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DEMICLOSED PRINCIPLE AND CONVERGENCE OF A HYBRID ALGORITHM FOR MULTIVALUED *-NONEXPANSIVE MAPPINGS

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Abstract. A demiclosed principle is proved for multivalued *-nonexpansive mappings. Moreover, strong convergence of an iterative algorithm is obtained for such mappings in a Banach space by using metric projections. The results of this paper improve and extend the corresponding results for single valued nonexpansive mappings which was studied by many authors.
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1. Introduction

The study of fixed points for multivalued contractions and nonexpansive mappings by using the Hausdorff metric was initiated by Markin [11] (see also [13]). Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [7] and references cited therein). Approximating fixed points of multivalued mappings have been studied by many authors. In particular, Sastry and Babu [15] introduced the analogs of Mann and Ishikawa iterates for multivalued mappings and proved convergence theorems for nonexpansive mappings whose domain is a compact convex subset of a Hilbert space. Recently, Panyanak [14] generalized results of Sastry and Babu to uniformly convex Banach spaces. Xu [16] introduced the class of multivalued *-nonexpansive mappings and showed that *-nonexpansiveness is different from nonexpansiveness for multivalued mappings.

Throughout this paper X denotes a real Banach space and C is a nonempty bounded closed convex subset of X. Also, $CB(X)$ denotes the set of nonempty closed bounded subsets of X and $\mathcal{K}(X)$ denotes the set of nonempty compact subsets of X. It is clear that $\mathcal{K}(X)$ is included in $CB(X)$. Let H be the Hausdorff metric on $CB(X)$, i.e.,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

where $A, B \in CB(X)$ and $d(x, B) = \inf\{\|x - y\| : y \in B\}$ is the distance from the point x to the set B.

A multivalued mapping $T : C \to CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in C$. T is said *-nonexpansive if for each $x, y \in C$ and $u_x \in Tx$ with $d(x, Tx) = \|x - u_x\|$ there exists $u_y \in Ty$ with $d(y, Ty) = \|y - u_y\|$ such that

$$\|u_x - u_y\| \leq \|x - y\|.$$

A point x is called a fixed point of T if $x \in Tx$. We denote the set of all fixed points of T by $F(T)$.

The mapping T is said to be demiclosed at y in X if $\{x_n\}$ is a sequence in C which converges weakly to x and $y_n \in Tx_n \to y$; then $y \in Tx$.

It is known that a multivalued nonexpansive mapping $T : C \to \mathcal{K}(C)$ has a fixed point if C is nonempty bounded closed convex subset of a uniformly convex Banach space X [10]. Some recent existence theorems for multivalued mappings can be found in [1, 8].

Remark 1.1 *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : C \to \mathcal{K}(C)$ be a multivalued *-nonexpansive mapping. Then T has a fixed point. To see this define $S : C \to \mathcal{K}(C)$ by*

$$Sx = \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\}.$$

Definition of *-nonexpansiveness implies that S is multivalued nonexpansive and by Lim's fixed point theorem [10] *S and hence T itself has a fixed point.*

On the other hand, using the metric projection, Matsushita and Takahashi [12] introduced an iterative algorithm for single valued nonexpansive mappings in Banach spaces as follows.

Let C be a nonempty subset of a reflexive, strictly convex and smooth Banach space X and $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. For a given $x_1 = x \in C$, compute the sequence $\{x_n\}$ by the iterative algorithm

$$\begin{cases} C_n = \overline{co}\{z \in C : \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\overline{co}D$ denotes the convex closure of the set D and $\{t_n\}$ is a sequence in $(0, 1)$ with $\lim_{n\to\infty} t_n = 0$. They proved that if T is nonexpansive and X is uniformly convex and smooth, then $\{x_n\}$ generated by (1.1) converges strongly to a fixed point of T.

Motivated by these facts, we introduce an analog to Matsushita and Takahashi's iterative algorithm for multivalued mappings. The algorithm is defined as follows.

Let C and X be as in (1.1), $T : C \to \mathcal{K}(C)$ be a mapping with $F(T) \neq \emptyset$, $x_1 = x \in C$ and compute the sequence $\{x_n\}$ by the iterative algorithm

$$\begin{cases} C_n = \overline{co}\{z \in C : d(z, Tz) \leq t_n d(x_n, Tx_n)\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{t_n\}$ is a sequence in $(0,1)$ with $\lim_{n\to\infty} t_n = 0$ and $P_{C_n \cap D_n}$ is the metric projection from X onto $C_n \cap D_n$.

