f_n \in \nabla(x_\varepsilon + z_n)$ for $n = 1, 2, \ldots$. Since $f_n(x_\varepsilon + z_n) = \|x_\varepsilon + z_n\| > 2 - 2\varepsilon$, then for every positive integer n, $f_n(z_n) > 1-2\varepsilon$ and $f_n(x_\varepsilon) > 1-2\varepsilon$. Since X is E-convex then it is reflexive and the unit ball B_{X^*} is weakly compact. For a subsequence of (f_n) (which we will denote again (f_n)), one has that there exists $f \in B_{X^*}$ such that $f_n \rightharpoonup f$. Since $z_n \rightharpoonup 0_X$ there exists a positive integer $k_1 > 1$ such that, for every $k \geq k_1$,

$$|f_1(z_k)| < \varepsilon, \quad |f(z_k)| < \frac{\varepsilon}{2} \quad \text{and} \quad |(f_k - f)(z_1)| < \frac{\varepsilon}{2}.$$

Analogously, there exists a positive integer $k_2 > k_1$ such that, for every $k \geq k_2$,

$$|f_{k_1}(z_k)| < \varepsilon, \quad |f(z_k)| < \frac{\varepsilon}{2} \quad \text{and} \quad |(f_k - f)(z_{k_1})| < \frac{\varepsilon}{2}.$$

By a simple induction we obtain a strictly increasing sequence of positive integers (k_i) such that, for $1 \leq i < j$,

$$|f_{k_i}(z_{k_j})| < \varepsilon, \quad |f(z_{k_i})| < \frac{\varepsilon}{2} \quad \text{and} \quad |(f_{k_j} - f)(z_{k_i})| < \frac{\varepsilon}{2}, \tag{3.5}$$

which, in turn implies that for $1 \leq i < j$,

$$|f_{k_j}(z_{k_i})| < \frac{\varepsilon}{2} + |f(z_{k_i})| < \varepsilon. \tag{3.6}$$

Then, for $i \neq j$,

$$\|f_{k_i} + f_{k_j}\| \geq (f_{k_i} + f_{k_j})(x_\varepsilon) > 2 - 4\varepsilon$$

and from (3.5) and (3.6) one follows also that

$$\begin{aligned} \|f_{k_i} - f_{k_j}\| &\geq (f_{k_i} - f_{k_j})\left(\tfrac{z_{k_i} - z_{k_j}}{\|z_{k_i} - z_{k_j}\|}\right) \\ &\geq \tfrac{1}{\|z_{k_i} - z_{k_j}\|}(f_{k_i}(z_{k_i}) + f_{k_j}(z_{k_j}) - f_{k_i}(z_{k_j}) - f_{k_j}(z_{k_i})) \\ &\geq \tfrac{1}{1+2\varepsilon}(f_{k_i}(z_{k_i}) + f_{k_j}(z_{k_j}) - f_{k_i}(z_{k_j}) - f_{k_j}(z_{k_i})) \\ &> \tfrac{2-6\varepsilon}{1+2\varepsilon} = 2 - \tfrac{10\varepsilon}{1+2\varepsilon}. \end{aligned}$$

This means that given $\eta \in (0,1)$ and $n \geq 2$ we have $f_1, \ldots, f_n \in S_{X^*}$ which are symmetrically $2-\eta$ separated, or, in other words, that the dual space X^* is not O-convex, a contradiction to the fact that X is E-convex.

□

Corollary 3.8. *Every E-convex Banach space satisfies the Prus-Sczepanik condition.*

Example 4.8 below shows that the converse of the above corollary is not true.

4. Separation between sufficient conditions for (FPP)

In this section we will be concerned with the following geometrical properties which are, in some sense, maximal.

(1) Asymptotic normal structure.
(2) Orthogonal convexity.
(3) Weak orthogonality.
(4) Prus-Szczepanik condition.

We begin with some further comments about properties (1) (2) and (3) of this list.

4.1. **Asymptotic normal structure (ANS).** After Kirk's above mentioned result of 1965 several researchers obtained a wide range of sufficient conditions for (NS), for instance, uniform convexity (Edelstein, 1963), $\varepsilon_0(X) < 1$ (Goebel, 1970), uniform convexity in every direction (Zizler, 1971), Opial condition (Gossez and Lami Dozo, 1972), k-uniform convexity (Sullivan, 1979), near uniform convexity (Van Dulst, 1981), uniform smoothness, (Turett, 1982).

