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THE METHOD OF SUCCESSIVE INTERPOLATIONS SOLVING INITIAL VALUE PROBLEMS FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. A new numerical method for initial value problems associated to second order functional differential equations is obtained. The method uses the fixed point technique, the trapezoidal quadrature rule, and a Birkhoff interpolation procedure. The convergence of the method is proved without smoothness conditions, the kernel function being only Lipschitzian in each argument. The interpolation procedure is used only on the points where the argument is modified. A stopping criterion of the algorithm is obtained and the accurancy of the method is illustrated on some numerical examples of pantograph type.
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1. Introduction

Many numerical methods solving initial value problems for second order differential equations were developed in the last fourty years. The tools are: Runge-Kutta procedures, the Nyström method, projection and collocation methods, spline functions methods and spline collocation methods. Some of them can be found in [11], [15], [16], [17], [18], [23], [25], [28]. Second order delay-functional differential equations appears in ship stabilization studies (see [24]). The existing numerical methods for initial value problems associated with second order functional differential equations are based on variational iterations (see [10], [33]) or use power series (see [32], [29]), Runge-Kutta-Nyström procedures (see [1], [26]), spline functions (see [7], [8], [12], [23]), Adomian decomposition methods (see [13]), trapezoidal Adams method for the transformed difference equation (see [22], [21]), $\theta-$methods (see [31], [34], [35]) and collocation methods (see [2], [5], [7], [8], [23]). Existence results for the solution of second order functional differential equations can be found in [14], [19], [20], [30] and the asymptotic behavior of the solutions was studied in [6], [14]. A recent classification, in the context of Lie groups, of the second order functional differential equations can be found in [27].

In the present paper we propose a new numerical method for initial value problems associated to second order differential equations with variable modification of the argument. The principles of this method firstly appeared in [3]. The method combines the Picard's sequence of successive approximations (given by the fixed point technique) with a quadrature rule and uses a procedure of piecewise Birkhoff interpolation only on the points where the modified argument appears. The interpolation procedure is repeated at each step of iteration by using the values computed at the previous step. All the procedures included in the algorithm are recurrent and therefore, easy to program. The method is developed as an alternative to the well-known spline functions, collocation, Runge-Kutta-Nyström methods in the situations uncovered by these methods, most of them requiring at least first order smoothness conditions, i.e. $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ to be continuous and bounded on $[0,a] \times \mathbb{R} \times \mathbb{R}$, while others require even fourth order smoothness conditions. Here the convergence of the method is obtained without smoothness conditions, the kernel function being only Lipschitzian in each argument.

Consider the initial value problem:

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [0,a] \\ x(0) = x_0, \quad x'(0) = v_0 \end{cases} \tag{1.1}$$

where $a > 0$, x_0, $v_0 \in \mathbb{R}$ and $\varphi : [0,a] \to \mathbb{R}$ is such that $0 \leq \varphi(t) \leq a$, $\forall t \in [0,a]$. The case $\varphi(t) = \lambda t$ corresponds to the second order pantograph equation. We suppose that φ is Lipschitzian and f is bounded and Lipschitzian in each argument. Under these conditions the proposed method is convergent and numerically stable. The *a priori* error estimate leads to the convergence and to the numerical stability of the method, whereas with the *a posteriori* error estimate a practical stopping criterion of the algorithm is obtained. The accuracy and the convergence of the method are tested on numerical examples of pantograph type.

The paper is organized as follows: in Section 2, some properties (uniform convergence and boundedness) of the sequence of successive approximations are proved. The numerical method and the corresponding algorithm used to approximate the solution of (1.1) are presented in Section 3. The convergence and numerical stability of the method are proved in Section 4. The numerical stability, the accuracy and the convergence of the proposed numerical method are illustrated and tested on numerical examples of pantograph type in Section 5. Section 6 contains some concluding remarks about the effectiveness of the proposed method.

2. The sequence of successive approximations

Consider the following conditions:

(i) $f \in C([0,a] \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C[0,a]$ and $0 \leq \varphi(t) \leq a$ for all $t \in [0,a]$

(ii) exist $\alpha, \beta > 0$ such that

$$|f(s,u,v) - f(s,u',v')| \leq \alpha |u - u'| + \beta |v - v'|, \quad \text{for all } s \in [0,a], u, u', v, v' \in \mathbb{R}$$

(iii) $\frac{a^2}{2}(\alpha+\beta)<1$
(iv) exist $\gamma, \delta > 0$ such that

$$|f(s,u,v)-f(s',u,v)| \leq \gamma |s-s'|, \quad \text{for all } s,s' \in [0,a], u,v \in \mathbb{R}$$

and

$$|\varphi(s)-\varphi(s')| \leq \delta |s-s'|, \quad \text{for all } s \in [0,a].$$

Let the function $x_0 : [0.a] \to \mathbb{R}$ given by $x_0(t) = x_0 + v_0 t$, $t \in [0,a]$. The context will avoid the confusion between the function x_0 and the number x_0. Since f and φ are continuous, there exists $M_0 \geq 0$ such that

$$M_0 = \max\{|f(s, x_0(s), x_0(\varphi(s)))| : s \in [0,a]\}.$$

The initial value problem (1.1) is equivalent in $C[0,a]$ with the following Volterra integral equation

$$x(t) = x_0 + v_0 t + \int_0^t (t-s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad t \in [0.a]. \tag{2.1}$$

We define the operator $A : C[0,a] \to C[0,a]$,

$$A(x)(t) = x_0 + v_0 t + \int_0^t (t-s) \cdot f(s, x(s), x(\varphi(s)))\, ds, \quad t \in [0.a]$$

and under the conditions (i), (ii), (iii), using the Picard-Banach fixed point technique we prove that the operator A is contraction having an unique fixed point $x^* \in C[0,a]$ which is the unique solution of the equation (2.1). After two times differentiation in the relation

$$x^*(t) = x_0 + v_0 t + \int_0^t (t-s) \cdot f(s, x^*(s), x^*(\varphi(s)))\, ds$$

we infer that $x^* \in C^2[0,a]$ and x^* is the solution of the initial value problem (1.1). Let the sequence of successive approximations $(x_k)_{k\in\mathbb{N}} \subset C[0,a]$ be

$$x_0(t) = x_0 + v_0 t, \quad t \in [0.a]$$

$$x_k(t) = x_0 + v_0 t + \int_0^t (t-s) \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))\, ds, \quad t \in [0.a], k \in \mathbb{N}^*. \tag{2.2}$$

According to the Banach's fixed point principle the sequence of successive approximations uniformly converges to x^* on $[0,a]$ and the following *apriori* and *a posteriori* error estimates hold:

$$|x_k(t) - x^*(t)| \leq \frac{(a^2/2)^k (\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)} \cdot \frac{M_0 a^2}{2}, \quad \text{for all } t \in [0,a],\ k \in \mathbb{N}^* \tag{2.3}$$

$$|x_k(t) - x^*(t)| \leq \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot |x_k(t) - x_{k-1}(t)| \leq$$

$$\leq \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \|x_k - x_{k-1}\|_C, \quad \text{for all } t \in [0,a],\ k \in \mathbb{N}^*. \tag{2.4}$$

Proposition 2.1. *On the bounded interval* $[0,a]$*, under the conditions (i)-(iii), the terms of the sequence of successive approximations are bounded.*

Proof. Let arbitrary $k \in \mathbb{N}^*$, $k \geq 2$. We get

$$|x_k(t) - x_{k-1}(t)| \leq$$

$$\leq \int_0^t (t-s) \cdot |f(s, x_{k-1}(s), x_{k-1}(\varphi(s))) - f(s, x_{k-2}(s), x_{k-2}(\varphi(s)))|\, ds \leq$$

$$\leq \int_0^t (t-s) \cdot (\alpha |x_{k-1}(s) - x_{k-2}(s)| + \beta |x_{k-1}(\varphi(s)) - x_{k-2}(\varphi(s))|)\, ds \leq$$

$$\leq (\alpha+\beta) \cdot \|x_{k-1} - x_{k-2}\|_C \int_0^t (t-s)\, ds \leq \frac{a^2}{2}(\alpha+\beta) \cdot \|x_{k-1} - x_{k-2}\|_C, \quad \forall t \in [0,a]$$

and by induction it follows

$$|x_k(t) - x_{k-1}(t)| \leq \left(a^2/2\right)^{k-1} (\alpha+\beta)^{k-1} \cdot \|x_1 - x_0\|_C, \quad \forall t \in [0,a].$$

So,

$$|x_k(t) - x_0(t)| \leq |x_k(t) - x_{k-1}(t)| + |x_{k-1}(t) - x_{k-2}(t)| + \ldots + |x_1(t) - x_0(t)| \leq$$

$$\leq [1 + \frac{a^2}{2}(\alpha+\beta) + \ldots + \left(a^2/2\right)^{k-1} (\alpha+\beta)^{k-1}] \cdot \|x_1 - x_0\|_C, \quad \forall t \in [0,a].$$

Thus

$$|x_k(t)| \leq |x_k(t) - x_0(t)| + |x_0(t)| \leq \frac{M_0 a^2}{2[1-(a^2/2)(\alpha+\beta)]} + R_0 = R, \quad \forall t \in [0,a]$$

where $R_0 = |x_0| + a|v_0|$. Then, $-R \leq |x_k(t)| \leq R$ for all $t \in [0,a]$, $k \in \mathbb{N}^*$. Moreover, we obtain $-R \leq |x^*(t)| \leq R$ for all $t \in [0,a]$ and considering the functions $F_k : [0.a] \to \mathbb{R}$ given by $F_k(s) = f(s, x_k(s), x_k(\varphi(s)))$, $s \in [0,a]$, $k \in \mathbb{N}$ it follows that $|F_k(s)| \leq M$, where $M = \max\{|f(s,u,v)| : s \in [0,a], u, v \in [-R,R]\}$. □

Remark 2.2. It is easy to see that $x_k \in C^2[0,a]$,

$$x_k''(t) = f(t, x_{k-1}(t), x_{k-1}(\varphi(t))), \quad t \in [0,a],\ k \in \mathbb{N}^*$$

and $|x_k''(t)| \leq M, \quad \forall t \in [0,a],\ k \in \mathbb{N}^*$.

