DEMICYCLOSED PRINCIPLE AND CONVERGENCE OF A HYBRID ALGORITHM FOR MULTIVALUED *
-NONEXPANSIVE MAPPINGS

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Abstract. A demiclosed principle is proved for multivalued *-nonexpansive mappings. Moreover,
strong convergence of an iterative algorithm is obtained for such mappings in a Banach space by
using metric projections. The results of this paper improve and extend the corresponding results for
single valued nonexpansive mappings which was studied by many authors.

Key Words and Phrases: Multivalued *-nonexpansive mapping, approximating fixed point, metric
projection, uniformly convex Banach space.

2010 Mathematics Subject Classification: 47H09, 47H10.

1. INTRODUCTION

The study of fixed points for multivalued contractions and nonexpansive mappings
by using the Hausdorff metric was initiated by Markin [11] (see also [13]). Later, an
interesting and rich fixed point theory for such maps was developed which has applica-
tions in control theory, convex optimization, differential inclusion and economics (see
[7] and references cited therein). Approximating fixed points of multivalued mappings
have been studied by many authors. In particular, Sastry and Babu [15] introduced
the analogs of Mann and Ishikawa iterates for multivalued mappings and proved con-
vergence theorems for nonexpansive mappings whose domain is a compact convex
subset of a Hilbert space. Recently, Panyanak [14] generalized results of Sastry and
Babu to uniformly convex Banach spaces. Xu [16] introduced the class of multival-
ued *-nonexpansive mappings and showed that *-nonexpansiveness is different from
nonexpansiveness for multivalued mappings.

Throughout this paper $X$ denotes a real Banach space and $C$ is a nonempty
bounded closed convex subset of $X$. Also, $CB(X)$ denotes the set of nonempty
closed bounded subsets of $X$ and $K(X)$ denotes the set of nonempty compact subsets
of $X$. It is clear that $K(X)$ is included in $CB(X)$. Let $H$ be the Hausdorff metric on
$CB(X)$, i.e.,

$$H(A,B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}.$$
where $A, B \in CB(X)$ and $d(x, B) = \inf\{\|x - y\| : y \in B\}$ is the distance from the point $x$ to the set $B$.

A multivalued mapping $T : C \rightarrow CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in C$. $T$ is said $\ast$-nonexpansive if for each $x, y \in C$ and $u_x \in Tx$ with $d(x, Tx) = \|x - u_x\|$ there exists $u_y \in Ty$ with $d(y, Ty) = \|y - u_y\|$ such that

$$\|u_x - u_y\| \leq \|x - y\|.$$

A point $x$ is called a fixed point of $T$ if $x \in Tx$. We denote the set of all fixed points of $T$ by $F(T)$.

The mapping $T$ is said to be demiclosed at $x$ in $X$ if $\{x_n\}$ is a sequence in $C$ which converges weakly to $x$ and $y_n \in Tx_n \rightharpoonup y$; then $y \in Tx$.

It is known that a multivalued nonexpansive mapping $T : C \rightarrow K(C)$ has a fixed point if $C$ is nonempty bounded closed convex subset of a uniformly convex Banach space $X$ [10]. Some recent existence theorems for multivalued mappings can be found in [1, 8].

**Remark 1.1** Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$ and $T : C \rightarrow K(C)$ be a multivalued $\ast$-nonexpansive mapping. Then $T$ has a fixed point. To see this define $S : C \rightarrow K(C)$ by

$$Sx = \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\}.$$ 

Definition of $\ast$-nonexpansiveness implies that $S$ is multivalued nonexpansive and by Lim’s fixed point theorem [10] $S$ and hence $T$ itself has a fixed point.

On the other hand, using the metric projection, Matsushita and Takahashi [12] introduced an iterative algorithm for single valued nonexpansive mappings in Banach spaces as follows.