However, in 1981 J.B. Baillon and R. Schöneberg, (see [1]) properly generalized (NS) as follows.

Definition 4.1. *A Banach space $(X, \|\cdot\|)$ has (ANS) if each nonempty bounded closed and convex subset C of X with diam$(C) > 0$ has the following property:*

For every sequence (x_n) in C with $\|x_n - x_{n+1}\| \to 0$ there exists a point $p \in C$ such that $\liminf \|p - x_n\| <$ *diam* (C).

They also showed the following.

Theorem 4.2. *If X is a reflexive Banach space with (ANS) then it has (FPP).*

For further information about (ANS) see also [4].

4.2. **Orthogonal convexity.** (See [17, 18]). This is a generalization of the uniform convexity. It was introduced by A. Jiménez-Melado in 1988. In order to give the definition of this concept, we need some further notation.

Let $x, y \in X$, and $\beta > 0$. $M_\beta(x, y) := B[x, \frac{1+\beta}{2} d(x, y)] \bigcap B[y, \frac{1+\beta}{2} d(x, y)]$.

If A is a nonempty bounded subset of X, $|A| := \sup\{\|x\| : x \in A\}$.

Definition 4.3. *A Banach space $(X, \|.\|)$ is orthogonally convex (OC) if for every weakly null sequence (x_n) with $D(x_n) > 0$, there exists $\beta > 0$ such that*

$$\limsup_n (\limsup_m \left|M_\beta(x_n, x_m)\right|) < D(x_n).$$

Every uniformly convex Banach space is (OC) but this notion is independent on (NS). Many Banach spaces as, for instance, c_0, ℓ_1, and the classical James space J are (OC) in spite of their very different geometry. In [22] was shown that (OC) Banach spaces have the Banach-Saks property.

Theorem 4.4. *Every (OC) Banach space enjoys (WFPP).*

4.3. **Property WORTH.**

Definition 4.5. *A Banach space has the (WORTH) property (Rosenthal, 1983; Sims 1988) if* $\lim_n \big|\|x_n - x\| - \|x_n + x\|\big| = 0$ *for all* $x \in X$ *and all weakly null sequence* (x_n) *in* X.

The classical spaces ℓ_p for $1 \le p < \infty$ and c_0 have this property. For reflexive spaces, (WORTH) and $R(X) < 2$ imply that X has (NS). (Sims, 1994). In [19] a coefficient which in some sense quantifies how far a Banach space is to enjoy (WORTH) was defined. This coefficient is

$$\mu(X) := \inf\{r > 0 : \limsup \|x_n + x\| \le r \limsup \|x_n - x\| : x_n \rightharpoonup 0_X \ \ x \in X\}.$$

Obviously $\mu(X) = 1 \Leftrightarrow X$ has (WORTH) and $1 \le \mu(X) \le 3$. The problem of whether reflexive spaces with (WORTH) property have (FPP) was raised by B. Sims. Many partial affirmative answers were obtained (see, for instance [11]). Very recently in 2010, H. Fetter and B. Gamboa, [9] solved this problem.

Theorem 4.6. *If* X *is reflexive and* $\mu(X) = 1$, *then* X *enjoys (FPP).*

4.4. **Independence of these sufficient conditions for (FPP).** Next, we shall prove that these geometrical properties are pairwise independent, in the sense that each one of these properties neither is implied by, nor implies, the other. To do this, we will consider several examples. Many of them are established in ℓ_2, the classical real space of all sequences $x = (x_n) = \big(x(n)\big)$ for which $\sum_{n=1}^{\infty} x_n^2 < \infty$. The Euclidean norm $\|x\|_2 := \sqrt{\sum_{n=1}^{\infty} x_n^2}$ is associated to the ordinary inner product $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n$. Also the "sup" norm $\|x\|_\infty = \sup\{|x(n)| : n = 1, \ldots\}$ will be sometimes considered. The standard Schauder basis of $(\ell_2, \|.\|_2)$ will be denoted by (e_n).

Example 4.7. *The James* E_β *spaces. For* $\beta \ge 1$, *let* E_β *be the space* ℓ_2 *endowed with the norm*

$$|x|_\beta := \max\{\|x\|_2, \beta\|x\|_\infty\}.$$

Some well known properties of these spaces are the following.

