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ITERATES OF BERNSTEIN TYPE OPERATORS ON A SQUARE WITH ONE CURVED SIDE VIA CONTRACTION PRINCIPLE

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Abstract. Given a function defined on a square with one curved side, we consider some Bernstein-type operators as well as their product and Boolean sum. Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of these operators.
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1. Weakly Picard operators

We recall some results regarding weakly Picard operators that will be used in the sequel (see, e.g., [15]).

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by

$F_A := \{x \in X \mid A(x) = x\}$-the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y,\ Y \neq \emptyset\}$-the family of the nonempty invariant subset of A

$A^0 := 1_X,\ A^1 := A,\ ...,\ A^{n+1} := A \circ A^n,\quad n \in \mathbb{N}.$

Definition 1.1. *The operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*
(i) $F_A = \{x^*\}$;
(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 1.2. *The operator A is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.*

Definition 1.3. *If A is a weakly Picard operator then we consider the operator A^∞, $A^\infty : X \to X$, defined by*

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Theorem 1.4. [15] *An operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A)$, $\forall \lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,* $\forall \lambda \in \Lambda$.

2. Bernstein type operators on a square with one curved side

In [4] there are introduced some Bernstein-type operators on a square with one curved side. In [3], [5] and [6] there have been introduced interpolation and Berstein-type operators on triangles with some curved sides.

Given $h > 0$, let D_h be the square with one curved side having the vertices $V_1 = (0,0)$, $V_2 = (h,0)$, $V_3 = (h,h)$ and $V_4 = (0,h)$, three straight sides Γ_1, Γ_2, along the coordinate axes and Γ_3 parallel to axis Ox, and the curved side Γ_4 which is defined by the function g, such that $g(h) = g(0) = h$ (see Figure 1).

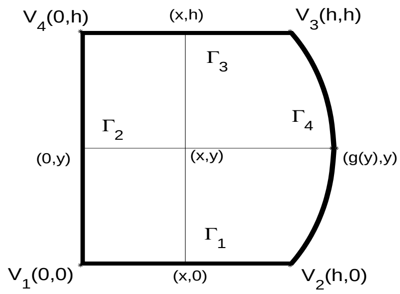

Figure 1. The square D_h.

Let F be a real-valued function defined on D_h and $(0,y)$, $(g(y),y)$, respectively, $(x,0)$, (x,h) be the points in which the parallel lines to the coordinate axes, passing through the point $(x,y) \in D_h$, intersect the sides Γ_2, Γ_4, respectively Γ_1 and Γ_3. We consider the uniform partitions of the intervals $[0, g(y)]$ and $[0,h]$, $y \in [0,h]$, $\Delta_m^x = \left\{ \frac{i}{m} g(y) \middle|\ i = \overline{0,m} \right\}$ and $\Delta_n^y = \left\{ \frac{j}{n} h \middle|\ j = \overline{0,n} \right\}$ and the Bernstein-type operators B_m^x and B_n^y defined by

$$\left(B_m^x F\right)(x,y) = \sum_{i=0}^{m} p_{m,i}(x,y)\, F\left(\tfrac{i}{m} g(y), y\right), \tag{2.1}$$

with

$$p_{m,i}(x,y) = \binom{m}{i} \left[\tfrac{x}{g(y)}\right]^i \left[1 - \tfrac{x}{g(y)}\right]^{m-i},$$

respectively,

$$\left(B_n^y F\right)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y)\, F\left(x, \tfrac{j}{n} h\right) \tag{2.2}$$

with

$$q_{n,j}\left(x,y\right)=\binom{n}{j}\left(\tfrac{y}{h}\right)^{j}\left(1-\tfrac{y}{h}\right)^{n-j}.$$

Theorem 2.1. [4] *If F is a real-valued function defined on D_h then we have*

(1) $B_m^x F = F$ *on* $\Gamma_2 \cup \Gamma_4$;
 $B_n^y F = F$ *on* $\Gamma_1 \cup \Gamma_3$,

(2) $\left(B_m^x e_{ij}\right)(x,y) = x^i y^j, \quad i=0,1;\ j\in\mathbb{N};$
 $\left(B_n^y e_{ij}\right)(x,y) = x^i y^j, \quad i\in\mathbb{N};\ j=0,1.$

Remark 2.2. *The interpolation properties of $B_m^x F$ and $B_n^y F$ are illustrated in Figures 2 and 3. The bold sides indicate the interpolation sets.*

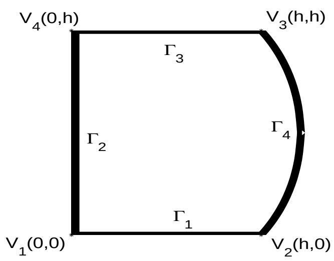

Figure 2. Interpolation domain for $B_m^x F$.