Let $C$ be a nonempty subset of a reflexive, strictly convex and smooth Banach space $X$ and $T : C \rightarrow C$ be a mapping with $F(T) \neq \emptyset$. For a given $x_1 = x \in C$, compute the sequence $\{x_n\}$ by the iterative algorithm

$$\begin{cases} 
 C_n = \overline{co}\{z \in C : \|z - Tx\| \leq t_n\|x_n - Tx_n\|\}, \\
 D_n = \{z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0\}, \\
 x_{n+1} = P_{C_n \cap D_n} x, \quad n \geq 1, 
\end{cases}$$

(1.1)

where $\overline{co}D$ denotes the convex closure of the set $D$ and $\{t_n\}$ is a sequence in $(0, 1)$ with $\lim_{n \rightarrow \infty} t_n = 0$. They proved that if $T$ is nonexpansive and $X$ is uniformly convex and smooth, then $\{x_n\}$ generated by (1.1) converges strongly to a fixed point of $T$.

Motivated by these facts, we introduce an analog to Matsushita and Takahashi’s iterative algorithm for multivalued mappings. The algorithm is defined as follows.

Let $C$ and $X$ be as in (1.1), $T : C \rightarrow K(C)$ be a mapping with $F(T) \neq \emptyset$, $x_1 = x \in C$ and compute the sequence $\{x_n\}$ by the iterative algorithm

$$\begin{cases} 
 C_n = \overline{co}\{z \in C : d(z, Tx) \leq t_n d(x_n, Tx_n)\}, \\
 D_n = \{z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0\}, \\
 x_{n+1} = P_{C_n \cap D_n} x, \quad n \geq 1, 
\end{cases}$$

(1.2)
where \( \{t_n\} \) is a sequence in \((0, 1)\) with \( \lim_{n \to \infty} t_n = 0 \) and \( P_{C_n \cap D_n} \) is the metric projection from \( X \) onto \( C_n \cap D_n \).

For a single valued mapping \( T \), the algorithm (1.2) reduces to (1.1). Note that 
\[
d(z, \{Tz\}) = \|z - Tz\|.
\]

The purpose of this paper is to establish strong convergence theorem of the iterative algorithm (1.2) for multivalued \(^*-\)nonexpansive mappings in a uniformly convex and smooth Banach space.

2. Preliminaries

When \( \{x_n\} \) is a sequence in \( X \), we denote strong convergence of \( \{x_n\} \) to \( x \in X \) by \( x_n \to x \) and weak convergence by \( x_n \rightharpoonup x \). A Banach space \( X \) is said to have Kadec-Klee property if for every sequence \( \{x_n\} \) in \( X \), \( x_n \rightharpoonup x \) and \( \|x_n\| \to \|x\| \) imply that \( x_n \to x \). Every uniformly convex Banach space has the Kadec-Klee property [2].

Let \( B_r \) denote the open ball of radius \( r \) centered at 0. A Banach space \( X \) is said to have convex approximation property if for each \( \epsilon > 0 \) there exists a positive integer \( p \) such that for every bounded subset \( M \) of \( X \),
\[
\text{co} M \subset \text{co}_p M + B_{\epsilon}.
\]

Let \( X^* \) be the dual of \( X \). We denote the value of \( x^* \in X^* \) at \( x \in X \) by \( \langle x, x^* \rangle \). The normalized duality mapping \( J \) from \( X \) to \( 2^{X^*} \) is defined by
\[
J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}
\]
for all \( x \in X \). It is known that a Banach space \( X \) is smooth if and only if the normalized duality mapping \( J \) is single valued. Some properties of duality mapping have been given in [2, 6]. Let \( C \) be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space \( X \). Then for any \( x \in X \) there exists a unique point \( x_0 \in C \) such that
\[
\|x_0 - x\| = \min_{y \in C} \|y - x\|.
\]
The mapping \( P_C : X \to C \) defined by \( P_Cx = x_0 \) is called the metric projection from \( X \) onto \( C \). Let \( x \in X \) and \( u \in C \). Then, it is known that \( u = P_Cx \) if and only if
\[
\langle u - y, J(x - u) \rangle \geq 0 \tag{2.2}
\]
for all \( y \in C \) (see [2, 3]).

3. Demiclosed principle

In this section, we shall obtain demiclosed principle for \(^*-\)nonexpansive mappings. To proceed in this direction, we first prove some Zarantonello-type inequalities for multivalued \(^*-\)nonexpansive mappings. For classical and contemporary developments related to the Zarantonello’s inequality, see [5, 9] where further references are provided.
Consistently with [4], we denote by $\Gamma$ the set of strictly increasing, continuous and convex functions $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$. A multivalued mapping $T : C \to \mathcal{K}(C)$ is said to be of type $(\gamma)$ if $\gamma \in \Gamma$ and

$$
\gamma(d(x(1-\lambda)y, T(\lambda x + (1-\lambda)y))) \leq \max\{d(x, Tx), d(y, Ty)\}
$$

(3.1)

for all $x, y \in C$ and $\lambda \in [0, 1]$.