(1) For all $\beta \ge 1$, $\mu(E_\beta) = 1$ [32].
(2) The spaces E_β have (ANS) if and only if $1 \le \beta < 2$, and they have (NS) if and only if $1 \le \beta < \sqrt{2}$. (See [1]).
(3) For every $\beta \ge 1$ one has that the space E_β is (OC). (See [18]).
(4) For all $\beta \ge 1$ one has that $M(E_\beta) > 1$. (See [5]). Hence all the E_β spaces satisfy (PSz) condition.

Proposition 4.8. *For* $\beta \ge \sqrt{2}$, *the space* E_β *fails to be* E*-convex.*

Proof. From a result due to S. Saejung,([31], Theorem 5), every E-convex Banach space X with $\mu(X) = 1$ has normal structure.

Since $\mu(E_\beta) = 1$, and E_β fails to have (NS) for $\beta \ge \sqrt{2}$, we would have a contradiction if E_β was E-convex for some $\beta \ge \sqrt{2}$. □

Notice that $\varepsilon_0(E_\beta) < 2$ for $1 \le \beta < \sqrt{2}$ (see [16], pag. 58) and hence the spaces E_β are E convex if and only if $\beta \in [1, \sqrt{2})$.

Example 4.9. *Let Y be the space ℓ_2 endowed with the norm*

$$\|x\| := \max\{\|x\|_2, \mathcal{M}(x)\},$$

where for $x \in \ell_2$, $\mathcal{M}(x) := \sup\{|x(i)| + |x(j)| : 1 \leq i < j\}$.
It is straightforward to check that Y has (WORTH) property, that is, that $\mu(Y) = 1$.

Proposition 4.10. *Y fails (OC).*

Proof. For $n, m \in \mathbb{N}$, $n \neq m$, $\|e_n\| = 1$, $\|e_n + e_m\| = \|e_n - e_m\| = 2$. Then, $D(e_n) = 2$. For $z = e_n + e_m$ one has $\|z\| = 2$ and

$$1 = \|z - e_n\| = \|z - e_m\| = \frac{1}{2}\|e_n - e_m\|.$$

Thus, $\forall \beta > 0$, $z \in M_\beta(e_m, e_n)$ if $m \neq n$, and

$$|M_\beta(e_m, e_n)| \geq \|z\| = 2.$$

which implies

$$\limsup_n (\limsup_m |M_\beta(e_n, e_m)|) \geq 2 = D(e_n).$$

Thus, Y is not (OC). □

Example 4.11. *Let VD be the real space ℓ_2 endowed with the norm*

$$|x| := \max\{\frac{1}{3}\|x\|_2, \sup\{|x_1 + x_n + x_{n+1}| : n \geq 2\}\}.$$

This space was introduced in 1982 by D. van Dulst. It is (OC) (see [18]), but it fails to have asymptotic normal structure [30].

Proposition 4.12. *The van Dulst space fails (PSz) condition*

Proof. Take $x_n = (0, \ldots, 0, \overset{4n+1}{\tfrac{1}{2}}, \tfrac{1}{2}, 0, \ldots)$. Note that $|x_n| = 1$ and that, for $m > n$,

$$|x_m - x_n| = |(0, \ldots, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, \frac{1}{2}, 0 \ldots)| = 1.$$

Then, $(x_n) \in \mathfrak{M}_{VD}$. Moreover, $e_1 \in S_{VD}$, and for every $n \geq 1$,

$$|e_1 + x_n| = |(1, 0, \ldots, 0, \frac{1}{2}, \frac{1}{2}, \ldots)| = 2.$$

Therefore, for every $\varepsilon \in (0, 1)$,

$$b_1(1, e_1) := \sup_{(y_n) \in \mathfrak{M}_{VD}} \liminf_n |e_1 + y_n| - |e_1| \geq \liminf_n |e_1 + x_n| - |e_1| = 2 - 1 > 1 - \varepsilon.$$

In the same way, it is clear that $(-x_n) \in \mathfrak{N}_{VD}$. For $n > 1$,

$$|e_1 + (-x_n)| = |(1, 0, \ldots, , -\frac{1}{2}, -\frac{1}{2}, 0, \ldots)| = \max\{\frac{1}{3}\sqrt{\frac{3}{2}}, 1\} = 1,$$

and

$$d(1, e_1) := \inf_{(y_n) \in \mathfrak{N}_{VD}} \limsup_n |e_1 + y_n| - |e_1| \leq \limsup_n |e_1 - x_n| - |v| = 1 - 1 = 0 < \varepsilon.$$