For a single valued mapping T, the algorithm (1.2) reduces to (1.1). Note that $d(z, \{Tz\}) = \|z - Tz\|$.

The purpose of this paper is to establish strong convergence theorem of the iterative algorithm (1.2) for multivalued *-nonexpansive mappings in a uniformly convex and smooth Banach space.

2. Preliminaries

When $\{x_n\}$ is a sequence in X, we denote strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. A Banach space X is said to have Kadec-Klee property if for every sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply that $x_n \to x$. Every uniformly convex Banach space has the Kadec-Klee property [2]. Let B_r denotes the open ball of radius r centered at 0. A Banach space X is said to have convex approximation property if for each $\epsilon > 0$ there exists a positive integer p such that for every bounded subset M of X,

$$coM \subset co_p M + B_\epsilon. \tag{2.1}$$

where coM denotes the convex hull of M and $co_p M$ denotes the set of convex combination of no more than p elements of M; or in other words, each convex combination of elements of M can be approximated by a convex combination of no more than p elements of M. It is known that every uniformly convex Banach space has the convex approximation property [5].

Let X^* be the dual of X. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle$. The normalized duality mapping J from X to 2^{X^*} is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in X$. It is known that a Banach space X is smooth if and only if the normalized duality mapping J is single valued. Some properties of duality mapping have been given in [2, 6]. Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space X. Then for any $x \in X$ there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y \in C} \|y - x\|.$$

The mapping $P_C : X \to C$ defined by $P_C x = x_0$ is called the *metric projection* from X onto C. Let $x \in X$ and $u \in C$. Then, it is known that $u = P_C x$ if and only if

$$\langle u - y, J(x - u) \rangle \geq 0 \tag{2.2}$$

for all $y \in C$ (see [2, 3]).

3. Demiclosed principle

In this section, we shall obtain demiclosed principle for *-nonexpansive mappings. To proceed in this direction, we first prove some Zarantonello-type inequalities for multivalued *-nonexpansive mappings. For classical and contemporary developments related to the Zarantonello's inequality, see [5, 9] where further references are provided.

Consistently with [4], we denote by Γ the set of strictly increasing, continuous and convex functions $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$. A multivalued mapping $T : C \to \mathcal{K}(C)$ is said to be of type (γ) if $\gamma \in \Gamma$ and

$$\gamma\left(d\left(\lambda x + (1-\lambda)y, T(\lambda x + (1-\lambda)y)\right)\right) \leq \max\{d(x,Tx), d(y,Ty)\} \tag{3.1}$$

for all $x, y \in C$ and $\lambda \in [0,1]$.

Lemma 3.1 *If X is uniformly convex then there exists $\gamma \in \Gamma$ such that every *-nonexpansive mapping $T : C \to \mathcal{K}(C)$ is of type (γ). Moreover, γ can be chosen to depend only on the diameter of C and not on T.*

Proof. Let δ be the modulus of uniform convexity of X. We know that $\delta : [0,2] \to [0,1]$ is continuous increasing with $\delta(0) = 0$, $\delta(t) > 0$ for $t > 0$ and

$$2\min\{\lambda, (1-\lambda)\}\delta\left(\|v-w\|\right) \leq 1 - \|\lambda v + (1-\lambda)w\|$$

for each $\lambda \in [0,1]$ and $v, w \in C$ with $\|v\| \leq 1$ and $\|w\| \leq 1$. Define

$$\alpha(t) = \frac{1}{2}\int_0^t \delta(s)ds, \qquad (0 \leq t \leq 2),$$

and $\alpha(t) = \alpha(2) + \frac{1}{2}\delta(2)(t-2)$ whenever $t > 2$. It is easy to verify that $\alpha \in \Gamma$, $\alpha(t) \leq \delta(t)$ for $0 \leq t \leq 2$ and

$$2\lambda(1-\lambda)\alpha\left(\|v-w\|\right) \leq 1 - \|\lambda v + (1-\lambda)w\|$$