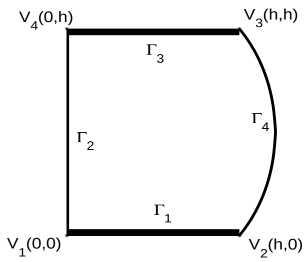

Figure 3. Interpolation domain for $B_y^n F$.

Let $P_{mn} = B_m^x B_n^y$, respectively, $Q_{nm} = B_n^y B_m^x$ be the products of the operators B_m^x and B_n^y. We have

$$\left(P_{mn}F\right)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}\left(x,y\right)q_{n,j}\left(i\tfrac{g(y)}{m},y\right)F\Big(i\tfrac{g(y)}{m},j\tfrac{h}{n}\Big), \tag{2.3}$$

respectively,

$$\left(Q_{nm}F\right)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}\left(x,j\tfrac{h}{n}\right)q_{n,j}\left(x,y\right)F\Big(\tfrac{i}{m}g(j\tfrac{h}{n}),j\tfrac{h}{n}\Big). \tag{2.4}$$

Theorem 2.3. [4] *If F is a real-valued function defined on D_h then:*

(1) $(P_{mn}F)(V_i) = F(V_i), \quad i=1,...,4;$
 $(Q_{nm}F)(V_i) = F(V_i), \quad i=1,...,4.$

(2) $\left(P_{mn}e_{ij}\right)(x,y) = x^i y^j, \quad i=0,1;\ j=0,1;$
 $\left(Q_{nm}e_{ij}\right)(x,y) = x^i y^j, \quad i=0,1;\ j=0,1.$

We consider the Boolean sums of the operators B_m^x and B_n^y, i.e.,

$$S_{mn} := B_m^x \oplus B_n^y = B_m^x + B_n^y - B_m^x B_n^y, \tag{2.5}$$

respectively,

$$T_{nm} := B_n^y \oplus B_m^x = B_n^y + B_m^x - B_n^y B_m^x. \tag{2.6}$$

3. Iterates of Bernstein type operators

Let F be a real-valued function defined on D_h, $h \in \mathbb{R}_+$.

Using the weakly Picard operators technique and the contraction principle, we obtain the following results regarding the convergence of the iterates of the Bernstein-type operators (2.1) and (2.2) and of their product and Boolean sum operators (2.3), (2.4), (2.5) and (2.6). The same approach for some other linear and positive operators lead to similar results in [1], [2], [16]-[18].

The limit behavior for the iterates of some classes of positive linear operators were also studied, for example, in [7]-[14].

Theorem 3.1. *The operators B_m^x and B_n^y are weakly Picard operators and*

$$\left(B_m^{x,\infty}F\right)(x,y) = F\left(0,y\right) + \frac{F(g(y),y) - F(0,y)}{g(y)}x, \tag{3.1}$$

$$\left(B_n^{y,\infty}F\right)(x,y) = F\left(x,0\right) + \frac{F(x,h) - F(x,0)}{h}y. \tag{3.2}$$

Proof. Taking into account the interpolation properties of B_m^x and B_n^y (from Theorem 2.1), let be

$$X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}} = \{F \in C(D_h) \mid F(0,y) = \varphi|_{\Gamma_2},\ F(g(y),y) = \varphi|_{\Gamma_4}\}, \quad \text{for } y \in [0,h],$$
$$X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}} = \{F \in C(D_h) \mid F(x,0) = \psi|_{\Gamma_1},\ F(x,h) = \psi|_{\Gamma_3}\}, \quad \text{for } x \in [0,h],$$

and denote by

$$F^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}(x,y) := \varphi|_{\Gamma_2} + \frac{\varphi|_{\Gamma_4} - \varphi|_{\Gamma_2}}{g(y)}x,$$
$$F^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}(x,y) := \psi|_{\Gamma_1} + \frac{\psi|_{\Gamma_3} - \psi|_{\Gamma_1}}{h}y,$$

with $\varphi, \psi \in C(D_h)$.