3. The numerical method

3.1. **The interpolation procedure.** To compute the terms of the sequence of successive approximations we use an interpolation procedure of Birkhoff lacunary type. Further, we present this procedure of piecewise Birkhoff interpolation which will be used in the iterative steps of the algorithm. The piecewise Birkhoff interpolation function $s : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and its restrictions to the subintervals $[t_{i-1}, t_i]$, $i = \overline{1, n}$ of a partition Δ of $[a, b]$

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$$

are s_i, $i = \overline{1, n}$. Considering the following interpolation conditions:

$$\begin{cases} s_i (t_{i-1}) = y_{i-1},\ s_i (t_i) = y_i \\ s_i'' (t_{i-1}) = y_{i-1}'',\ s_i'' (t_i) = y_i'' \end{cases},\ i = \overline{1, n}, \tag{3.1}$$

for given values y_i, y_i'', $i = \overline{0, n}$, we uniquely obtain

$$s_i (t) = \frac{t_i - t}{h_i} \cdot y_{i-1} + \frac{t - t_{i-1}}{h_i} \cdot y_i - \frac{(t - t_{i-1}) (t_i - t) [(t_i - t) + h_i]}{6h_i} \cdot y_{i-1}'' -$$

$$- \frac{(t - t_{i-1}) (t_i - t) [(t - t_{i-1}) + h_i]}{6h_i} \cdot y_i'', \quad \forall t \in [t_{i-1}, t_i], \quad i = \overline{1, n} \tag{3.2}$$

where $h_i = t_i - t_{i-1}$, $i = \overline{1, n}$.

In regard to the error of the interpolation procedure, the following result for lacunary Birkhoff polynomial interpolation of two times differentiable functions on open intervals holds (see Lemma 5 in [4]).

Lemma 3.1. *([4]): Let $f : [a, b] \to \mathbb{R}$ be a two times differentiable function on (a, b), $f \in C[a, b]$ and $f', f'' \in C(a, b)$ with finite limits $\lim\limits_{x \to a, x > a} f''(x) \overset{notation}{=} f''(a)$, $\lim\limits_{x \to b, x < b} f''(x) \overset{notation}{=} f''(b)$. The interpolation conditions*

$$P(a) = f(a),\ P(b) = f(b),\ P''(a) = f''(a),\ P''(b) = f''(b)$$

uniquely determine the Birkhoff interpolation polynomial $P : [a, b] \to \mathbb{R}$ and the error estimation in the interpolation formula $f(x) = P(x) + R(x)$, $x \in [a, b]$ is given in the inequality

$$|R(x)| \le \frac{(b - a)^2}{4} \cdot \|f''\|_C, \quad \forall x \in [a, b]$$

where $\|f''\|_C = \max\{|f''(a)|, |f''(b)|, \sup\limits_{x \in (a,b)} |f''(x)|\}$.

3.2. **The algorithm.** In order to compute the terms of the sequence of successive approximations consider the uniform partition of $[0, a]$ given by the knots $t_i = \frac{i \cdot a}{n}$, $i = \overline{0, n}$. Let $h = \frac{a}{n}$. On these knots, the relations (2.2) become

$$x_k (t_i) = x_0 + v_0 t_i + \int_0^{t_i} (t_i - s) \cdot f (s, x_{k-1} (s), x_{k-1} (\varphi (s))) ds, \quad i = \overline{0, n}, k \in \mathbb{N}^*. \tag{3.3}$$

We define the functions $F_{k,i} : [0,a] \to \mathbb{R}$, $i = \overline{0,n}$, $k \in \mathbb{N}$, given by $F_{k,i}(s) = (t_i - s) \cdot f(s, x_k(s), x_k(\varphi(s)))$.

Proposition 3.2. *Under the conditions (i)-(iv) the functions x_k'' and F_k, $k \in \mathbb{N}^*$ are Lipschitzian with the same Lipschitz constant $\overline{L} = \gamma + (\alpha + \delta\beta)(|v_0| + 2aM)$. Moreover, the functions $F_{k,.i}$, $i = \overline{0,n}$, $k \in \mathbb{N}$, are Lipschitzian with the same constant $L = M + a\overline{L}$.*

Proof. For arbitrary $t, t' \in [0,a]$ we have,

$$|x_0(t) - x_0(t')| \leq |v_0| \cdot |t - t'|$$

$$|x_k(t) - x_k(t')| \leq |v_0| \cdot |t - t'| + \left| \int_0^t f(s, x_{k-1}(s), x_{k-1}(\varphi(s))) \cdot [(t - s) - (t' - s)] ds \right|$$

$$+ \left| \int_0^t (t' - s) f(s, x_{k-1}(s), x_{k-1}(\varphi(s))) ds - \int_0^{t'} (t' - s) f(s, x_{k-1}(s), x_{k-1}(\varphi(s))) ds \right|$$

$$\leq |v_0| \cdot |t - t'| + M \int_0^t |t - t'| ds + \int_t^{t'} aM ds \leq (|v_0| + 2aM) |t - t'|, \quad \forall k \in \mathbb{N}^*$$

and

$$|F_0(t) - F_0(t')| \leq [\gamma + (\alpha + \delta\beta)|v_0|] \cdot |t - t'|$$

$$|F_k(t) - F_k(t')| \leq \gamma |t - t'| + \alpha |x_k(t) - x_k(t')| + \beta |x_k(\varphi(t)) - x_k(\varphi(t'))| \leq$$

$$\leq \gamma |t - t'| + \alpha (|v_0| + 2aM) |t - t'| + \beta (|v_0| + 2aM) \cdot |\varphi(t) - \varphi(t')| \leq$$

$$\leq [\gamma + (\alpha + \delta\beta)(|v_0| + 2aM)] \cdot |t - t'| = \overline{L} \cdot |t - t'|, \quad \forall k \in \mathbb{N}^*.$$

So,

$$|F_{0,i}(t) - F_{0,i}(t')| \leq M |t - t'| + a[\gamma + (\alpha + \delta\beta)|v_0|] \cdot |t - t'|$$

$$|F_{k,i}(t) - F_{k,i}(t')| \leq |f(t, x_k(t), x_k(\varphi(t)))| \cdot |(t_i - t) - (t_i - t')| +$$

$$+ |t_i - t'| \cdot |f(t, x_k(t), x_k(\varphi(t))) - f(t', x_k(t'), x_k(\varphi(t')))| \leq M |t - t'| +$$

$$+ a\overline{L} \cdot |t - t'| = L \cdot |t - t'|, \quad \forall k \in \mathbb{N}^*, \forall t, t' \in [0,a].$$

□

In the computation of the integrals from (3.3) we apply the trapezoidal quadrature rule with recent remainder estimation obtained for Lipschitzian functions in [9]

$$\int_a^b F(t)dt = \frac{(b-a)}{2n} \cdot [\sum_{i=0}^{n-1} F\left(a + \frac{i(b-a)}{n}\right) + F\left(a + \frac{(i+1)(b-a)}{n}\right)] + R_n(F) \quad (3.4)$$

$$|R_n(F)| \leq \frac{L(b-a)^2}{4n} \quad (3.5)$$

where $L > 0$ is the Lipschitz constant of F.