**Lemma 3.1** If $X$ is uniformly convex then there exists $\gamma \in \Gamma$ such that every $\gamma$-nonexpansive mapping $T : C \to \mathcal{K}(C)$ is of type $(\gamma)$. Moreover, $\gamma$ can be chosen to depend only on the diameter of $C$ and not on $T$.

**Proof.** Let $\delta$ be the modulus of uniform convexity of $X$. We know that $\delta : [0, 2] \to [0, 1]$ is continuous increasing with $\delta(0) = 0$, $\delta(t) > 0$ for $t > 0$ and

$$
2 \min\{\lambda, (1-\lambda)\} \delta(\|v-w\|) \leq 1 - \|\lambda v + (1-\lambda)w\|
$$

for each $\lambda \in [0, 1]$ and $v, w \in C$ with $\|v\| \leq 1$ and $\|w\| \leq 1$. Define

$$
\alpha(t) = \frac{1}{2} \int_0^t \delta(s)ds, \quad (0 \leq t \leq 2),
$$

and $\alpha(t) = \alpha(2) + \frac{1}{2} \delta(2)(t-2)$ whenever $t > 2$. It is easy to verify that $\alpha \in \Gamma$, $\alpha(t) \leq \delta(t)$ for $0 \leq t \leq 2$ and

$$
2\lambda(1-\lambda)\alpha(\|v-w\|) \leq 1 - \|\lambda v + (1-\lambda)w\|
$$

for each $\lambda \in [0, 1]$ and $v, w \in C$ with $\|v\| \leq 1$ and $\|w\| \leq 1$. Note that $\lambda(1-\lambda) \leq \min\{\lambda, (1-\lambda)\}$. Since $T$ is compact, then $d(x, Tx) = \|x-u_x\|$ for some $u_x \in Tx$. By the $\gamma$-nonexpansiveness of $T$ there exist $u_z \in Tz$ with $d(z, Tu_z) = \|z-u_z\|$ and $u_y \in Ty$ with $d(y, Ty) = \|y-u_y\|$ such that $\|u_x - u_z\| \leq \|z-x\|$ and $\|u_z - u_y\| \leq \|z-y\|$ and therefore $\|u_x - u_y\| \leq \|x-y\|$ where $z = \lambda x + (1-\lambda)y$. The case that $x = y$ or $\lambda = 0, 1$ is trivial. Now, take

$$
v = (u_y - u_z)/\lambda \|x-y\| \quad \text{and} \quad w = (u_z - u_x)/(1-\lambda) \|x-y\|.
$$

Noting that the function $s \to \alpha(s)/s$ is increasing and $\lambda(1-\lambda)\|x-y\| \leq d/4$ for $d = \text{diam}(C)$, we conclude that

$$
\frac{d}{2} \alpha\left(\frac{4}{d} \lambda u_x + (1-\lambda)u_y - u_z\right) \leq \|x-y\| - \|u_x - u_y\|.
$$

Put $\beta(t) = (d/2)\alpha(4t/d)$. Taking $t = \max\{d(x, Tx), d(y, Ty)\}$ and considering

$$
\|x-y\| - \|u_x - u_y\| \leq \|x-u_x\| + \|y-u_y\| \leq 2t,
$$

we have

$$
\beta(\|\lambda u_x + (1-\lambda)u_y - u_z\|) \leq 2t.
$$

As $\beta^{-1}$ is increasing, we obtain

$$
d(z, Tz) = \|z - u_z\| \leq \|\lambda u_x + (1-\lambda)u_y - u_z\| + \|z - \lambda u_x - (1-\lambda)u_y\|
\leq \beta^{-1}(2t) + \lambda \|x - u_x\| + (1-\lambda)\|y - u_y\|
\leq \beta^{-1}(2t) + t.
$$