□

Proposition 4.13. *$\mu(VD) \geq 2$.*

Proof. Take $x_n := \big(0, \dots, 0, \overset{(2n+1)}{\tfrac{1}{2}}, \tfrac{1}{2}, 0, \dots\big)$, one has $x_n \rightharpoonup 0_{\ell_2}$ and $|x_n| = 1$ $(n = 1, 2, \dots)$. Since $e_1 \in S_{VD}$, and for $n > 1$,

$$\begin{aligned} |e_1 + x_n| &= \big|\big(1, 0, \dots, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, \dots\big)\big| = 2. \\ |e_1 - x_n| &= \big|\big(1, 0, \dots, 0, \tfrac{-1}{2}, \tfrac{-1}{2}, 0, \dots\big)\big| = 1 . \end{aligned}$$

It follows that $\mu(VD) \geq 2$. □

Example 4.14. *The Banach space $\ell_2 \oplus_1 \ell_2$.*

This space is (UNC) and hence it satisfies the (PSz) condition (See [28]).

Proposition 4.15. *The space $\ell_2 \oplus_1 \ell_2$ is E-convex but it fails to be (OC).*

Proof. (See [10]). Indeed, $(\ell_2, \|\cdot\|_2)$ is P-convex, and from Theorem 1.5 in [2] P-convexity is preserved under ℓ_∞-direct sums. Then, $\ell_2 \oplus_1 \ell_2 = (\ell_2 \oplus_1 \ell_2)^*$ is P-convex and hence O-convex. To see that this space fails to be (OC), for k positive integer put

$$v_{2k} = (0_{\ell_2}, e_{2k}), \quad v_{2k+1} = (e_{2k+1}, 0_{\ell_2}).$$

It is obvious that the sequence (v_n) is weakly convergent to $(0_{\ell_2}, 0_{\ell_2})$, and that $\|v_n\| = 1$ for $n = 1, 2, \dots$.

For $n < m$ one has that $\|v_n - v_m\| = 2$ whenever n and m have different parity while $\|v_n - v_m\| = \sqrt{2}$ if m and n have the same parity. Thus, $D(v_n) = \limsup_m [\limsup_n \|v_m - v_n\|] = 2$. If $\beta > 0$ and m, n have different parity, since

$$\|(v_m + v_n) - v_n\| = 1 = \frac{1}{2}\|v_m - v_n\|, \quad \|(v_m + v_n) - v_m\| = 1 = \frac{1}{2}\|v_m - v_n\|,$$

we have that $w_{m,n} = v_m + v_n \in M_\beta(v_n, v_m)$. Given that $\|w_{m,n}\| = \|v_n + v_m\| = \|e_n\|_2 + \|e_m\|_2 = 2$, then

$$|M_\beta(v_n, v_m)| \geq \|w_{m,n}\| = 2$$

and therefore for every $\beta > 0$,

$$\limsup_m [\limsup_n |M_\beta(v_n, v_m)|] \geq 2 = D(v_n),$$

which implies that $\ell_2 \oplus_1 \ell_2$ fails to be (OC). □

Example 4.16. *The Bynum spaces. For $x \in \ell_2$ let*

$$\|x\|_{2,1} := \|x^+\|_2 + \|x^-\|_2, \quad \|x\|_{2,\infty} := \max\{\|x^+\|_2, \|x^-\|_2\}.$$

The spaces $\ell_{2,1} := (\ell_2, \|\cdot\|_{2,1})$ and $\ell_{2,\infty} := (\ell_2, \|\cdot\|_{2,\infty})$ were introduced by L.E. Bynum.

Well known features of these spaces are the following.

(1) $\ell_{2,1}^* = \ell_{2,\infty}$. The space $\ell_{2,1}$ has (NS) (and hence (ANS)) [16]. But $\ell_{2,\infty}$ fails to have (ANS). (L. Bynum, private communication).
(2) $\varepsilon_0(\ell_{2,\infty}) = 1$. Hence $M(\ell_{2,\infty}) > 1$ and this space satisfy (PSz) condition. $\varepsilon_0(\ell_{2,\infty}) = 1$ also directly implies that $\ell_{2,\infty}$ is E-convex.
(3) $\ell_{2,\infty}$ is (OC) (See [17]).
(4) $\mu(\ell_{2,1}) = \mu(\ell_{2,\infty}) = \sqrt{2}$. (See [20]).