We have the following properties:

(i) $X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$ and $X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$ are closed subsets of $C(D_h)$;

(ii) $X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$ is an invariant subset of B_m^x and $X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$ is an invariant subset of B_n^y, for $\varphi, \psi \in C(D_h)$ and $n, m \in \mathbb{N}^*$;

(iii) $C(D_h) = \underset{\varphi\in C(D_h)}{\cup} X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$ and $C(D_h) = \underset{\psi\in C(D_h)}{\cup} X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$ are partitions of $C(D_h)$;

(iv) $F^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}} \in X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}} \cap F_{B_m^x}$ and $F^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}} \in X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}} \cap F_{B_n^y}$, where $F_{B_m^x}$ and $F_{B_n^y}$ denote the fixed points sets of B_m^x and B_n^y.

The statements (i) and (iii) are obvious.

(ii) By linearity of Bernstein operators and Theorem 2.1, it follows that $\forall F^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}} \in X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$ and $\forall F^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}} \in X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$ we have

$$B^x_m F^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}(x,y) = F^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}(x,y),$$
$$B^y_n F^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}(x,y) = F^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}(x,y).$$

So, $X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$ and $X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$ are invariant subsets of B^x_m and, respectively, of B^y_n, for $\varphi,\psi \in C(D_h)$ and $n,m \in \mathbb{N}^*$;

(iv) We prove that

$$B^x_m|_{X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}} : X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}} \to X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}} \text{ and } B^y_n|_{X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}} : X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}} \to X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$$

are contractions for $\varphi,\psi \in C(D_h)$ and $n,m \in \mathbb{N}^*$.

Let $F, G \in X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$. From (2.1) we have

$$\begin{aligned}
&|B^x_m(F)(x,y) - B^x_m(G)(x,y)| = |B^x_m(F-G)(x,y)| \le \\
&\le \left|1 - \left(1 - \frac{x}{g(y)}\right)^m - \left(\frac{x}{g(y)}\right)^m\right| \cdot \|F-G\|_\infty \le \\
&\le \left(1 - \frac{1}{2^{m-1}}\right)\|F-G\|_\infty ,
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm. So,

$$\|B^x_m(F)(x,y) - B^x_m(G)(x,y)\|_\infty \le \left(1 - \frac{1}{2^{m-1}}\right)\|F-G\|_\infty , \forall F, G \in X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}},$$

i.e., $B^x_m|_{X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}}$ is a contraction for $\varphi \in C(D_h)$.

Analogously we have

$$\|B^y_n(F)(x,y) - B^y_n(G)(x,y)\|_\infty \le \left(1 - \frac{1}{2^{n-1}}\right)\|F-G\|_\infty , \ \forall F, G \in X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}},$$

i.e., $B^y_n|_{X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}}$ is a contraction for $\psi \in C(D_h)$.

On the other hand, $\varphi|_{\Gamma_2} + \frac{\varphi|_{\Gamma_4} - \varphi|_{\Gamma_2}}{g(y)}(\cdot) \in X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$, $\psi|_{\Gamma_1} + \frac{\psi|_{\Gamma_3} - \psi|_{\Gamma_1}}{h}(\cdot) \in X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$ are fixed points of B^x_m and B^y_n, i.e.,

$$B^x_m(\varphi|_{\Gamma_2} + \frac{\varphi|_{\Gamma_4} - \varphi|_{\Gamma_2}}{g(y)}(\cdot)) = \varphi|_{\Gamma_2} + \frac{\varphi|_{\Gamma_4} - \varphi|_{\Gamma_2}}{g(y)}(\cdot),$$
$$B^y_n(\psi|_{\Gamma_1} + \frac{\psi|_{\Gamma_3} - \psi|_{\Gamma_1}}{h}(\cdot)) = \psi|_{\Gamma_1} + \frac{\psi|_{\Gamma_3} - \psi|_{\Gamma_1}}{h}(\cdot).$$