Applying the quadrature rule (3.4)-(3.5) to the integrals from (3.3) we obtain the following numerical method:

$$x_0(t_i) = x_0 + v_0 t_i, \quad \text{for all } i = \overline{0,n} \quad (3.6)$$

$$x_k(t_0) = x_0, \quad x_k(t_i) = x_0 + v_0 t_i + \int\limits_0^{t_i} F_{k-1,i}(s)ds =$$

$$= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, x_{k-1}(t_{j-1}), x_{k-1}(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, x_{k-1}(t_j), x_{k-1}(\varphi(t_j)))] + R_{k,i}, \quad \text{for all } i = \overline{1,n},\ k \in \mathbb{N}^*. \tag{3.7}$$

Since the functions $F_{k.i}$, $i = \overline{0,n}$, $k \in \mathbb{N}$ are Lipschitzian with the same constant L, for the remainder estimation in (3.7) we have

$$|R_{k,i}| \leq \frac{La^2}{4n}, \quad \text{for all } i = \overline{1,n}, \quad k \in \mathbb{N}^*. \tag{3.8}$$

The relations (3.6)-(3.7) lead to the following algorithm:

$$x_0(t_i) = x_0 + v_0 t_i, \quad \text{for all } i = \overline{0,n} \text{ and } x_k(t_0) = x_0, \quad k \in \mathbb{N}^* \tag{3.9}$$

$$x_1(t_i) = x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, x_0(t_{j-1}), x_0(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, x_0(t_j), x_0(\varphi(t_j)))] + R_{1,i} = \overline{x_1(t_i)} + R_{1,i}, \quad \text{for all } i = \overline{1,n} \tag{3.10}$$

and

$$x_2(t_i) = x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, \overline{x_1(t_{j-1})} + R_{1,j-1}, x_1(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, \overline{x_1(t_j)} + R_{1,j}, x_1(\varphi(t_j)))] + R_{2,i} =$$

$$= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, \overline{x_1(t_{j-1})}, s_1(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, \overline{x_1(t_j)}, s_1(\varphi(t_j)))] + \overline{R_{2,i}} = \overline{x_2(t_i)} + \overline{R_{2,i}}, \quad \text{for all } i = \overline{1,n}, \tag{3.11}$$

where $s_1 : [0,a] \to \mathbb{R}$ is the piecewise Birkhoff interpolation function inspired by the construction in (3.1), (3.2). The function s_1 interpolates the values $\overline{x_1(t_i)}$, $i = \overline{0,n}$ and has the restrictions to the intervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$,

$$s_1^{(i)}(t) = \frac{t_i - t}{h} \cdot \overline{x_1(t_{i-1})} + \frac{t - t_{i-1}}{h} \cdot \overline{x_1(t_i)} -$$

$$- \frac{(t - t_{i-1})(t_i - t)[(t - t_{i-1}) + h]}{6h} \cdot f(t_i, x_0(t_i), x_0(\varphi(t_i))) -$$

$$- \frac{(t - t_{i-1})(t_i - t)[(t_i - t) + h]}{6h} \cdot f(t_{i-1}, x_0(t_{i-1}), x_0(\varphi(t_{i-1}))), \tag{3.12}$$

$t \in [t_{i-1}, t_i]$, $i = \overline{1,n}$. By induction, for $k \geq 3$ we obtain:

$$x_k(t_i) = x_0 + v_0 t_i + \frac{a}{2n} \cdot$$

$$\cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, \overline{x_{k-1}(t_{j-1})} + \overline{R_{k-1,j-1}}, x_{k-1}(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, \overline{x_{k-1}(t_j)} + \overline{R_{k-1,j}}, x_{k-1}(\varphi(t_j)))] + R_{k,i} =$$

$$= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot f(t_{j-1}, \overline{x_{k-1}(t_{j-1})}, s_{k-1}(\varphi(t_{j-1}))) +$$

$$+ (t_i - t_j) \cdot f(t_j, \overline{x_{k-1}(t_j)}, s_{k-1}(\varphi(t_j)))] + \overline{R_{k,i}} = \overline{x_k(t_i)} + \overline{R_{k,i}}, \quad \forall i = \overline{1,n} \quad (3.13)$$

where $s_{k-1} : [0,a] \to \mathbb{R}$ is the piecewise Birkhoff interpolation function inspired by the construction in (3.1), (3.2). The function s_{k-1} interpolates the values $\overline{x_{k-1}(t_i)}$, $i = \overline{0,n}$ and has the restrictions to the intervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$:

$$s_{k-1}^{(i)}(t) = \frac{t_i - t}{h} \cdot \overline{x_{k-1}(t_{i-1})} + \frac{t - t_{i-1}}{h} \cdot \overline{x_{k-1}(t_i)} - \frac{(t - t_{i-1})(t_i - t)[(t_i - t) + h]}{6h} \cdot$$

$$\cdot \left(\overline{x_{k-1}(t_{i-1})}\right)'' - \frac{(t - t_{i-1})(t_i - t)[(t - t_{i-1}) + h]}{6h} \cdot \left(\overline{x_{k-1}(t_i)}\right)'', \quad (3.14)$$

$t \in [t_{i-1}, t_i]$, $\quad i = \overline{1,n}$, where

$$\left(\overline{x_{k-1}(t_i)}\right)'' = f\left(t_i,\ \overline{x_{k-2}(t_i)}, s_{m-2}(\varphi(t_i))\right), \quad i = \overline{0,n}. \quad (3.15)$$

4. The convergence analysis

4.1. **The error estimation.**

Theorem 4.1. *Under the conditions (i)-(iv), if* $a^2(\alpha + \beta) < 1$, *then the values of the unique solution* x^* *of the initial value problem (1.1) is approximated on the knots* $t_i = \frac{i \cdot a}{n}$, $i = \overline{0,n}$ *by the sequence* $(\overline{x_k(t_i)})_{k \in \mathbb{N}^*}$. *The a priori error estimate is:*

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le \frac{\left(a^2/2\right)^k (\alpha + \beta)^k}{1 - (a^2/2)(\alpha + \beta)} \cdot \frac{M_0 a^2}{2} + \frac{a^2 L}{4n[1 - a^2(\alpha + \beta)]} +$$

$$+ \frac{2a^2 \beta M h^2 + \left(5M + 3\overline{M}\right) h^2}{8\left[1 - a^2(\alpha + \beta)\right]}, \quad \forall i = \overline{1,n}, \forall k \in \mathbb{N}^*, k \ge 2 \quad (4.1)$$

where the constant $\overline{M}$ *is given in (4.8).*

Proof. From (2.3), (3.10), (3.11), and (3.13) it results that

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le |x^*(t_i) - x_k(t_i)| +$$

$$+ \left|x_k(t_i) - \overline{x_k(t_i)}\right| = |x^*(t_i) - x_k(t_i)| + \left|\overline{R_{k,i}}\right|, \quad \forall k \in \mathbb{N}^*, i = \overline{1,n}$$

and according to (3.8) we have

$$\left|x_1(t_i) - \overline{x_1(t_i)}\right| = |R_{1,i}| \le \frac{La^2}{4n}, \quad \forall i = \overline{1,n} \quad (4.2)$$

$$\left|\overline{x_1(t_i)}\right| \le \left|x_1(t_i) - \overline{x_1(t_i)}\right| + |x_1(t_i)| \le R + \frac{La^2}{4n}, \quad \forall i = \overline{1,n}.$$

Because $x_k(t_i) \ne \overline{x_k(t_i)}$, $\forall k \in \mathbb{N}^*, i = \overline{1,n}$ we infer that s_k interpolates the values $\overline{x_k(t_i)}$, $i = \overline{0,n}$, but not the function x_k. Therefore we define for any k the function

V_{k-1}, $k \in \mathbb{N}^*$, $V_{k-1} : [0,a] \to \mathbb{R}$ given by its restrictions to the subintervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$ as follows:

$$V_{k-1}(t) = x_{k-1}(t) + [\overline{x_{k-1}(t_i)} - x_{k-1}(t_i)] \cdot \frac{t - t_{i-1}}{h} + [\overline{x_{k-1}(t_{i-1})} - x_{k-1}(t_{i-1})] \cdot \frac{t_i - t}{h} -,$$

$$- \frac{(t - t_{i-1})(t_i - t)[(t_i - t) + h]}{6h} \cdot [f\left(t_{i-1}, \overline{x_{k-2}(t_{i-1})}, s_{m-2}(\varphi(t_{i-1}))\right) - x''_{k-1}(t_{i-1})] -$$

$$- \frac{(t - t_{i-1})(t_i - t)[(t - t_{i-1}) + h]}{6h} \cdot [f\left(t_i, \overline{x_{k-2}(t_i)}, s_{m-2}(\varphi(t_i))\right) - x''_{k-1}(t_i)].$$

On the one hand, we see that $V_{k-1}(t_i) = \overline{x_{k-1}(t_i)}$, $\forall i = \overline{0,n}$, that is V_{k-1} interpolates the values $\overline{x_{k-1}(t_i)}$, $i = \overline{0,n}$ and it is continuous. So, s_{k-1} interpolates the function V_{k-1} for any $k \in \mathbb{N}^*$ and $V_{k-1} \in C[t_{i-1}, t_i] \cap C^2(t_{i-1}, t_i)$ for any $i = \overline{1,n}$, $V''_{k-1}(t_i) = f\left(t_i, \overline{x_{k-2}(t_i)}, s_{m-2}(\varphi(t_i))\right) = (s_{k-1})''(t_i)$, $\forall i = \overline{0,n}$. Consequently, V_{k-1} and s_{k-1} are in the context of Lemma 3.1 on each interval $[t_{i-1}, t_i]$, $i = \overline{1,n}$.