Example 4.17. *The Banach space* $Z := (\ell_2, |\cdot|_Z)$ *where*

$$|x|_Z := \max\{\|x\|_2, \mathcal{M}(x)\} + \sum_{i=1}^{\infty} 2^{-i}|x(i)|$$

satisfies Opial condition (see [17]). Since it is reflexive, then it has normal structure, and hence (ANS). On the other hand, for the sequence (e_n) of unit vectors in ℓ_2 we have that it is weakly null and that $D(e_n) = A_\lambda[(e_n)]$ for all $\lambda > 0$, what shows that Z fails (OC).

Example 4.18. *Let* V *be the space* ℓ_2 *endowed with the norm*

$$\|x\|_V := \max\{\|x\|_2, \sqrt{2}A(x)\}.$$

where $A(x) := \sup\{|x_1 + x_n| : n \geq 2\}$.

Proposition 4.19. *The space* V *fails (PSz) condition.*

Proof. For each positive integer n put $v_n := \frac{1}{\sqrt{2}}e_n$. Note that $\|v_n\|_V = 1$ and that, for $m > n > 1$, $\|v_m - v_n\|_V = 1$. Then $(v_n) \in \mathfrak{M}_V$.

Moreover, $v_1 \in S_V$, and for every $n \geq 1$, $\|v_1 + v_n\|_V = 2$. Therefore, for every $\varepsilon \in (0,1)$,

$$\begin{aligned} b_1(1, v_1) &:= \sup_{(y_n)\in\mathfrak{M}_V} \liminf_n \|v_1 + y_n\|_V - \|v_1\|_V \\ &\geq \liminf_n \|v_1 + v_n\|_V - \|v_1\|_V = 2 - 1 > 1 - \varepsilon. \end{aligned}$$

In the same way, since $(-v_n) \in \mathfrak{N}_V$, and for $n \geq 2$,

$$\|v_1 + (-v_n)\|_V = \|(\frac{1}{\sqrt{2}}, 0, \ldots, 0, -\frac{1}{\sqrt{2}}, 0, \ldots)\|_V = \max\{1, \sqrt{2}\frac{1}{\sqrt{2}}\} = 1$$

and

$$\begin{aligned} d(1, v_1) &:= \inf_{(y_n)\in\mathfrak{N}_V} \limsup_n \|v_1 + y_n\|_V - \|v_1\|_V \\ &\leq \limsup_n \|v_1 + (-v_n)\|_V - \|v_1\|_V = 0 < \varepsilon. \end{aligned}$$

□

To prove that the space V has asymptotic normal structure, we closely will follow Lemma 3 in [1]. First, we need the following result, also from [1].

Lemma 4.20. *Let* K *a closed bounded and convex subset of* ℓ_2 *and let* (x_n) *be a sequence in* K*. Then, there exists a unique point* $z \in K$ *(called the* $\|\cdot\|_2$*-asymptotic-center of* (x_n) *in* K*) which minimizes the functional* $x \mapsto \limsup_{n\to\infty} \|x_n - x\|_2$*. This point satisfies*

(a)

$$\limsup_{n\to\infty} \|x_n - z\|_2^2 + \|z - x\|_2^2 \leq \limsup_{n\to\infty} \|x_n - x\|_2^2$$

for all $x \in K$*, and*

(b)

$$2\limsup_{n\to\infty} \|x_n - z\|_2^2 \leq \limsup_{p\to\infty}\big(\limsup_{n\to\infty} \|x_n - x_p\|_2^2\big).$$

Proposition 4.21. *V has asymptotic normal structure.*

Proof. Assume for a contradiction that there exists a closed, bounded and convex subset K of ℓ_2, with $\operatorname{diam}_{\|\cdot\|_V}(K) = d > 0$ and a sequence (x_n) in K, with $x_n - x_{n+1} \to 0_{\ell_2}$, and such that for every $x \in K$

$$\|x_n - x\|_V \to d.$$

Since K is weakly compact we can suppose, passing to a subsequence if necessary, that $x_n \rightharpoonup x \in K$. Let $z \in K$ be the $\|\cdot\|_2$-asymptotic-center of (x_n) in K. We claim that $z = x$.