for each $\lambda \in [0,1]$ and $v, w \in C$ with $\|v\| \leq 1$ and $\|w\| \leq 1$. Note that $\lambda(1-\lambda) \leq \min\{\lambda, (1-\lambda)\}$. Since Tx is compact, then $d(x,Tx) = \|x - u_x\|$ for some $u_x \in Tx$. By the *-nonexpansiveness of T there exist $u_z \in Tz$ with $d(z,Tz) = \|z - u_z\|$ and $u_y \in Ty$ with $d(y,Ty) = \|y - u_y\|$ such that $\|u_x - u_z\| \leq \|x - z\|$ and $\|u_z - u_y\| \leq \|z - y\|$ and therefore $\|u_x - u_y\| \leq \|x - y\|$ where $z = \lambda x + (1-\lambda)y$. The case that $x = y$ or $\lambda = 0, 1$ is trivial. Now, take

$$v = (u_y - u_z)/\lambda\|x-y\| \quad \text{and} \quad w = (u_z - u_x)/(1-\lambda)\|x-y\|.$$

Noting that the function $s \to \alpha(s)/s$ is increasing and $\lambda(1-\lambda)\|x-y\| \leq d/4$ for $d = diam(C)$, we conclude that

$$\frac{d}{2}\alpha\left(\frac{4}{d}\|\lambda u_x + (1-\lambda)u_y - u_z\|\right) \leq \|x-y\| - \|u_x - u_y\|.$$

Put $\beta(t) = (d/2)\alpha(4t/d)$. Taking $t = \max\{d(x,Tx), d(y,Ty)\}$ and considering

$$\|x-y\| - \|u_x - u_y\| \leq \|x - u_x\| + \|y - u_y\| \leq 2t,$$

we have

$$\beta\left(\|\lambda u_x + (1-\lambda)u_y - u_z\|\right) \leq 2t.$$

As β^{-1} is increasing, we obtain

$$\begin{aligned} d(z,Tz) &= \|z - u_z\| \leq \|\lambda u_x + (1-\lambda)u_y - u_z\| + \|z - \lambda u_x - (1-\lambda)u_y\| \\ &\leq \beta^{-1}(2t) + \lambda\|x - u_x\| + (1-\lambda)\|y - u_y\| \\ &\leq \beta^{-1}(2t) + t. \end{aligned}$$

If γ denotes the inverse function $t \to \beta^{-1}(2t) + t$, we get the assertion. $\square$

Lemma 3.2 *Suppose $\gamma \in \Gamma$. Then for each positive integer p there exists $\gamma_p \in \Gamma$ such that for any $T : C \to \mathcal{K}(C)$ of type (γ), any $x_1, \ldots, x_p$ in C and any positive real numbers $\lambda_1, \ldots, \lambda_p$ with $\lambda_1 + \cdots + \lambda_p = 1$,*

$$\gamma_p \left(d\left(\sum_{i=1}^{p} \lambda_i x_i, T\left(\sum_{i=1}^{p} \lambda_i x_i \right) \right) \right) \le \max_{1 \le i \le p} d(x_i, Tx_i). \tag{3.2}$$

Proof. We prove lemma by induction on p. We set $\gamma_2 = \gamma$ and suppose $\gamma_p \in \Gamma$ has been chosen. Define the function $\gamma_{p+1} : [0, \infty) \to [0, \infty)$ by $\gamma_{p+1}(t) = \gamma_2(\gamma_p(t))$. It is easy to see that $\gamma_{p+1} \in \Gamma$. We must verify (3.2) for $p+1$. Let $T : C \to \mathcal{K}(C)$ be of type (γ) and fix $x_1, \ldots, x_{p+1}$ in C and positive real numbers $\lambda_1, \ldots, \lambda_{p+1}$ with $\lambda_1 + \cdots + \lambda_{p+1} = 1$. The case $\lambda_{p+1} = 1$ is trivial. We take $u_i = (1 - \lambda_{p+1})x_i + \lambda_{p+1} x_{p+1}$ and $\mu_i = \lambda_i/(1 - \lambda_{p+1})$ for all $i \in \{1, 2, \ldots, p\}$. Then $\sum_{i=1}^{p} \mu_i = 1$ and $\sum_{i=1}^{p} \mu_i u_i = \sum_{i=1}^{p+1} \lambda_i x_i$. Since T is of type (γ), we have

$$\gamma_2(d(u_i, Tu_i)) \le \max\{d(x_i, Tx_i), d(x_{p+1}, Tx_{p+1})\}. \tag{3.3}$$