On the other hand, from (3.11) and (3.13) we have the estimates:

$$\left|\overline{R_{2,i}}\right| = \left|x_2(t_i) - \overline{x_2(t_i)}\right| \le |R_{2,i}| + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot$$

$$\cdot \left| f(t_{j-1}, \overline{x_1(t_{j-1})} + R_{1,j-1}, x_1(\varphi(t_{j-1}))) - f(t_{j-1}, \overline{x_1(t_{j-1})}, s_1(\varphi(t_{j-1}))) \right| +$$

$$+ (t_i - t_j) \cdot \left| f(t_j, \overline{x_1(t_j)} + R_{1,j}, x_1(\varphi(t_j))) - f(t_j, \overline{x_1(t_j)}, s_1(\varphi(t_j))) \right|] \le$$

$$\le |R_{2,i}| + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot (\alpha |R_{1,j-1}| + \beta |x_1(\varphi(t_{j-1})) - s_1(\varphi(t_{j-1}))|) +$$

$$+ (t_i - t_j) \cdot (\alpha |R_{1,j}| + \beta |x_1(\varphi(t_j)) - s_1(\varphi(t_j))|)], \quad \forall i = \overline{1,n}. \tag{4.3}$$

For $k \ge 3$ it follows analogously

$$\left|\overline{R_{k,i}}\right| = \left|x_k(t_i) - \overline{x_k(t_i)}\right| \le |R_{k,i}| + \frac{a}{2n} \cdot \sum_{j=1}^{i} [(t_i - t_{j-1}) \cdot$$

$$\cdot \left(\alpha \left|\overline{R_{k-1,j-1}}\right| + \beta |x_{k-1}(\varphi(t_{j-1})) - s_{k-1}(\varphi(t_{j-1}))|\right) + (t_i - t_j) \cdot$$

$$\cdot \left(\alpha \left|\overline{R_{k-1,j}}\right| + \beta |x_{k-1}(\varphi(t_j)) - s_{k-1}(\varphi(t_j))|\right)], \quad \forall i = \overline{1,n} \tag{4.4}$$

These suggest the necessity to estimate $|x_{k-1}(t) - s_{k-1}(t)|$ for $t \in [0,a]$ and $k \ge 2$. Recurently, we obtain

$$|x_1(t) - s_1(t)| \le |x_1(t) - V_1(t)| + |V_1(t) - s_1(t)| \le \frac{t - t_{i-1}}{h} \cdot |R_{1,i}| + \frac{t_i - t}{h} \cdot |R_{1,i-1}| +$$

$$+ \left| \frac{(t - t_{i-1})(t_i - t)[(t_i - t) + h]}{6h} \right| \cdot |f(t_{i-1}, x_0(t_{i-1}), x_0(\varphi(t_{i-1}))) - x''_1(t_{i-1})| +$$

$$+ \left| \frac{(t - t_{i-1})(t_i - t)[(t - t_{i-1}) + h]}{6h} \right| \cdot |f(t_i, x_0(t_i), x_0(\varphi(t_i))) - x''_1(t_i)| +$$

$$+ \frac{h^2}{4} \cdot \max\{[|x''_1(t)| + \frac{t_i - t}{h} \cdot |f(t_{i-1}, x_0(t_{i-1}), x_0(\varphi(t_{i-1}))) - x''_1(t_{i-1})| +$$

$$+\frac{t-t_{i-1}}{h}\cdot\left|f\left(t_i,x_0\left(t_i\right),x_0\left(\varphi\left(t_i\right)\right)\right)-x_1''\left(t_i\right)\right|],\left|f\left(t_i,x_0\left(t_i\right),x_0\left(\varphi\left(t_i\right)\right)\right)\right|,$$

$$,\left|f\left(t_{i-1},x_0\left(t_{i-1}\right),x_0\left(\varphi\left(t_{i-1}\right)\right)\right)\right|\}\leq\frac{La^2}{4n}+\frac{Mh^2}{4},\quad\forall t\in[0,a],$$

$$\left|s_1\left(t\right)\right|\leq\left|x_1\left(t\right)-s_1\left(t\right)\right|+\left|x_1\left(t\right)\right|\leq R+\frac{La^2}{4n}+\frac{Mh^2}{4},\quad\forall t\in[0,a],$$

$$\left|\overline{R_{2,i}}\right|\leq\left|R_{2,i}\right|+\frac{a}{2n}\cdot\sum_{j=1}^{i}[(t_i-t_{j-1})\cdot(\alpha\left|R_{1,j-1}\right|+\beta\left|x_1\left(\varphi\left(t_{j-1}\right)\right)-s_1(\varphi(t_{j-1}))\right|)+$$

$$+\left(t_i-t_j\right)\cdot(\alpha\left|R_{1,j}\right|+\beta\left|x_1\left(\varphi\left(t_j\right)\right)-s_1(\varphi(t_j))\right|)]\leq\frac{La^2}{4n}+a^2\alpha\cdot\frac{La^2}{4n}+$$

$$+a^2\beta\left(\frac{La^2}{4n}+\frac{Mh^2}{4}\right)=\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4},\quad\forall i=\overline{1,n},$$

$$\left|\overline{x_2\left(t_i\right)}\right|\leq R+\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4},\quad\forall i=\overline{1,n}$$

and

$$\left|x_2\left(t\right)-s_2\left(t\right)\right|\leq\left|x_2\left(t\right)-V_2\left(t\right)\right|+\left|V_2\left(t\right)-s_2\left(t\right)\right|\leq\max\{\left|\overline{R_{2,i}}\right|:i=\overline{0,n}\}+$$

$$+\left|\frac{\left(t-t_{i-1}\right)\left(t_i-t\right)\left[\left(t_i-t\right)+h\right]}{6h}\right|\cdot\left|f\left(t_{i-1},\ \overline{x_1(t_{i-1})},s_1\left(\varphi\left(t_{i-1}\right)\right)\right)-x_2''\left(t_{i-1}\right)\right|+$$

$$+\left|\frac{\left(t-t_{i-1}\right)\left(t_i-t\right)\left[\left(t-t_{i-1}\right)+h\right]}{6h}\right|\cdot\left|f\left(t_i,\ \overline{x_1(t_i)},s_1\left(\varphi\left(t_i\right)\right)\right)-x_2''\left(t_i\right)\right|+$$

$$+\frac{h^2}{4}\cdot\max\{[\left|x_2''\left(t\right)\right|+\frac{t_i-t}{h}\cdot\left|f\left(t_{i-1},\ \overline{x_1(t_{i-1})},s_1\left(\varphi\left(t_{i-1}\right)\right)\right)-x_2''\left(t_{i-1}\right)\right|+$$

$$+\frac{t-t_{i-1}}{h}\cdot\left|f\left(t_i,\ \overline{x_1(t_i)},s_1\left(\varphi\left(t_i\right)\right)\right)-x_2''\left(t_i\right)\right|],\left|f\left(t_i,\ \overline{x_1(t_i)},s_1\left(\varphi\left(t_i\right)\right)\right)\right|,$$

$$,\left|f\left(t_{i-1},\ \overline{x_1(t_{i-1})},s_1\left(\varphi\left(t_{i-1}\right)\right)\right)\right|\}\leq\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}+$$

$$+\left(M+M_1\right)\cdot\frac{h^2}{8}+\frac{h^2}{4}\cdot\max\{2M+M_1,M_1,M_1\}=$$

$$=\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}+\frac{\left(5M+3M_1\right)h^2}{8}\tag{4.5}$$

where

$$M_1=\max\{\left|f\left(t,u,v\right)\right|:t\in[0,a],u,v\in[-\left(R+\frac{La^2}{4n}+\frac{Mh^2}{4}\right),R+\frac{La^2}{4n}+\frac{Mh^2}{4}]\}.$$

Moreover,

$$\left|s_2\left(t\right)\right|\leq\left|x_2\left(t\right)-s_2\left(t\right)\right|+\left|x_2\left(t\right)\right|\leq R+\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}+$$

$$+\frac{\left(5M+3M_1\right)h^2}{8}\leq R+\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+\frac{a^2\beta Mh^2}{4}+\frac{\left(5M+3M_1\right)h^2}{8},\ \forall t\in[0,a],$$

$$\left|\overline{R_{3,i}}\right|\leq\left|R_{3,i}\right|+\frac{a}{2n}\cdot\sum_{j=1}^{i}[(t_i-t_{j-1})\cdot\left(\alpha\left|\overline{R_{2,j-1}}\right|+\beta\left|x_2\left(\varphi\left(t_{j-1}\right)\right)-s_2(\varphi(t_{j-1}))\right|\right)+$$

$$+\left(t_i-t_j\right)\cdot\left(\alpha\left|\overline{R_{2,j}}\right|+\beta\left|x_2\left(\varphi\left(t_j\right)\right)-s_2(\varphi(t_j))\right|\right)]\leq\frac{La^2}{4n}+$$
$$+a^2\alpha\cdot\left(\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}\right)+a^2\beta\cdot$$
$$\cdot\left(\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{La^2}{4n}+\frac{a^2\beta Mh^2}{4}+\frac{\left(5M+3M_1\right)h^2}{8}\right)\leq$$
$$\leq\left[1+a^2\left(\alpha+\beta\right)+a^4\left(\alpha+\beta\right)^2\right]\cdot\frac{La^2}{4n}+\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{a^2\beta Mh^2}{4}+$$
$$+\frac{a^2\beta\left(5M+3M_1\right)h^2}{8}\leq\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+$$
$$+\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{a^2\beta Mh^2}{4}+\frac{a^2\beta\left(5M+3M_1\right)h^2}{8},\quad\forall i=\overline{1,n}\tag{4.6}$$

and

$$\left|\overline{x_3\left(t_i\right)}\right|\leq R+\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+$$
$$+\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{a^2\beta Mh^2}{4}+\frac{a^2\beta\left(5M+3M_1\right)h^2}{8},\quad\forall i=\overline{1,n}.$$