From the contraction principle, $F^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}(x,y) := \varphi|_{\Gamma_2} + \frac{\varphi|_{\Gamma_4} - \varphi|_{\Gamma_2}}{g(y)} x$ is the unique fixed point of B^x_m in $X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}$ and $B^x_m|_{X^{(1)}_{\varphi|_{\Gamma_2},\varphi|_{\Gamma_4}}}$ is a Picard operator, with

$$(B^{x,\infty}_m F)(x,y) = F(0,y) + \frac{F(g(y),y) - F(0,y)}{g(y)} x,$$

and, similarly, $F^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}(x,y) := \psi|_{\Gamma_1} + \frac{\psi|_{\Gamma_3} - \psi|_{\Gamma_1}}{h} y$ is the unique fixed point of B^y_n in $X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}$ and $B^y_n|_{X^{(2)}_{\psi|_{\Gamma_1},\psi|_{\Gamma_3}}}$ is a Picard operator, with

$$(B^{y,\infty}_n F)(x,y) = F(x,0) + \frac{F(x,h) - F(x,0)}{h} y.$$

Consequently, taking into account (ii), by Theorem 1.4 it follows that the operators B^x_m and B^y_n are weakly Picard operators. □

Theorem 3.2. *The operators P_{mn} and Q_{nm} are weakly Picard operators and*

$$(P^{\infty}_{mn} F)(x,y) = F(0,0) + \frac{F(h,0) - F(0,0)}{g(y)} x + \frac{F(0,h) - F(0,0)}{h} y + \frac{F(0,0) - F(0,h) - F(h,0) + F(h,h)}{g(y)h} xy, \tag{3.3}$$

$$(Q^{\infty}_{nm} F)(x,y) = F(0,0) + \frac{F(h,0) - F(0,0)}{g(y)} x + \frac{F(0,h) - F(0,0)}{h} y + \frac{F(0,0) - F(0,h) - F(h,0) + F(h,h)}{g(y)h} xy. \tag{3.4}$$

Proof. Let

$$X_{\alpha,\beta,\gamma,\delta} = \{F \in C(D_h) \mid F(0,0) = \alpha,\ F(0,h) = \beta, F(h,h) = \gamma, F(h,0) = \delta\}$$

and denote by

$$F_{\alpha,\beta,\gamma,\delta}(x,y) := \alpha + \frac{\delta - \alpha}{g(y)} x + \frac{\beta - \alpha}{h} y + \frac{\alpha - \beta - \delta + \gamma}{g(y)h} xy$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

We remark that

(i) $X_{\alpha,\beta,\gamma,\delta}$ is closed subset of $C(D_h)$;
(ii) $X_{\alpha,\beta,\gamma,\delta}$ is an invariant subset of P_{mn} and Q_{nm}, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $n, m \in \mathbb{N}^*$;
(iii) $C(D_h) = \bigcup_{\alpha,\beta,\gamma,\delta} X_{\alpha,\beta,\gamma,\delta}$ is a partition of $C(D_h)$;
(iv) $F_{\alpha,\beta,\gamma,\delta} \in X_{\alpha,\beta,\gamma,\delta} \cap F_{P_{mn}}$ and $F_{\alpha,\beta,\gamma,\delta} \in X_{\alpha,\beta,\gamma,\delta} \cap F_{Q_{nm}}$, where $F_{P_{mn}}$ and $F_{Q_{nm}}$ denote the fixed points sets of P_{mn} and Q_{nm}.

The statements (i) and (iii) are obvious.

(ii) Similarly with the proof of Theorem 3.1, by linearity of Bernstein operators and Theorem 2.3, it follows that $X_{\alpha,\beta,\gamma,\delta}$ is an invariant subset of P_{mn} and, respectively, of Q_{nm}, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $n, m \in \mathbb{N}^*$;

(iv) We prove that

$$P_{mn}|_{X_{\alpha,\beta,\gamma,\delta}} : X_{\alpha,\beta,\gamma,\delta} \to X_{\alpha,\beta,\gamma,\delta} \text{ and } Q_{nm}|_{X_{\alpha,\beta,\gamma,\delta}} : X_{\alpha,\beta,\gamma,\delta} \to X_{\alpha,\beta,\gamma,\delta}$$

are contractions for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $n, m \in \mathbb{N}^*$. Let $F, G \in X_{\alpha,\beta,\gamma,\delta}$. From [2, Lemma 8] it follows that

$$\begin{aligned} &|P_{mn}(F)(x,y) - P_{mn}(G)(x,y)| = |P_{mn}(F-G)(x,y)| \leq \\ &\leq \left(1 - \frac{1}{2^{m+n-2}}\right) \|F-G\|_\infty . \end{aligned}$$