Indeed, for every positive integer n we have that

$$\|x_n - z\|_2^2 = \|x_n - x\|_2^2 + \|x - z\|_2^2 - 2\langle x_n - x, x - z\rangle.$$

Since $x_n - x \rightharpoonup 0$

$$\limsup_{n\to\infty} \|x_n - z\|_2^2 = \limsup_{n\to\infty} \|x_n - x\|_2^2 + \|x - z\|_2^2. \tag{4.1}$$

On the other hand, from Lemma 4.20(a) we have that

$$\limsup_{n\to\infty} \|x_n - z\|_2^2 + \|x - z\|_2^2 \le \limsup_{n\to\infty} \|x_n - x\|_2^2.$$

Bearing in mind (4.1) from this last inequality we get that

$$\limsup_{n\to\infty} \|x_n - z\|_2^2 + \|x - z\|_2^2 \le \limsup_{n\to\infty} \|x_n - x\|_2^2 - \|x - z\|_2^2$$

and hence $\|x - z\|_2 = 0$, that is, $z = x$ as we claimed.

By Lemma 4.20(b) we have that $\limsup_{n\to\infty} \|x_n - z\|_2^2 \le \frac{d^2}{2}$ and since $\|x_n - z\|_V^2 \to d^2$ we have that $[\sqrt{2}\,A(x_n - z)]^2 \to d^2$, that is,

$$\sup_{j\ge 2} \big|x(1) + x_n(j) - z(1) - z(j)\big| \to \frac{d}{\sqrt{2}}.$$

Given that $x_n, z \in \ell_2$, we have that the above supremum is attained, that is, for each positive integer n there exists a positive integer j_n such that

$$A(x_n - z) = \big|x(1) + x_n(j_n) - z(1) - z(j_n)\big|.$$

Since $x_n - z \rightharpoonup 0_{\ell_2}$, and hence $x_n(1) - z(1) \to 0$, and $x_n - x_{n+1} \to 0_{\ell_2}$,

$$\begin{aligned} &A(x_n - z)^2 - 2|x_n(1) - z(1)|.|x_n(j_n) - z(j_n)| \\ &\qquad + \big(A(x_{n+1} - z) - A(x_n - x_{n+1}) - |x_n(1) - z(1)|\big)^2 \longrightarrow d^2. \end{aligned}$$

Therefore, given $r \in (\frac{d^2}{2}, d^2)$ there exists a positive integer n_0 such that for every $n \ge n_0$ we have that

$$\left.\begin{aligned} &\|x_n - z\|_2^2 \le r \\ &A(x_{n+1} - z) - A(x_n - x_{n+1}) - |x_n(1) - z(1)| \ge 0 \\ &A(x_n - z)^2 - 2\,|x_n(1) - z(1)|\,|x_n(j_n) - z(j_n)| \\ &\qquad + \big(A(x_{n+1} - z) - A(x_n - x_{n+1}) - |x_n(1) - z(1)|\big)^2 > r \end{aligned}\right\}.$$

Put $k = j_{n_0}$. We claim that $j_n = k$ for every $n \geq n_0$. Otherwise there exists $n \geq n_0$ such that $j_n \neq j_{n+1}$ and then

$$\begin{aligned}
r \quad &\geq \|x_n - z\|_2^2 \\
&\geq |x_n(1) - z(1)|^2 + |x_n(j_n) - z(j_n)|^2 + |x_n(j_{n+1}) - z(j_{n+1})|^2 \\
&\geq |x_n(1) - z(1) + x_n(j_n) - z(j_n)|^2 - 2|x_n(1) - z(1)||x_n(j_n) - z(j_n)| \\
&\qquad\qquad + |x_n(j_{n+1}) - z(j_{n+1})|^2 \\
&= |x_n(1) - z(1) + x_n(j_n) - z(j_n)|^2 - 2|x_n(1) - z(1)||x_n(j_n) - z(j_n)| \\
&\quad + |x_{n+1}(j_{n+1}) - z(j_{n+1}) + x_{n+1}(1) - z(1) + \\
&\qquad + x_n(j_{n+1}) - x_{n+1}(j_{n+1}) + x_n(1) - x_{n+1}(1) - x_n(1) + z(1)|^2
\end{aligned} \tag{4.2}$$