If $\gamma$ denotes the inverse function $t \to \beta^{-1}(2t) + t$, we get the assertion. \qed
Lemma 3.2 Suppose \( \gamma \in \Gamma \). Then for each positive integer \( p \) there exists \( \gamma_p \in \Gamma \) such that for any \( T : C \to K(C) \) of type \( \gamma \), any \( x_1, \ldots, x_p \) in \( C \) and any positive real numbers \( \lambda_1, \ldots, \lambda_p \) with \( \lambda_1 + \cdots + \lambda_p = 1 \),

\[
\gamma_p \left( d \left( \sum_{i=1}^{p} \lambda_i x_i, T \left( \sum_{i=1}^{p} \lambda_i x_i \right) \right) \right) \leq \max_{1 \leq i \leq p} d(x_i, Tx_i). \tag{3.2}
\]

Proof. We prove lemma by induction on \( p \). We set \( \gamma_2 = \gamma \) and suppose \( \gamma_p \in \Gamma \) has been chosen. Define the function \( \gamma_{p+1} : [0, \infty) \to [0, \infty) \) by \( \gamma_{p+1}(t) = \gamma_2(\gamma_p(t)) \). It is easy to see that \( \gamma_{p+1} \in \Gamma \). We must verify (3.2) for \( p + 1 \). Let \( T : C \to K(C) \) be of type \( \gamma \) and fix \( x_1, \ldots, x_{p+1} \) in \( C \) and positive real numbers \( \lambda_1, \ldots, \lambda_{p+1} \) with \( \lambda_1 + \cdots + \lambda_{p+1} = 1 \). The case \( \lambda_{p+1} = 1 \) is trivial. We take \( u_i = (1-\lambda_{p+1}) x_i + \lambda_{p+1} x_{p+1} \) and \( \mu_i = \lambda_i/(1-\lambda_{p+1}) \) for all \( i \in \{1, 2, \ldots, p\} \). Then \( \sum_{i=1}^{p} \mu_i = 1 \) and \( \sum_{i=1}^{p} \mu_i u_i = \sum_{i=1}^{p} \lambda_i x_i \). Since \( T \) is of type \( \gamma \), we have

\[
\gamma_2(\gamma_p(u_i, T u_i)) \leq \max \{d(x_i, Tx_i), d(x_{p+1}, T x_{p+1})\}. \tag{3.3}
\]

It follows from the induction hypothesis that

\[
\gamma_p \left( d \left( \sum_{i=1}^{p} \mu_i u_i, T \left( \sum_{i=1}^{p} \mu_i u_i \right) \right) \right) \leq \max_{1 \leq i \leq p} d(u_i, Tu_i).
\]

This together with (3.3) and definition of \( \gamma_{p+1} \) implies that

\[
\gamma_{p+1} \left( d \left( \sum_{i=1}^{p+1} \lambda_i x_i, T \left( \sum_{i=1}^{p+1} \lambda_i x_i \right) \right) \right) = \gamma_{p+1} \left( d \left( \sum_{i=1}^{p} \mu_i u_i, T \left( \sum_{i=1}^{p} \mu_i u_i \right) \right) \right) \leq \max_{1 \leq i \leq p} \gamma_2(d(u_i, Tu_i)) \leq \max_{1 \leq i \leq p+1} d(x_i, Tx_i).
\]

This completes the proof. \( \square \)

Theorem 3.3 If \( X \) is uniformly convex then there exists a strictly increasing continuous convex function \( \gamma : [0, \infty) \to [0, \infty) \) with \( \gamma(0) = 0 \) depending only on the diameter of \( C \) such that

\[
\gamma \left( d \left( \sum_{i=1}^{n} \lambda_i x_i, T \left( \sum_{i=1}^{n} \lambda_i x_i \right) \right) \right) \leq \max_{1 \leq i \leq n} d(x_i, Tx_i)
\]

holds for any \( \ast \)-nonexpansive mapping \( T : C \to K(C) \), any elements \( x_1, \ldots, x_n \) in \( C \) and any numbers \( \lambda_1, \ldots, \lambda_n \geq 0 \) with \( \lambda_1 + \cdots + \lambda_n = 1 \). (Note that \( \gamma \) does not depend on \( T \).)

Proof. Since \( T \) is \( \ast \)-nonexpansive, it follows from Lemma 3.1 that \( T \) is of type \( \alpha \) for some \( \alpha \in \Gamma \). Then we can determine \( \gamma_p \in \Gamma \) for \( p = 2, 3, \ldots \) by Lemma 3.2.