Let

$$R'=R+\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+\frac{a^2\beta Mh^2}{4}+\frac{\left(5M+3M_1\right)h^2}{8}$$

and

$$M'=\max\{M,M_1,\max\{\left|f\left(t,u,v\right)\right|:t\in[0,a],u,v\in[-R',R']\}\}.$$

Similar as (4.5) we obtain

$$\left|x_3\left(t\right)-s_3\left(t\right)\right|\leq\max\{\left|\overline{R_{3,i}}\right|:i=\overline{0,n}\}+\left(M+M'\right)\cdot\frac{h^2}{8}+$$
$$+\frac{h^2}{4}\cdot\max\{2M+M',M',M'\}\leq\max\{\left|\overline{R_{3,i}}\right|:i=\overline{0,n}\}+\frac{\left(5M+3M'\right)h^2}{8}$$

and from (4.6) we infer that

$$\left|x_3\left(t\right)-s_3\left(t\right)\right|\leq\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{a^2\beta Mh^2}{4}+$$
$$+\left[1+a^2\beta\right]\cdot\frac{\left(5M+3M'\right)h^2}{8},\quad\forall t\in[0,a]$$

and

$$\left|s_3\left(t\right)\right|\leq R+\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+\left[1+a^2\left(\alpha+\beta\right)\right]\cdot\frac{a^2\beta Mh^2}{4}+\left[1+a^2\beta\right]\cdot$$
$$\cdot\frac{\left(5M+3M'\right)h^2}{8}\leq R+\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+\frac{2a^2\beta Mh^2+\left(5M+3M'\right)h^2}{8\left[1-a^2\left(\alpha+\beta\right)\right]},\ \forall t\in[0,a].$$

Let

$$\overline{R}=R+\frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]}+\frac{2a^2\beta Mh^2+\left(5M+3M'\right)h^2}{8\left[1-a^2\left(\alpha+\beta\right)\right]}$$

and by induction for $k \geq 3$ we obtain

$$\left|\overline{R_{k,i}}\right| \leq \frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]} + \frac{2a^2\beta Mh^2 + \left(5M+3\overline{M}\right)h^2}{8\left[1-a^2\left(\alpha+\beta\right)\right]},\ \forall i=\overline{1,n}, \forall k \in \mathbb{N}^*,\ k \geq 2 \tag{4.7}$$

and

$$\left|\overline{x_k\left(t_i\right)}\right| \leq R + \frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]} + \frac{2a^2\beta Mh^2 + \left(5M+3\overline{M}\right)h^2}{8\left[1-a^2\left(\alpha+\beta\right)\right]}, \quad \forall i=\overline{1,n},\ k \geq 2$$

$$\left|x_{k-1}\left(t\right) - s_{k-1}\left(t\right)\right| \leq \frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]} + \frac{2a^2\beta Mh^2 + \left(5M+3\overline{M}\right)h^2}{8\left[1-a^2\left(\alpha+\beta\right)\right]} +$$

$$+\frac{\left(5M+3\overline{M}\right)h^2}{8}, \quad \forall t \in [0,a], \forall k \in \mathbb{N}^*, \forall k \geq 2$$

$$\left|s_{k-1}\left(t\right)\right| \leq R + \frac{La^2}{4n\left[1-a^2\left(\alpha+\beta\right)\right]} + \frac{\left(5M+3\overline{M}\right)h^2}{8} +$$

$$+\frac{2a^2\beta Mh^2 + \left(5M+3\overline{M}\right)h^2}{8\left[1-a^2\left(\alpha+\beta\right)\right]}, \quad \forall t \in [0,a], , \forall k \in \mathbb{N}^*, \forall k \geq 2$$

where

$$\overline{M} = \max\{M', \max\{\left|f\left(t,u,v\right)\right| : t \in [0,a], u,v \in [-\overline{R},\overline{R}]\}\}. \tag{4.8}$$

□

Remark 4.2. From the estimate (4.1) it follows the consistency of the method. The conditions in Theorem 4.1 differ from the conditions for the existence and uniqueness of the solution only by the inequality $a^2\left(\alpha+\beta\right) < 1$ and by the Lipschitz requirements (iv). Supplementary boundedness and smoothness conditions are not necessary.

Remark 4.3. Under the conditions of Theorem 4.1 we obtain similar continuous approximation of the solution interpolating the computed values $\overline{x_k\left(t_i\right)}$, $i=\overline{0,n}$ by the same procedure as in (3.14)-(3.15). So, we obtain the continuous approximation of the solution, $s_k : [0,a] \to \mathbb{R}$ given by its restrictions to the intervals $[t_{i-1},t_i]$, $i=\overline{1,n}$:

$$s_k^{(i)}(t) = \frac{t_i - t}{h} \cdot \overline{x_k(t_{i-1})} + \frac{t-t_{i-1}}{h} \cdot \overline{x_k(t_i)} - \frac{\left(t-t_{i-1}\right)\left(t_i-t\right)\left[\left(t_i-t\right)+h\right]}{6h} \cdot$$

$$\cdot \left(\overline{x_k(t_{i-1})}\right)'' - \frac{\left(t-t_{i-1}\right)\left(t_i-t\right)\left[\left(t-t_{i-1}\right)+h\right]}{6h} \cdot \left(\overline{x_k(t_i)}\right)'',\ \forall t \in [t_{i-1},t_i],\ i=\overline{1,n}, \tag{4.9}$$

with

$$\left(\overline{x_k(t_i)}\right)'' = f\left(t_i,\ \overline{x_{k-1}(t_i)}, s_{m-1}\left(\varphi\left(t_i\right)\right)\right), \quad i=\overline{0,n}. \tag{4.10}$$

Moreover, the approximations of the second derivative on the knots $t_i = \frac{i \cdot a}{n}$, $i=\overline{0,n}$, are computed in (4.10).

Corollary 4.4. *The error estimates in (4.9) and (4.10) are:*

$$|x^*(t)-s_k(t)| \le \frac{\left(a^2/2\right)^k(\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)}\cdot\frac{M_0a^2}{2}+\frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]}+$$
$$+\frac{2a^2\beta Mh^2+\left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]}+\frac{\left(5M+3\overline{M}\right)h^2}{8},\quad \forall t\in[0,a],\quad \forall k\in\mathbb{N}^*, \tag{4.11}$$

$$\left|(x^*(t_i))''-\left(\overline{x_k(t_i)}\right)''\right| \le \alpha\Big[\frac{\left(a^2/2\right)^k(\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)}\cdot\frac{M_0a^2}{2}+$$
$$+\frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]}+\frac{2a^2\beta Mh^2+\left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]}\Big]+\beta\Big[\frac{\left(a^2/2\right)^k(\alpha+\beta)^k}{1-(a^2/2)(\alpha+\beta)}\cdot\frac{M_0a^2}{2}+$$
$$+\frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]}+\frac{2a^2\beta Mh^2+\left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]}+\frac{\left(5M+3\overline{M}\right)h^2}{8}\Big] \tag{4.12}$$

for all $i=\overline{0,n}$ *and* $k\in\mathbb{N}^*$.