It follows from the induction hypothesis that

$$\gamma_p \left(d\left(\sum_{i=1}^{p} \mu_i u_i, T\left(\sum_{i=1}^{p} \mu_i u_i \right) \right) \right) \le \max_{1 \le i \le p} d(u_i, Tu_i).$$

This together with (3.3) and definition of γ_{p+1} implies that

$$\begin{aligned} \gamma_{p+1} \left(d\left(\sum_{i=1}^{p+1} \lambda_i x_i, T\left(\sum_{i=1}^{p+1} \lambda_i x_i \right) \right) \right) &= \gamma_{p+1} \left(d\left(\sum_{i=1}^{p} \mu_i u_i, T\left(\sum_{i=1}^{p} \mu_i u_i \right) \right) \right) \\ &\le \max_{1 \le i \le p} \gamma_2(d(u_i, Tu_i)) \\ &\le \max_{1 \le i \le p+1} d(x_i, Tx_i). \end{aligned}$$

This completes the proof. □

Theorem 3.3 *If X is uniformly convex then there exists a strictly increasing continuous convex function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ depending only on the diameter of C such that*

$$\gamma \left(d\left(\sum_{i=1}^{n} \lambda_i x_i, T\left(\sum_{i=1}^{n} \lambda_i x_i \right) \right) \right) \le \max_{1 \le i \le n} d(x_i, Tx_i)$$

holds for any *-nonexpansive mapping $T : C \to \mathcal{K}(C)$, any elements $x_1, \ldots, x_n$ in C and any numbers $\lambda_1, \ldots, \lambda_n \ge 0$ with $\lambda_1 + \cdots + \lambda_n = 1$. (Note that γ does not depend on T.)

Proof. Since T is *-nonexpansive, it follows from Lemma 3.1 that T is of type (α) for some $\alpha \in \Gamma$. Then we can determine $\gamma_p \in \Gamma$ for $p = 2, 3, \ldots$ by Lemma 3.2. Let $t > 0$ be arbitrary. Since every uniformly convex Banach space has the convex approximation property and C is bounded, there exists $q \ge 2$ such that

$$coM \subset co_q M + B_t \tag{3.4}$$

for all $M \subseteq C$. We put $\delta(t) = 2t + \gamma_q^{-1}(t)$ and suppose $x_1, \ldots, x_n \in C$ satisfy $\max_{1\le i\le n} d(x_i, Tx_i) \le t$. By taking $M = \{x_1, \ldots, x_n\}$ in (3.4), it follows that for each numbers $\lambda_1, \ldots, \lambda_n \ge 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ there exist numbers $\mu_1, \ldots, \mu_q \ge 0$ with $\mu_1 + \cdots + \mu_q = 1$ and $i_1, \ldots, i_q \in \{1, 2, \ldots, n\}$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{j=1}^{q} \mu_j x_{i_j} \right\| \le t.$$

Put $y = \sum_{i=1}^{n} \lambda_i x_i$ and $z = \sum_{j=1}^{q} \mu_j x_{i_j}$. Since Ty is compact, there exists $u_y \in Ty$ such that $d(y, Ty) = \|y - u_y\|$. It follows from *-nonexpansiveness of T that there exists $u_z \in Tz$ with $d(z, Tz) = \|z - u_z\|$ such that $\|u_y - u_z\| \le \|y - z\|$. Hence

$$\begin{aligned} d\left(\sum_{i=1}^{n} \lambda_i x_i, T\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right) &= \|y - u_y\| \le \|y - z\| + \|z - u_z\| + \|u_z - u_y\| \\ &\le 2t + d\left(\sum_{j=1}^{q} \mu_j x_{i_j}, T\left(\sum_{j=1}^{q} \mu_j x_{i_j}\right)\right) \\ &\le 2t + \gamma_q^{-1}(t) = \delta(t). \end{aligned}$$