Let \( t > 0 \) be arbitrary. Since every uniformly convex Banach space has the convex approximation property and \( C \) is bounded, there exists \( q \geq 2 \) such that

\[
coM \subset co_q M + B_t \tag{3.4}
\]
for all $M \subseteq C$. We put $\delta(t) = 2t + \gamma^{-1}_q(t)$ and suppose $x_1, \ldots, x_n \in C$ satisfy $\max_{1 \leq i \leq n} d(x_i, T x_i) \leq t$. By taking $M = \{x_1, \ldots, x_n\}$ in (3.4), it follows that for each numbers $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ there exist numbers $\mu_1, \ldots, \mu_q \geq 0$ with $\mu_1 + \cdots + \mu_q = 1$ and $i_1, \ldots, i_q \in \{1, 2, \ldots, n\}$ such that

$$
\left\| \sum_{i=1}^n \lambda_i x_i - \sum_{j=1}^q \mu_j x_{i_j} \right\| \leq t.
$$

Put $y = \sum_{i=1}^n \lambda_i x_i$ and $z = \sum_{j=1}^q \mu_j x_{i_j}$. Since $Ty$ is compact, there exists $u_y \in Ty$ such that $d(y, Ty) = \|y - u_y\|$. It follows from $^*$-nonexpansiveness of $T$ that there exists $u_z \in Tz$ with $d(z, Tz) = \|z - u_z\|$ such that $\|u_y - u_z\| \leq \|y - z\|$. Hence

$$
d \left( \sum_{i=1}^n \lambda_i x_i, T \left( \sum_{i=1}^n \lambda_i x_i \right) \right) = \|y - u_y\| \leq \|y - z\| + \|z - u_z\| + \|u_z - u_y\|
$$

$$
\leq 2t + d \left( \sum_{j=1}^q \mu_j x_{i_j}, T \left( \sum_{j=1}^q \mu_j x_{i_j} \right) \right)
$$

$$
\leq 2t + \gamma^{-1}_q(t) = \delta(t).
$$

We may suppose that $\delta : (0, \infty) \to (0, \infty)$ is strictly increasing. If not, we would pass to $\delta'$ where

$$
\delta'(t) = \inf_{\eta \in [t, \infty)} \delta(\eta) + t.
$$

Let $\beta(t) = \min\{t, \delta^{-1}(t)\}$. Thus $\beta : (0, \infty) \to (0, \infty)$ is strictly increasing and $\lim_{t \to 0^+} \beta(t) = 0$. Moreover, we may assume that $\beta$ is continuous by passing to $\beta'$ where

$$
\beta'(t) = \frac{1}{t} \int_0^t \beta(s) ds, \quad (t > 0).
$$

Now, we define the function $\gamma : [0, \infty) \to [0, \infty)$ by $\gamma(0) = 0$ and

$$
\gamma(t) = \frac{1}{d} \int_0^t \beta(s) ds, \quad (t > 0),
$$

where $d$ is the diameter of $C$. It is easy to verify that $\gamma \in \Gamma$ and $\gamma(t) \leq \beta(t) \leq \delta^{-1}(t)$ for all $t \in [0, d]$. By taking $t = \max_{1 \leq i \leq n} d(x_i, T x_i)$ we obtain

$$
\gamma \left( d \left( \sum_{i=1}^n \lambda_i x_i, T \left( \sum_{i=1}^n \lambda_i x_i \right) \right) \right) \leq t = \max_{1 \leq i \leq n} d(x_i, T x_i).
$$

This completes the proof. \hfill \Box

**Theorem 3.4 (Demiclosed principle)** Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$ and $T : C \to K(C)$ be a $^*$-nonexpansive mapping. Then $(I - T)$ is demiclosed at 0.
Proof. Let \( \{x_n\} \) be a sequence in \( C \) such that \( x_n \to x \) and \( x_n - y_n \to 0 \), where \( y_n \in Tx_n \). Thus for each \( \epsilon > 0 \) there exist positive integers \( n_0 \) and \( n_1, \ldots, n_k \geq n_0 \) and real numbers \( \lambda_1, \ldots, \lambda_k \geq 0 \) with \( \lambda_1 + \cdots + \lambda_k = 1 \) such that
\[
\left\| x - \sum_{i=1}^{k} \lambda_i x_{n_i} \right\| \leq \epsilon \quad \text{and} \quad d(x_{n_i}, Tx_{n_i}) \leq \|x_{n_i} - y_{n_i}\| \leq \epsilon \quad \text{for all } i = 1, \ldots, k.
\]