Proof. For $k\in\mathbb{N}^*$ we have

$$|x^*(t)-s_k(t)| \le |x^*(t)-x_k(t)|+|x_k(t)-s_k(t)|$$

and according to the inequalities (2.3) and (4.7) the estimate (4.11) follows. In addition to this,

$$\left|(x^*(t_i))''-\left(\overline{x_k(t_i)}\right)''\right| = \left|f\left(t_i,x^*(t_i),x^*(\varphi(t_i))\right)-f\left(t_i,\ \overline{x_{k-1}(t_i)},s_{m-1}(\varphi(t_i))\right)\right|$$

$$\le \alpha\left|x^*(t_i)-\overline{x_{k-1}(t_i)}\right|+\beta\left|x^*(\varphi(t_i))-s_{m-1}(\varphi(t_i))\right|,\quad \forall i=\overline{0,n},\quad \forall k\in\mathbb{N}^*$$

and from (2.3), (4.7) and (4.11) the estimate (4.12) follows. □

Remark 4.5. We see that the '*a posteriori*' (2.4) and 'a priori' (4.1) estimates can offer a practical stopping criterion of the algorithm. This can be stated as follows: for given $\varepsilon'>0$ and $n\in\mathbb{N}^*$ (previously chosen) we determine the first natural number $k\in\mathbb{N}^*$ for which,

$$\left|\overline{x_k(t_i)}-\overline{x_{k-1}(t_i)}\right|<\varepsilon'\quad \text{for all } i=\overline{1,n}$$

and we stop to this k, retaining the approximations $\overline{x_k(t_i)}$, $i=\overline{0,n}$ of the solution. The demonstration of this criterion is the following.
We denote:

$$\Omega=\frac{La^2}{4n\left[1-a^2(\alpha+\beta)\right]}+\frac{2a^2\beta Mh^2+\left(5M+3\overline{M}\right)h^2}{8\left[1-a^2(\alpha+\beta)\right]}.$$

For each $i=\overline{1,n}$ we have

$$\left|x^*(t_i)-\overline{x_k(t_i)}\right| \le |x^*(t_i)-x_k(t_i)|+\left|x_k(t_i)-\overline{x_k(t_i)}\right| \le$$
$$\le \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)}\cdot|x_k(t_i)-x_{k-1}(t_i)|+\left|\overline{R_{k,i}}\right|$$

and

$$|x_k(t_i)-x_{k-1}(t_i)| \le \left|x_k(t_i)-\overline{x_k(t_i)}\right|+\left|\overline{x_k(t_i)}-\overline{x_{k-1}(t_i)}\right|+$$

$$+\left|\overline{x_{k-1}(t_i)} - x_{k-1}(t_i)\right| = \left|\overline{R_{k,i}}\right| + \left|\overline{R_{k-1,i}}\right| + \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right|.$$

So,

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le \left|\overline{R_{k,i}}\right| + \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| +$$

$$+\frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left(\left|\overline{R_{k,i}}\right| + \left|\overline{R_{k-1,i}}\right|\right).$$

Then

$$\left|x^*(t_i) - \overline{x_k(t_i)}\right| \le \Omega \cdot \frac{1+(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} + \frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right|.$$

For given $\varepsilon > 0$ we require

$$\Omega \cdot \frac{1+(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} < \frac{\varepsilon}{2} \tag{4.13}$$

and

$$\frac{(a^2/2)(\alpha+\beta)}{1-(a^2/2)(\alpha+\beta)} \cdot \left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \frac{\varepsilon}{2}.$$

Since

$$\Omega \le \frac{La^2}{4n[1-a^2(\alpha+\beta)]} + \frac{2a^4\beta M + \left(5M + 3\overline{M}\right)a^2}{8n[1-a^2(\alpha+\beta)]}$$

we can chose the smallest natural number n,

$$n > \frac{[1+(a^2/2)(\alpha+\beta)] \cdot [2La^2 + 2a^4\beta M + \left(5M+3\overline{M}\right)a^2]}{4\varepsilon[1-(a^2/2)(\alpha+\beta)] \cdot [1-a^2(\alpha+\beta)]}$$

for which the inequality (4.13) holds. Afterwards we find the smallest natural number k for which

$$\left|\overline{x_k(t_i)} - \overline{x_{k-1}(t_i)}\right| < \frac{\varepsilon}{2} \cdot \frac{1-(a^2/2)(\alpha+\beta)}{(a^2/2)(\alpha+\beta)} \overset{notation}{=} \varepsilon'$$

for all $i = \overline{1,n}$. This is the last iterative step to be made. With these we obtain $\left|x^*(t_i) - \overline{x_k(t_i)}\right| < \varepsilon$, for all $i = \overline{1,n}$.

Remark 4.6. From the error estimate (2.3) we see that the sequence of successive approximations $(x_k)_{k\in\mathbb{N}^*}$ uniformly converges on $[0,a]$ to the solution x^*. From the error estimates (4.1) it follows that

$$\left|x_k(t_i) - \overline{x_k(t_i)}\right| = O(h) + O\left(h^2\right), \quad \forall i = \overline{1,n-1}, \quad \forall k \in \mathbb{N}^*$$

and the second and the third terms go to zero when $h = \frac{a}{n}$ does. These lead to the convergence of the computed values $\overline{x_k(t_i)}$ to the values of the exact solution, $x^*(t_i)$, $i = \overline{1,n}$.

4.2. **The numerical stability.** In order to obtain the numerical stability of the method we consider the initial value problem with the same second order differential equation, but with modified initial conditions:

$$\left\{ \begin{array}{c} x''(t) = f(t, x(t), x(\varphi(t))), \quad t \in [0, a] \\ x(0) = x_0', \quad x'(0) = v_0' \end{array} \right. \tag{4.14}$$

such that $|x_0 - x_0'| < \epsilon$ and $|v_0 - v_0'| < \varepsilon$ for small $\epsilon > 0, \varepsilon > 0$.

For the initial value problem (4.14) the sequence of successive approximations on the knots $t_i = \frac{i \cdot a}{n}$, $i = \overline{0, n}$ is:

$$y_0(t_i) = x_0' + v_0' t_i, \quad i = \overline{0, n}, \quad y_k(t_0) = x_0'$$

$$y_k(t_i) = x_0' + v_0' t + \int_0^{t_i} (t_i - s) \cdot f(s, y_{k-1}(s), y_{k-1}(\varphi(s)))\, ds, \quad i = \overline{1, n}, k \in \mathbb{N}^*.$$

The effective computed values are

$$y_0(t_i) = x_0' + v_0' t_i, \quad i = \overline{0, n}, \quad y_k(t_0) = x_0'$$

and $\overline{y_k(t_i)}, i = \overline{1, n}, k \in \mathbb{N}^*$ with $y_k(t_i) = \overline{y_k(t_i)} + \overline{R'_{k,i}}, \quad \forall i = \overline{1, n}, k \in \mathbb{N}^*$. We see that

$$|x_0(t) - y_0(t)| \leq |x_0 - x_0'| + |v_0 - v_0'|\, a < \epsilon + a\varepsilon, \quad \forall t \in [0, a].$$

Definition 4.7. *We say that the proposed numerical method is numerically stable if there exist $p, q \in \mathbb{N}^*$ and the constants $K_1, K_2, K_3, K_4 > 0$, such that*

$$\left| \overline{x_k(t_i)} - \overline{y_k(t_i)} \right| \leq K_1 \epsilon + K_2 \varepsilon + K_3 \cdot h^p + K_4 \cdot h^q = K_1 \epsilon + K_2 \varepsilon + O(h^p) + O(h^q)$$

for all $i = \overline{1, n}$, $k \in \mathbb{N}^*$.

Theorem 4.8. *Under the conditions of Theorem 4.1 the proposed method of successive interpolations is numerically stable.*

Proof. We have:

$$\left| \overline{x_k(t_i)} - \overline{y_k(t_i)} \right| \leq \left| \overline{x_k(t_i)} - x_k(t_i) \right| + |x_k(t_i) - y_k(t_i)| + \left| y_k(t_i) - \overline{y_k(t_i)} \right| \leq$$

$$\leq |x_k(t_i) - y_k(t_i)| + \left| \overline{R_{k,i}} \right| + \left| \overline{R'_{k,i}} \right|, \quad \forall i = \overline{1, n-1}, \forall k \in \mathbb{N}^*$$

and,

$$\left| \overline{R_{k,i}} \right| \leq \frac{La^2}{4n\left[1 - a^2(\alpha + \beta)\right]} + \frac{2a^4 \beta M + \left(5M + 3\overline{M}\right) a^2}{8n^2\left[1 - a^2(\alpha + \beta)\right]}, \ \forall i = \overline{1, n}, \forall k \in \mathbb{N}^*$$

$$\left| \overline{R'_{k,i}} \right| \leq \frac{L' a^2}{4n\left[1 - a^2(\alpha + \beta)\right]} + \frac{2a^4 \beta M + \left(5M + 3\overline{M}\right) a^2}{8n^2\left[1 - a^2(\alpha + \beta)\right]}, \ \forall i = \overline{1, n}, \forall k \in \mathbb{N}^*$$

where

$$\overline{L}' = \gamma + (\alpha + \delta\beta)(|v_0'| + 2aM)$$

and $L' = M + a\overline{L}'$. In inductive manner, according to the condition $\frac{a^2}{2}(\alpha + \beta) < 1$, we get:

$$|x_0(t) - y_0(t)| < \epsilon + a\varepsilon, \quad \forall t \in [0, a]$$

$$|x_k(t_0) - y_k(t_0)| \leq |x_0 - x_0'| < \epsilon, \quad \forall k \in \mathbb{N}^*$$

and

$$|x_k(t) - y_k(t)| \leq |x_0(t) - y_0(t)| +$$

$$+\int_0^t (t-s) \cdot |f(s, x_{k-1}(s), x_{k-1}(\varphi(s))) - f(s, y_{k-1}(s), y_{k-1}(\varphi(s)))| \, ds \leq$$

$$\leq \epsilon + a\varepsilon + \int_0^t (t-s) \cdot [\alpha |x_{k-1}(s) - y_{k-1}(s)| + \beta |x_{k-1}(\varphi(s)) - y_{k-1}(\varphi(s))|] ds \leq$$

$$\leq \epsilon + a\varepsilon + (\alpha+\beta)\frac{a^2}{2} \cdot \|x_{k-1} - y_{k-1}\|_C \leq \ldots \leq$$

$$\leq [1 + (\alpha+\beta)\frac{a^2}{2} + \ldots + \left((\alpha+\beta)\frac{a^2}{2}\right)^k] \cdot (\epsilon + a\varepsilon) =$$

$$= \frac{1 - \left((\alpha+\beta)\frac{a^2}{2}\right)^{k+1}}{1 - (\alpha+\beta)\frac{a^2}{2}} \cdot (\epsilon + a\varepsilon) \leq \frac{(\epsilon + a\varepsilon)}{1 - (\alpha+\beta)\frac{a^2}{2}}, \quad \forall t \in [0,a], \quad \forall k \in \mathbb{N}^*.$$