Since $A(x_{n+1} - z) - A(x_n - x_{n+1}) - |x_n(1) - z(1)| \geq 0$, that is

$$\big|x(1)+x_{n+1}(j_{n+1})-z(1)-z(j_{n+1})\big| - \big|x(1)+x_n(j_n)-z(1)-z(j_n)\big| - \big|x_n(1)-z(1)\big| \geq 0$$

then,

$$\begin{aligned}
0 \quad &\leq \big|x(1) + x_{n+1}(j_{n+1}) - z(1) - z(j_{n+1})\big| \\
&\qquad - \big|x(1) + x_n(j_n) - z(1) - z(j_n)\big| - \big|x_n(1) - z(1)\big| \\
&\leq \big|x(1) + x_{n+1}(j_{n+1}) - z(1) - z(j_{n+1})\big| \\
&\qquad - \big|\big(x(1) + x_n(j_n) - z(1) - z(j_n)\big) + \big(x_n(1) - z(1)\big)\big| \\
&\leq \big|x(1) + x_{n+1}(j_{n+1}) - z(1) - z(j_{n+1}) \\
&\qquad - \big(x(1) + x_n(j_n) - z(1) - z(j_n)\big) - \big(x_n(1) - z(1)\big)\big|
\end{aligned}$$

and hence

$$\begin{aligned}
0 \quad &\leq \Big(\big|x(1) + x_{n+1}(j_{n+1}) - z(1) - z(j_{n+1})\big| \\
&\qquad - \big|x(1) + x_n(j_n) - z(1) - z(j_n)\big| - \big|x_n(1) - z(1)\big|\Big)^2 \\
&\leq \big|x(1) + x_{n+1}(j_{n+1}) - z(1) - z(j_{n+1}) \\
&\qquad - \big(x(1) + x_n(j_n) - z(1) - z(j_n)\big) - \big(x_n(1) - z(1)\big)\big|^2.
\end{aligned}$$

Bearing this in mind, it follows from (4.2) that

$$\begin{aligned}
r \quad &\geq |x_n(1) - z(1) + x_n(j_n) - z(j_n)|^2 - 2|x_n(1) - z(1)||x_n(j_n) - z(j_n)| \\
&\quad + |x_{n+1}(j_{n+1}) - z(j_{n+1}) + x_{n+1}(1) - z(1) + \\
&\qquad + x_n(j_{n+1}) - x_{n+1}(j_{n+1}) + x_n(1) - x_{n+1}(1) - x_n(1) + z(1)|^2 \\
&\geq |x_n(1) - z(1) + x_n(j_n) - z(j_n)|^2 - 2|x_n(1) - z(1)||x_n(j_n) - z(j_n)| \\
&\quad + \Big(\big|x(1) + x_{n+1}(j_{n+1}) - z(1) - z(j_{n+1})\big| \\
&\qquad - \big|x(1) + x_n(j_n) - z(1) - z(j_n)\big| - \big|x_n(1) - z(1)\big|\Big)^2 \\
&= |x_n(1) - z(1) + x_n(j_n) - z(j_n)|^2 - 2|x_n(1) - z(1)||x_n(j_n) - z(j_n)| \\
&\quad + \Big(A(x_{n+1} - z) - A(x_n - x_{n+1}) - |x_n(1) - z(1)|\Big)^2 > r,
\end{aligned}$$

a contradiction which proves our claim.

Then $j_n = k$ for every $n \geq n_0$. Since $x_n - z \rightharpoonup 0$ we have that $x_n(1) - z(1) \to 0$ and $x_n(k) - z(k) \to 0$ as $n \to \infty$. By the above we get that

$$\begin{aligned}\tfrac{d^2}{2} &= \lim_{n\to\infty} A(x_n - z)^2 \\ &= \lim_{n\to\infty} |x_n(1) + x_n(j_n) - z(1) - z(j_n)|^2 \\ &= \lim_{n\to\infty} |x_n(1) + x_n(k) - z(1) - z(k)|^2 = 0.\end{aligned}$$

Consequently, $d = 0$, which contradicts our assumption. □

Example 4.22. *Let* $1 < p < \infty$ *and* $\beta \geq 1$. *Let us consider the Banach space* $X_{p,\beta} := \big(\mathbb{R} \oplus_1 E_\beta\big) \oplus_\infty \ell_p$.