Suppose \( z = \sum_{i=1}^{k} \lambda_i x_{n_i} \) and \( d(x, Tx) = \|x - u_x\| \) for some \( u_x \in Tx \). Since \( T \) is nonexpansive there exists \( u_z \in Tx \) with \( d(z, Tz) = \|z - u_z\| \) such that \( \|u_z - u_x\| \leq \|z - x\| \). It follows from Theorem 3.3 that
\[
d(x, Tx) = \|x - u_x\| \leq \|x - z\| + \|z - u_z\| + \|u_z - u_x\| \\
\leq 2\epsilon + \gamma^{-1}(\epsilon).
\]
Since \( Tx \) is closed it follows \( x \in Tx \). \( \square \)

Using Lemma 3.1 and Theorem 3.4 we obtain following result related to the structure of \( F(T) \).

Corollary 3.5 Let \( C \) be a nonempty bounded closed convex subset of a uniformly convex Banach space \( X \) and \( T : C \to K(C) \) be a \(^*-\)nonexpansive mapping. Then \( F(T) \) is convex and closed.

4. Strong convergence theorem

In this section, we study the iterative algorithm (1.2) for finding fixed points of multivalued \(^*-\)nonexpansive mappings in a uniformly convex and smooth Banach space. We first prove that the sequence \( \{x_n\} \) generated by (1.2) is well-defined. Then, we prove that \( \{x_n\} \) converges strongly to \( P_{F(T)}x \), where \( P_{F(T)} \) is the metric projection from \( X \) onto \( F(T) \).

Lemma 4.1 Let \( X \) be a reflexive, strictly convex and smooth Banach space, \( C \subseteq X \) and let \( T : C \to K(C) \) be a \(^*-\)nonexpansive mapping. If \( F(T) \neq \emptyset \), then the sequence \( \{x_n\} \) generated by (1.2) is well-defined.

Proof. It is easy to check that \( C_n \cap D_n \) is closed and convex and \( F(T) \subseteq C_n \) for each \( n \in \mathbb{N} \). Moreover \( D_1 = C \) and so \( F(T) \subseteq C_1 \cap D_1 \). Suppose \( F(T) \subseteq C_k \cap D_k \) for \( k \in \mathbb{N} \). Since \( x_{k+1} = P_{C_k \cap D_k} x \), it follows from (2.2) that
\[
\langle x_{k+1} - y, J(x - x_{k+1}) \rangle \geq 0,
\]
for all \( y \in C_k \cap D_k \) and so for all \( y \in F(T) \), that is \( F(T) \subseteq C_{k+1} \cap D_{k+1} \). By mathematical induction, we obtain that \( F(T) \subseteq C_n \cap D_n \) for all \( n \in \mathbb{N} \). Therefore, \( \{x_n\} \) is well-defined. \( \square \)

In order to prove our main result, the following lemma is needed.