So, we obtain:

$$\left|\overline{x_k(t_i)} - \overline{y_k(t_i)}\right| \leq |x_k(t_i) - y_k(t_i)| + \frac{La^2}{4n[1 - a^2(\alpha+\beta)]} +$$

$$+ \frac{2a^4\beta M + \left(5M + 3\overline{M}\right)a^2}{8n^2[1 - a^2(\alpha+\beta)]} + \frac{L'a^2}{4n[1 - a^2(\alpha+\beta)]} + \frac{2a^4\beta M + \left(5M + 3\overline{M}\right)a^2}{8n^2[1 - a^2(\alpha+\beta)]} \leq$$

$$\leq \frac{(\epsilon + a\varepsilon)}{1 - (\alpha+\beta)\frac{a^2}{2}} + \frac{La + L'a}{4[1 - a^2(\alpha+\beta)]} \cdot \left(\frac{a}{n}\right) + \frac{2a^2\beta M + \left(5M + 3\overline{M}\right)}{4[1 - a^2(\alpha+\beta)]} \cdot \left(\frac{a}{n}\right)^2 =$$

$$= K_1\epsilon + K_2\varepsilon + K_3 \cdot h + K_4 \cdot h^2 = K_1\epsilon + K_2\varepsilon + O(h) + O\left(h^2\right), \ \forall i = \overline{1, n-1}, \ \forall k \in \mathbb{N}^*$$

Of course, in the same conditions the continuous dependence, by the data f and φ, of the solution can be obtained by using an analogous technique. □

Corollary 4.9. *The proposed method of successive interpolations for the initial value problem associated to functional differential equations of second order is convergent.*

Proof. The convergence results from Theorem 4.1, Remark 4.6 and Theorem 4.8. □

5. Numerical examples

Example 5.1. Consider the following initial value problem:

$$\begin{cases} x''(t) = 1 + 2\left(1 + t^2/8\right)\cos(t/2) - 2\cos(t/2)\cdot x(t/2), \quad t \in [0, \frac{\pi}{4}] \\ x(0) = 1, \quad x'(0) = 1 \end{cases}.$$

Here, $a = \frac{\pi}{4}$, $\varphi(t) = \lambda t$ with $\lambda = \frac{1}{2}$ and $f(t,u,v) = 1+2\left(1+\frac{t^2}{8}\right)\cos\left(\frac{t}{2}\right) - 2\cos\left(\frac{t}{2}\right)\cdot v$. The exact solution is $x^*(t) = \frac{t^2}{2} + \sin t + 1$, $t \in [0, \frac{\pi}{4}]$. Applying the above presented algorithm with $n = 10$ and $\varepsilon' = 10^{-15}$ we get $k = 7$ (the number of iterations). The values of $x^*(t_i)$ and $\overline{x_7(t_i)}$, $i = \overline{0,10}$ are in Table 1, where a comparison between $x^*(t_i)$ and $\overline{x_7(t_i)}$ for each $i = \overline{0,10}$ reveals the accuracy of the method in the fourth column (here $e_i = \left|x^*(t_i) - \overline{x_7(t_i)}\right|$, $i = \overline{0,10}$). The maximum error on the knots is less than 6.9×10^{-4}.

Table 1

t_i	$\overline{x_7(t_i)}$	$x^*(t_i)$	e_i	d_i
0	1	1	0	0.1
0.07853981634	1.081624067715	1.08154334710	8.072061e-005	0.105
0.15707963268	1.168932165347	1.16877147054	1.606948e-004	0.220
0.23561944901	1.261442806522	1.26120362623	2.391803e-004	0.330
0.31415926535	1.358680462801	1.35836501638	3.154464e-004	0.437
0.39269908169	1.460178491566	1.45978971675	3.887748e-004	0.536
0.47123889804	1.565482019342	1.56502354925	4.584701e-004	0.622
0.54977871438	1.674150738983	1.67362688211	5.238569e-004	0.759
0.62831853072	1.785761633819	1.78517734031	5.842935e-004	0.796
0.70685834706	1.899911575268	1.89927240973	6.391655e-004	0.812
0.78539816340	2.016219820718	2.01553191872	6.879020e-004	0.853

In order to test the numerical stability of the method we consider $\epsilon = \varepsilon = 0.1$ and we represent the differences between the effective computed values $d_i = \left|\overline{x_k(t_i)} - \overline{y_k(t_i)}\right|$, $i = \overline{0,10}$ in the fifth column. So as to illustrate and to test the convergence we put $n = 100$, $\varepsilon' = 10^{-15}$ and we can see how decrease e_i, $i = \overline{0,n}$ when h decreases. The number of iterations is $k = 7$. In order to test the numerical stability we consider again $\epsilon = \varepsilon = 0.1$. The differences d_i are those presented in the fifth column. The results are in Table 2 with the knots and the corresponding values being selected by tens, such that the knots are the same as in Table 1. It can be observed that the order of effective error becomes $O\left(10^{-6}\right)$. For $n = 1000$, $\varepsilon' = 10^{-15}$ we have $k = 7$ iterations and the order of effective error is $O\left(10^{-8}\right)$, the errors $e_i = \left|x^*(t_i) - \overline{x_7(t_i)}\right|$ for $i = \overline{0,1000}$, $i = 100 \cdot k$, $k = \overline{1,9}$, being presented in Table 3 (on the same knots as in Table 1). The results in Tables 1, 2, and 3 confirm the convergence of the algorithm, that is $e_i \to 0$ when $h \to 0$. In the implementation of the algorithm we have used Visual C^{++} and the data were considered with 10^{-20} precision in the computational process.

Table 2

t_i	$\overline{x_7(t_i)}$	$x^*(t_i)$	e_i	d_i
0	1	1	0	0.1
0.07853981634	1.081544153323	1.08154334710	8.062203e-007	0.109
0.15707963268	1.168773075523	1.16877147054	1.604982e-006	0.124
0.23561944901	1.261206015109	1.26120362623	2.388876e-006	0.172
0.31415926535	1.358368166972	1.35836501638	3.150592e-006	0.235
0.39269908169	1.459793599717	1.45978971675	3.882969e-006	0.414
0.47123889804	1.565028128292	1.56502354925	4.579041e-006	0.593
0.54977871438	1.673632114191	1.67362688211	5.232084e-006	0.644
0.62831853072	1.785183175975	1.78517734031	5.835661e-006	0.718
0.70685834706	1.899278793397	1.89927240973	6.383665e-006	0.765
0.78539816340	2.015538789077	2.01553191872	6.870357e-006	0.843

Table 3

t_i	e_i
0.07853981634	8.062104e-009
0.15707963268	1.604962e-008
0.23561944901	2.388846e-008
0.31415926535	3.150553e-008
0.39269908169	3.882921e-008
0.47123889804	4.578985e-008
0.54977871438	5.232019e-008
0.62831853072	5.835589e-008
0.70685834706	6.383585e-008
0.78539816340	6.870270e-008

Example 5.2. For the following initial value problem:

$$\begin{cases} x''(t) = 4e^{-\frac{t}{2}} \cdot \sin\left(\frac{t}{2}\right) \cdot x\left(\frac{t}{2}\right), & t \in [0, \frac{\pi}{4}] \\ x(0) = 1, \quad x'(0) = -1 \end{cases}.$$

the exact solution is $x^*(t) = e^{-t} \cdot \cos t$ and applying the above presented algorithm with $n = 10$ and $\varepsilon' = 10^{-15}$ we get $k = 6$ (the number of iterations). Here, the kernel function is

$$f(t, u, v) = 4e^{-\frac{t}{2}} \cdot \sin\left(\frac{t}{2}\right) \cdot v.$$

The values of $x^*(t_i)$ and $\overline{x_6(t_i)}$, $i = \overline{0, 10}$ are in Table 4 together with the effective errors $e_i = \left| x^*(t_i) - \overline{x_6(t_i)} \right|$, $i = \overline{0, 10}$. The maximum error on the knots is less than 1.19×10^{-3}.