So,

$$\|P_{mn}(F)(x,y) - P_{mn}(G)(x,y)\|_\infty \leq \left(1 - \frac{1}{2^{m+n-2}}\right) \|F-G\|_\infty , \forall F, G \in X_{\alpha,\beta,\gamma,\delta},$$

i.e., $P_{mn}|_{X_{\alpha,\beta,\gamma,\delta}}$ is a contraction for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Analogously, we have

$$\|Q_{nm}(F)(x,y) - Q_{nm}(G)(x,y)\|_\infty \leq \left(1 - \frac{1}{2^{m+n-2}}\right) \|F-G\|_\infty , \ \forall F, G \in X_{\alpha,\beta,\gamma,\delta},$$

i.e., $Q_{nm}|_{X_{\alpha,\beta,\gamma,\delta}}$ is a contraction for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

We have that

$$F_{\alpha,\beta,\gamma,\delta}(x,y) := \alpha + \frac{\delta-\alpha}{g(y)}x + \frac{\beta-\alpha}{h}y + \frac{\alpha-\beta-\delta+\gamma}{g(y)h}xy$$

and

$$\begin{aligned} P_{mn} &\left(\alpha + \frac{\delta-\alpha}{g(y)}x + \frac{\beta-\alpha}{h}y + \frac{\alpha-\beta-\delta+\gamma}{g(y)h}xy\right) \\ &= \alpha + \frac{\delta-\alpha}{g(y)}x + \frac{\beta-\alpha}{h}y + \frac{\alpha-\beta-\delta+\gamma}{g(y)h}xy, \end{aligned}$$

$$\begin{aligned} Q_{nm} &\left(\alpha + \frac{\delta-\alpha}{g(y)}x + \frac{\beta-\alpha}{h}y + \frac{\alpha-\beta-\delta+\gamma}{g(y)h}xy\right) \\ &= \alpha + \frac{\delta-\alpha}{g(y)}x + \frac{\beta-\alpha}{h}y + \frac{\alpha-\beta-\delta+\gamma}{g(y)h}xy. \end{aligned}$$

From the contraction principle we have that $F_{\alpha,\beta,\gamma,\delta}$ is the unique fixed point of P_{mn} in $X_{\alpha,\beta,\gamma,\delta}$ and $P_{mn}|_{X_{\alpha,\beta,\gamma,\delta}}$ is a Picard operator and, respectively, $F_{\alpha,\beta,\gamma,\delta}$ is the unique fixed point of Q_{nm} in $X_{\alpha,\beta,\gamma,\delta}$ and $Q_{nm}|_{X_{\alpha,\beta,\gamma,\delta}}$ is a Picard operator, so (3.3) and (3.4) hold. Consequently, taking into account (ii), by Theorem 1.4 it follows that the operators P_{mn} and Q_{nm} are weakly Picard operators. □

Theorem 3.3. *The operator S_{mn} is weakly Picard operator and*

$$\begin{aligned}(S_{mn}^{\infty}F)(x,y) &= F(0,y)+F(x,0)-F(0,0)\\ &+\frac{F(g(y),y)-F(0,y)-F(h,0)+F(0,0)}{g(y)}x\\ &+\frac{F(x,h)-F(x,0)-F(0,h)+F(0,0)}{h}y\\ &-\frac{F(0,0)-F(0,h)-F(h,0)+F(h,h)}{g(y)h}xy.\end{aligned}$$

Proof. The proof follows the same steps as in the previous theorems but using the following inequality

$$\|S_{mn}(F)(x,y)-S_{mn}(G)(x,y)\|_{\infty}\leq\left[1-\left(\frac{1}{2^{m-1}}+\frac{1}{2^{n-1}}-\frac{1}{2^{m+n-2}}\right)\right]\|F-G\|_{\infty},$$

in order to prove that S_{mn} is a contraction. □

Remark 3.4. *We have an analogous result for the operator T_{nm}.*

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