Proposition 4.23. *For* $1 < p < \infty$ *and* $\beta \geq \sqrt{2}$ *the space* $X_{p,\beta}$ *enjoys property (WORTH) but it fails (PSz) condition.*

Proof. For the sequence (w_n) in E_β given by $w_n = \frac{1}{\beta} e_n$, take the sequences $\big(z_m\big)$ and $\big(y_m\big)$ in $S_{X_{p,\beta}}$ defined as

$$z_m = \big((0, w_m), 0_{\ell_p}\big), \quad y_m = \big((0, 0_{E_\beta}), e_m\big),$$

where $\big(e_m\big)_m$ is the standard basis of ℓ_p. For $x = \big((1, 0_{E_\beta}), 0_{\ell_p}\big) \in S_{X_{p,\beta}}$ and positive integers m, n we have that

$$\begin{aligned}\|x + z_m\|_{X_{p,\beta}} &= \|\big((1, w_m), 0_{\ell_p}\big)\|_{X_{p,\beta}} = \max\big\{1 + |w_m|_{(\beta)}, 0\big\} = 2, \\ \|x + y_m\|_{X_{p,\beta}} &= \|\big((1, 0_{E_\beta}), e_m\big)\|_{X_{p,\beta}} = \max\big\{1, \|e_m\|_p\big\} = 1, \\ \|z_n - z_m\|_{X_{p,\beta}} &= \|\big((0, w_m - w_n), 0_{\ell_p}\big)\|_{X_{p,\beta}} = |w_n - w_m|_\beta = 1.\end{aligned}$$

Since $z_m \rightharpoonup 0_{X_{p,\beta}}$ and $y_m \rightharpoonup 0_{X_{p,\beta}}$ we get that $(z_n) \in \mathfrak{M}_{X_{p,\beta}}$ and $(y_n) \in \mathfrak{N}_{X_{p,\beta}}$. Therefore we have for every $\varepsilon \in (0, 1)$ that

$$d(1, x) = \inf_{(\xi_n)\in\mathfrak{N}_X} \limsup_{n\to\infty} \|x + \xi_n\|_{X_{p,\beta}} - 1 \leq \limsup_{n\to\infty} \|x + y_n\|_{X_{p,\beta}} - 1 = 0 \leq \varepsilon$$

and

$$b_1(1, x) = \sup_{(\xi_n)\in\mathfrak{M}_X} \liminf_{n\to\infty} \|x + \xi_n\|_{X_{p,\beta}} - 1 \geq \liminf_{n\to\infty} \|x + z_n\|_{X_{p,\beta}} - 1 = 1 \geq 1 - \varepsilon.$$

Thus, $X_{p,\beta}$ fails condition (PSz).

On the other hand, $X_{p,\beta}$ has property (WORTH) because it is well known that the spaces ℓ_p and E_β have property (WORTH) and Kato and Tamura proved in [21] that this property is preserved under ℓ_q^n-direct sums for $1 \leq q \leq \infty$. □

4.5. **Summary.** Next, we summarize the above results in the following table:

fails ↑ ←has $\boxed{X}$	ANS	OC	PS	WORTH
ANS	*	Z	V	$\ell_{2,1}$
OC	VD	*	VD	$\ell_{2,\infty}$
PS	$\ell_{2,\infty}$	$\ell_2 \oplus_1 \ell_2$	*	$\ell_{2,\infty}$
WORTH	E_β, $\beta \geq 2$	Y	$X_{p,\beta}$	*

5. Final remarks

All the above examples are either equivalent renormings of ℓ_2 or isometric to such renormings, that is, they are Banach spaces of the form $(\ell_2, \|.\|)$ with $\|\cdot\|_2 \le \|\cdot\| \le b\|\cdot\|_2$. The problem if such spaces $(\ell_2, \|\cdot\|)$ have the (FPP) remains open. In these framework, it is known that $(\ell_2, \|\cdot\|)$ enjoys (FPP) provided that $b < \sqrt{\frac{5+\sqrt{17}}{2}}$. It is easy to give examples showing that this condition nor implies nor is implied by (ANS), (PSz), (WORTH) and (OC).

It is worth noting that one can find in the literature several sufficient conditions for (FPP) which we not considered here. Because of its complexity, we have omitted in our study those which are not directly depending on the geometry of the underlying space. For instance, some of these conditions are specific for spaces with a suitable basis, and they are stated in terms of certain constants depending on such a basis. For similar reasons, we have not payed attention to those sufficient conditions for (WFPP) which are stated in terms of the properties of the ultrapowers of the Banach space where the fixed point problem is considered (as, for example, the so called super fixed point property, or property (AMC) in [13]).

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