Lemma 4.2 Let \( C \) be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space \( X \), let \( T : C \to K(C) \) be a \(^*-\)nonexpansive mapping and let \( \{x_n\} \) be the sequence generated by (1.2). Then,
\[
\lim_{n \to \infty} d(x_n, Tx_n) = 0.
\]
Proof. Since \( x_{n+1} = P_{C_n \cap D_n} x \), then \( x_{n+1} \in C_n \). Since \( t_n > 0 \), there exist \( y_1, \ldots, y_m \in C \) and \( \lambda_1, \ldots, \lambda_m \geq 0 \) with \( \lambda_1 + \cdots + \lambda_m = 1 \) such that
\[
\left\| x_{n+1} - \sum_{i=1}^{m} \lambda_i y_i \right\| < t_n, \tag{4.1}
\]
and \( d(y_i, T y_i) \leq t_n d(x_n, T x_n) \) for all \( i \in \{1, \ldots, m\} \). Put \( q = P_{F(T)} x \) and \( r_0 = 2 \sup_{n \geq 1} \| x_n - q \| \). Since \( q \in T q \) and \( T \) is \(^*\)-nonexpansive, there exists \( u_{x_n} \in T x_n \) with \( d(x_n, T x_n) = \| x_n - u_{x_n} \| \) such that \( \| q - u_{x_n} \| \leq \| q - x_n \| \). It follows from boundedness of \( C \) that
\[
d(y_i, T y_i) \leq t_n d(x_n, T x_n) = t_n \| x_n - u_{x_n} \| \leq t_n (\| x_n - q \| + \| q - u_{x_n} \|) \leq r_0 t_n, \tag{4.2}
\]
for all \( i \in \{1, \ldots, m\} \). Let \( z = \sum_{i=1}^{m} \lambda_i y_i \) and \( u_{x_{n+1}} \in T x_{n+1} \) with \( d(x_{n+1}, T x_{n+1}) = \| x_{n+1} - u_{x_{n+1}} \| \). There exists \( u_z \in T z \) with \( d(z, T z) = \| z - u_z \| \) such that
\[
\| u_z - u_{x_{n+1}} \| \leq \| z - x_{n+1} \|.
\]
It follows from Theorem 3.3, (4.1) and (4.2) that
\[
d(x_{n+1}, T x_{n+1}) = \| x_{n+1} - u_{x_{n+1}} \|
\leq \left\| x_{n+1} - \sum_{i=1}^{m} \lambda_i y_i \right\| + d \left( \sum_{i=1}^{m} \lambda_i y_i, T \left( \sum_{i=1}^{m} \lambda_i y_i \right) \right) + \| u_z - u_{x_{n+1}} \|
\leq 2t_n + \gamma^{-1} \left( \max_{1 \leq i \leq m} d(y_i, T y_i) \right)
\leq 2t_n + \gamma^{-1} (r_0 t_n).
\]
Since \( \lim_{n \to \infty} t_n = 0 \), it follows from the last inequality that \( \lim_{n \to \infty} d(x_n, T x_n) = 0 \). This completes the proof.

**Theorem 4.3** Let \( C \) be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space \( X \), let \( T : C \to K(C) \) be a \(^*\)-nonexpansive mapping and let \( \{x_n\} \) be the sequence generated by (1.2). Then \( \{x_n\} \) converges strongly to the element \( P_{F(T)} x \) of \( F(T) \).

**Proof.** Put \( q = P_{F(T)} x \). Since \( F(T) \subset C \cap D_n \) and \( x_{n+1} = P_{C_n \cap D_n} x \), we have that
\[
\| x - x_{n+1} \| \leq \| x - q \| \tag{4.3}
\]
for all \( n \in \mathbb{N} \). Since \( \{x_n\} \) is bounded, there exists \( \{x_n\} \subset \{x_n\} \) such that \( x_{n_i} \to p \). It follows from Lemma 4.2 and Lemma 3.4 (demiclosedness of \((I - T)\)) that \( p \in F(T) \).

From the weakly lower semicontinuity of norm and (4.3), we obtain
\[
\| x - q \| \leq \| x - p \| \leq \liminf_{i \to \infty} \| x - x_{n_i} \| \leq \limsup_{i \to \infty} \| x - x_{n_i} \| \leq \| x - q \|.
\]
This together with the uniqueness of \( P_{F(T)} x \), implies \( q = p \), and hence \( x_{n_i} \to q \).

Therefore, we obtain \( x_n \to q \). By using the same argument as in proof above, we have
\[
\lim_{n \to \infty} \| x_n - q \| = \| x - q \|.
\]
Since \( X \) is uniformly convex, using the Kadec-Klee property, we have that \( x - x_n \to x - q \). It follows that \( x_n \to q \). This completes the proof. \( \square \)
Since every single valued nonexpansive mapping can be viewed as a multivalued $^\ast$-nonexpansive mapping, we obtain the strong convergence theorem of the iterative algorithm (1.1) due to Matsushita and Takahashi [12, Theorem 3.1].

**Corollary 4.4** Let $C$ be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space $X$, let $T$ be a nonexpansive self-mapping of $C$, and let $\{x_n\}$ be the sequence generated by (1.1). Then $\{x_n\}$ converges strongly to the element $P_{F(T)}x$ of $F(T)$.

**References**


Received: May 20, 2011; Accepted: March 29, 2012.