Table 4

t_i	$\overline{x_6}\,(t_i)$	$x^*\,(t_i)$	e_i
0.000000000000	1.0000000000000	1.000000000000	0.000000000000
0.078539816339	0.9214601836602	0.921615432534	1.552489e-004
0.157079632679	0.8438151332115	0.844114011816	2.988786e-004
0.235619449019	0.7678191950125	0.768251274197	4.320792e-004
0.314159265358	0.6940981233705	0.694654238841	5.561155e-004
0.392699081698	0.6231606123730	0.623832938257	6.723259e-004
0.471238898038	0.5554095452595	0.556191606964	7.820617e-004
0.549778714378	0.4911527424040	0.492039450356	8.867080e-004
0.628318530717	0.4306133141265	0.431600931989	9.876179e-004
0.706858347057	0.3739393829595	0.375025531956	1.086149e-003
0.785398163397	0.3212133450620	0.322396941944	1.183597e-003

Now, in order to illustrate the convergence we consider $n = 100$, that is $h = 0.01$ and $n = 1000$, that is $h = 0.001$. The results are in Table 5 and Table 6, where the data were selected from ten to ten and hundred to hundred, respectively. For $n = 100$ the number of iterations were 6 and for $n = 1000$ this number is 6.

Table 5

t_i	$\overline{x_6}\,(t_i)$	$x^*\,(t_i)$	e_i
0.000000000000	1.000000000000	1.000000000000	0.0000000000
0.078539816339	0.921613879326	0.921615432534	1.553208e-006
0.157079632679	0.844111021458	0.844114011816	2.990358e-006
0.235619449019	0.768246950901	0.768251274197	4.323296e-006
0.314159265358	0.694648674111	0.694654238841	5.564731e-006
0.392699081698	0.623826210307	0.623832938257	6.727951e-006
0.471238898038	0.556183780397	0.556191606964	7.826568e-006
0.549778714378	0.492030576048	0.492039450356	8.874308e-006
0.628318530717	0.431591047159	0.431600931989	9.884830e-006
0.706858347057	0.375014660385	0.375025531956	1.087157e-005
0.785398163397	0.322385094321	0.322396941944	1.184762e-005

Table 6

t_i	$\overline{x_6}\,(t_i)$	$x^*\,(t_i)$	e_i
0.000000000000	1.00000000000000000	1.000000000000	0.0000000000
0.078539816339	0.92161541700187566	0.921615432534	1.553215e-008
0.157079632679	0.84411398191278664	0.844114011816	2.990374e-008
0.235619449019	0.76825123096466941	0.768251274197	4.323321e-008
0.314159265358	0.69465418319366345	0.694654238841	5.564767e-008
0.392699081698	0.62383287097786977	0.623832938257	6.727997e-008
0.471238898038	0.55619152869849453	0.556191606964	7.826627e-008
0.549778714378	0.49203936161275225	0.492039450356	8.874380e-008
0.628318530717	0.43160083314019043	0.431600931989	9.884916e-008
0.706858347057	0.37502542323991056	0.375025531956	1.087167e-007
0.785398163397	0.32239682346743259	0.322396941944	1.184774e-007

Example 5.3. Consider the initial value problem:

$$\begin{cases} x''(t) = \frac{1}{2} \cdot x(t) + \frac{1}{2} \cdot e^{t/2} \cdot x\left(\frac{t}{2}\right), & t \in [0, 0.75] \\ x(0) = 1, \quad x'(0) = 1 \end{cases}.$$

Here, $a = \frac{3}{4}$, $\varphi(t) = \lambda t$ with $\lambda = \frac{1}{2}$ and $f(t, u, v) = \frac{1}{2} \cdot u + \frac{1}{2} \cdot e^{t/2} \cdot v$.. The exact solution is $x^*(t) = e^t$ and applying the above presented algorithm with $n = 10$ and $\varepsilon' = 10^{-15}$ we get $k = 8$ (the number of iterations). The values of $x^*(t_i)$ and $\overline{x_8(t_i)}$, $i = \overline{0, 10}$ are in Table 7 together with the effective errors $e_i = \left| x^*(t_i) - \overline{x_8(t_i)} \right|$, $i = \overline{0, 10}$. The maximum error on the knots is less than 9.33×10^{-4}. In order to illustrate the convergence we consider $n = 100$, that is $h = 0.01$ and $n = 1000$, that is $h = 0.001$. The results are in Table 8 and Table 9, where the data were selected from ten to ten and hundred to hundred, respectively. For $n = 100$ the number of iterations were 9 and for $n = 1000$ this number is 8. The order of effective error are $O\left(10^{-6}\right)$ in Table 8 and $O\left(10^{-8}\right)$ in Table 9.

Table 7

t_i	$\overline{x_8(t_i)}$	$x^*(t_i)$	e_i
0.000	1.000000000000000	1.000000000000000	0.000000e+000
0.075	1.077812500000000	1.077884150884631	7.165088e-005
0.150	1.161687791028313	1.161834242728283	1.464517e-004
0.250	1.252097770563387	1.252322716191864	2.249456e-004
0.300	1.349551088223366	1.349858807576003	3.077194e-004
0.375	1.454596017661094	1.454991414618201	3.953970e-004
0.450	1.567823530419636	1.568312185490168	4.886551e-004
0.525	1.689870632585210	1.690458848379091	5.882158e-004
0.600	1.821423936077122	1.822118800390508	6.948643e-004
0.675	1.963223535265628	1.964032975969847	8.094407e-004
0.750	2.116067156192846	2.117000016612674	9.328604e-004

Table 8

t_i	$\overline{x_9(t_i)}$	$x^*(t_i)$	e_i
0.000	1.000000000000000	1.000000000000000	0.0000000000
0.075	1.077883433744976	1.077884150884631	7.171397e-007
0.150	1.161832776945772	1.161834242728283	1.465783e-006
0.255	1.290459043596952	1.290461620872889	2.577276e-006
0.300	1.349855727772528	1.349858807576003	3.079803e-006
0.375	1.454987457317024	1.454991414618201	3.957301e-006
0.450	1.568307294866200	1.568312185490168	4.890624e-006
0.525	1.690452961348848	1.690458848379091	5.887030e-006
0.600	1.822111846046275	1.822118800390508	6.954344e-006
0.675	1.964024874959785	1.964032975969847	8.101010e-006
0.750	2.116990680462500	2.117000016612674	9.336150e-006

Table 9

t_i	$\overline{x_8}\,(t_i)$	$x^*\,(t_i)$	e_i
0.000	1.000000000000000	1.000000000000000	0.0000000000
0.075	1.077884143713172	1.077884150884631	7.171459e-009
0.150	1.161834228070331	1.161834242728283	1.465795e-008
0.255	1.290461595099910	1.290461620872889	2.577298e-008
0.300	1.349858776777707	1.349858807576003	3.079830e-008
0.375	1.454991375044856	1.454991414618201	3.957334e-008
0.450	1.568312136583521	1.568312185490168	4.890665e-008
0.525	1.690458789508301	1.690458848379091	5.887079e-008
0.600	1.822118730846495	1.822118800390508	6.954401e-008
0.675	1.964032894959086	1.964032975969847	8.101076e-008
0.750	2.116999923250417	2.117000016612674	9.336226e-008

We see that for the proposed method the order of effective error is $O\left(10^{-4}\right)$, (more exactly, varies between 6.88×10^{-4} and 1.19×10^{-3}), for stepsize $h = 0.1$, $O\left(10^{-6}\right)$ for stepsize $h = 0.01$ and $O\left(10^{-8}\right)$ for stepsize $h = 0.001$.

6. Conclusions

The aim of the present paper is to provide a new numerical method for initial value problems associated to second order differential equations with deviating argument This method combines the Picard sequence of successive approximations, the trapezoidal quadrature rule, and the piecewise Birkhoff cubic interpolation. The interpolation procedure is used only on the points where the modification of the argument appears. The algorithm has a recurrent form easy to program. By using the error estimates a practical stopping criterion of the algorithm is obtained. Moreover, the numerical stability of the method is proved and tested.

The main results of the paper consist in Theorems 4.1 and 4.8 that demonstrate the convergence and the numerical stability of the method. The proposed method is convergent even in the case of Lipschitzian kernel function (in each argument), therefore the smoothness conditions and boundedness conditions are not necessary. That extends the applicability of the method. The above presented numerical examples illustrate the accuracy of the method.The convergence and numerical stability are tested for stepsize $h = 0.1$, $h = 0.01$, and $h = 0.001$, and the order of effective error is $O\left(10^{-4}\right)$, $O\left(10^{-6}\right)$, and $O\left(10^{-8}\right)$ respectively. The principle of the method, i.e. the use, in the numerical integration, of an interpolation procedure only on the points where the argument is modified, gives its generality, the method being applicable to other types of functional equations with modified argument.

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