We may suppose that $\delta : (0, \infty) \to (0, \infty)$ is strictly increasing. If not, we would pass to δ' where

$$\delta'(t) = \inf_{\eta \in [t, \infty)} \delta(\eta) + t.$$

Let $\beta(t) = \min\{t, \delta^{-1}(t)\}$. Thus $\beta : (0, \infty) \to (0, \infty)$ is strictly increasing and $\lim_{t\to 0^+} \beta(t) = 0$. Moreover, we may assume that β is continuous by passing to β' where

$$\beta'(t) = \frac{1}{t} \int_0^t \beta(s) ds, \qquad (t > 0).$$

Now, we define the function $\gamma : [0, \infty) \to [0, \infty)$ by $\gamma(0) = 0$ and

$$\gamma(t) = \frac{1}{d} \int_0^t \beta(s) ds, \qquad (t > 0),$$

where d is the diameter of C. It is easy to verify that $\gamma \in \Gamma$ and $\gamma(t) \le \beta(t) \le \delta^{-1}(t)$ for all $t \in [0, d]$. By taking $t = \max_{1\le i\le n} d(x_i, Tx_i)$ we obtain

$$\gamma\left(d\left(\sum_{i=1}^{n} \lambda_i x_i, T\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right)\right) \le t = \max_{1\le i\le n} d(x_i, Tx_i).$$

This completes the proof. $\square$

Theorem 3.4 *(**Demiclosed principle**) Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : C \to \mathcal{K}(C)$ be a *-nonexpansive mapping. Then $(I - T)$ is demiclosed at 0.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ and $x_n - y_n \to 0$, where $y_n \in Tx_n$. Thus for each $\epsilon > 0$ there exist positive integers n_0 and $n_1, \ldots, n_k \geq n_0$ and real numbers $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \cdots + \lambda_k = 1$ such that

$$\left\| x - \sum_{i=1}^{k} \lambda_i x_{n_i} \right\| \leq \epsilon \quad \text{and} \quad d\left(x_{n_i}, Tx_{n_i}\right) \leq \|x_{n_i} - y_{n_i}\| \leq \epsilon \quad \text{for all } i = 1, \ldots, k.$$

Suppose $z = \sum_{i=1}^{k} \lambda_i x_{n_i}$ and $d(x, Tx) = \|x - u_x\|$ for some $u_x \in Tx$. Since T is *-nonexpansive there exists $u_z \in Tz$ with $d(z, Tz) = \|z - u_z\|$ such that $\|u_z - u_x\| \leq \|z - x\|$. It follows from Theorem 3.3 that

$$\begin{aligned} d(x, Tx) &= \|x - u_x\| \leq \|x - z\| + \|z - u_z\| + \|u_z - u_x\| \\ &\leq 2\epsilon + \gamma^{-1}(\epsilon). \end{aligned}$$

Since Tx is closed it follows $x \in Tx$. □

Using Lemma 3.1 and Theorem 3.4 we obtain following result related to the structure of $F(T)$.

Corollary 3.5 *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : C \to \mathcal{K}(C)$ be a *-nonexpansive mapping. Then $F(T)$ is convex and closed.*

4. Strong convergence theorem

In this section, we study the iterative algorithm (1.2) for finding fixed points of multivalued *-nonexpansive mappings in a uniformly convex and smooth Banach space. We first prove that the sequence $\{x_n\}$ generated by (1.2) is well-defined. Then, we prove that $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from X onto $F(T)$.

Lemma 4.1 *Let X be a reflexive, strictly convex and smooth Banach space, $C \subseteq X$ and let $T : C \to \mathcal{K}(C)$ be a *-nonexpansive mapping. If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.2) is well-defined.*

Proof. It is easy to check that $C_n \cap D_n$ is closed and convex and $F(T) \subset C_n$ for each $n \in \mathbb{N}$. Moreover $D_1 = C$ and so $F(T) \subset C_1 \cap D_1$. Suppose $F(T) \subset C_k \cap D_k$ for $k \in \mathbb{N}$. Since $x_{k+1} = P_{C_k \cap D_k} x$, it follows from (2.2) that

$$\langle x_{k+1} - y, J(x - x_{k+1}) \rangle \geq 0,$$

for all $y \in C_k \cap D_k$ and so for all $y \in F(T)$, that is $F(T) \subset C_{k+1} \cap D_{k+1}$. By mathematical induction, we obtain that $F(T) \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well-defined. □

In order to prove our main result, the following lemma is needed.

Lemma 4.2 *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space X, let $T : C \to \mathcal{K}(C)$ be a *-nonexpansive mapping and let $\{x_n\}$ be the sequence generated by (1.2). Then,*

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0.$$

Proof. Since $x_{n+1} = P_{C_n \cap D_n} x$, then $x_{n+1} \in C_n$. Since $t_n > 0$, there exist $y_1, \dots, y_m \in C$ and $\lambda_1, \dots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ such that

$$\left\| x_{n+1} - \sum_{i=1}^{m} \lambda_i y_i \right\| < t_n, \tag{4.1}$$

and $d(y_i, Ty_i) \leq t_n d(x_n, Tx_n)$ for all $i \in \{1, \dots, m\}$. Put $q = P_{F(T)} x$ and $r_0 = 2 \sup_{n \geq 1} \|x_n - q\|$. Since $q \in Tq$ and T is *-nonexpansive, there exists $u_{x_n} \in Tx_n$ with $d(x_n, Tx_n) = \|x_n - u_{x_n}\|$ such that $\|q - u_{x_n}\| \leq \|q - x_n\|$. It follows from boundedness of C that

$$d(y_i, Ty_i) \leq t_n d(x_n, Tx_n) = t_n \|x_n - u_{x_n}\| \leq t_n (\|x_n - q\| + \|q - u_{x_n}\|) \leq r_0 t_n, \tag{4.2}$$

for all $i \in \{1, \dots, m\}$. Let $z = \sum_{i=1}^{m} \lambda_i y_i$ and $u_{x_{n+1}} \in Tx_{n+1}$ with $d(x_{n+1}, Tx_{n+1}) = \|x_{n+1} - u_{x_{n+1}}\|$. There exists $u_z \in Tz$ with $d(z, Tz) = \|z - u_z\|$ such that

$$\|u_z - u_{x_{n+1}}\| \leq \|z - x_{n+1}\|.$$

It follows from Theorem 3.3, (4.1) and (4.2) that

$$\begin{aligned} d\left(x_{n+1}, Tx_{n+1}\right) &= \|x_{n+1} - u_{x_{n+1}}\| \\ &\leq \left\| x_{n+1} - \sum_{i=1}^{m} \lambda_i y_i \right\| + d\left(\sum_{i=1}^{m} \lambda_i y_i, T\left(\sum_{i=1}^{m} \lambda_i y_i \right) \right) + \|u_z - u_{x_{n+1}}\| \\ &\leq 2t_n + \gamma^{-1} \left(\max_{1 \leq i \leq m} d(y_i, Ty_i) \right) \\ &\leq 2t_n + \gamma^{-1}(r_0 t_n). \end{aligned}$$

Since $\lim_{n\to\infty} t_n = 0$, it follows from the last inequality that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. This completes the proof. □

Theorem 4.3 *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space X, let $T : C \to \mathcal{K}(C)$ be a *-nonexpansive mapping and let $\{x_n\}$ be the sequence generated by (1.2). Then $\{x_n\}$ converges strongly to the element $P_{F(T)} x$ of $F(T)$.*

Proof. Put $q = P_{F(T)} x$. Since $F(T) \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n} x$, we have that

$$\|x - x_{n+1}\| \leq \|x - q\| \tag{4.3}$$

for all $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p$. It follows from Lemma 4.2 and Lemma 3.4 (demiclosedness of $(I - T)$) that $p \in F(T)$. From the weakly lower semicontinuity of norm and (4.3), we obtain

$$\|x - q\| \leq \|x - p\| \leq \liminf_{i\to\infty} \|x - x_{n_i}\| \leq \limsup_{i\to\infty} \|x - x_{n_i}\| \leq \|x - q\|.$$

This together with the uniqueness of $P_{F(T)} x$, implies $q = p$, and hence $x_{n_i} \rightharpoonup q$. Therefore, we obtain $x_n \rightharpoonup q$. By using the same argument as in proof above, we have

$$\lim_{n\to\infty} \|x - x_n\| = \|x - q\|.$$

Since X is uniformly convex, using the Kadec-Klee property, we have that $x - x_n \to x - q$. It follows that $x_n \to q$. This completes the proof. □

Since every single valued nonexpansive mapping can be viewed as a multivalued *-nonexpansive mapping, we obtain the strong convergence theorem of the iterative algorithm (1.1) due to Matsushita and Takahashi [12, Theorem 3.1].

Corollary 4.4 *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space X, let T be a nonexpansive self-mapping of C, and let $\{x_n\}$ be the sequence generated by (1.1). Then $\{x_n\}$ converges strongly to the element $P_{F(T)}x$ of $F(T)$.*

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