DENSITY OF THE SET OF RENORMINGS IN $c_0$ WITHOUT ASYMPTOTICALLY ISOMETRIC COPIES OF $c_0$ AND FAILING TO HAVE THE FIXED POINT PROPERTY

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Dedicated to K. Goebel on the occasion of his retirement, and to L. Ciric, W.A. Kirk and I. A. Rus on the occasion of their 75th birthday

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Abstract. It is proved that the family of all equivalent norms in $c_0$ without asymptotically isometric copies of $c_0$ and failing to have the fixed point property is dense in the set of all renormings of $c_0$.

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1. Introduction

A Banach space is said to have the fixed point property (FPP) if every nonexpansive mapping defined from a closed convex bounded subset into itself has a fixed point. Recall that a mapping defined on a subset of a Banach space is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in the domain of $T$. It is well-known that the geometry of the Banach space plays a fundamental role to assure the FPP (see the monographs [8], [10] and the references therein). It is also classical that the sequence Banach spaces $\ell_1$ and $c_0$ fail to have the FPP. In 1997, P. Dowling, C. Lennard and B. Turett [2] proved that a Banach space fails to have the FPP whenever this space contains a sequence which generates either an asymptotically isometric copy of $c_0$ or an asymptotically isometric copy of $\ell_1$ (see also Chapter 9 in [10]). In the last years, asymptotically isometric copies of $c_0$ and $\ell_1$ have become very helpful in the development of the fixed point theory for nonexpansive mappings. Actually, many classes of nonreflexive Banach spaces are known to fail the FPP because it is proved that they contain one of such copies (see for instance [3], [5], [6] or [13]).

It is noteworthy that the non-expansiveness condition strongly depends on the given norm in the Banach space $X$. If $T$ is nonexpansive for a norm $\| \cdot \|$, $T$ may fail

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to be nonexpansive for another equivalent norm $\| \cdot \|$ in $X$. In fact, it was proved by P.K. Lin [12] that $\ell_1$ can be renormed to have the fixed point property for nonexpansive mappings and it is still an open problem whether there exists an equivalent norm in $c_0$ satisfying such condition. On the other hand, not all equivalent norms in $c_0$ contain an asymptotically isometric copy of $c_0$ [2]. In fact, in [9] a dense family of equivalent norms in $c_0$ failing to have an asymptotically isometric copy of $c_0$ is given. In [1] T. Domínguez-Benavides proved that the set of all equivalent norms in $c_0$ which fail to have the FPP is also dense in the set of all renormings of $c_0$. The main object of this paper is to unify these two results. That is, we will prove that for every equivalent norm in $c_0$, we can obtain a renorming of $c_0$ as close as the first one and we like and which fails both, to have the FPP and to have an asymptotically isometric copy of $c_0$, which improves the density results given for the Banach space $c_0$ in [1] and [9].

2. Preliminaries

In this section we introduce the definitions, notation and the known results that we will use in the proof of the main theorem.

**Definition 2.1.** A Banach space $(X, \| \cdot \|)$ is said to have an asymptotically isometric copy of $c_0$ if there are a null sequence $(\epsilon_n)$ in $(0, 1)$ and a sequence $(x_n)$ in $X$ so that

$$\sup_{n \in \mathbb{N}} (1 - \epsilon_n) |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sup_{n \in \mathbb{N}} |t_n|,$$

for all $(t_n) \in c_0$. In this case we say that $X$ contains a $c_0$-a.i. copy.

Given $(X, \| \cdot \|)$ a Banach space we define by $\mathcal{P}(X)$ the set of all equivalent norms on $X$ endowed with the metric [7]:

$$\rho(p, q) = \sup\{|p(x) - q(x)| : \|x\| \leq 1\}, \quad \text{if } p, q \in \mathcal{P}(X).$$

We consider the Banach space $c_0$ endowed with its usual norm given by $\|x\|_{\infty} = \sup_n |t_n|$ whenever $x = (t_n)_{n} \in c_0$.

Similarly to [9], we set:

$$\mathcal{P}_{FPP}(c_0) := \{p \in \mathcal{P}(c_0) : (c_0, p) has \text{ the FPP } \},$$

and

$$\mathcal{P}_0(c_0) := \{p \in \mathcal{P}(c_0) : (c_0, p) fails \text{ to have a } c_0\text{-a.i. copy } \}.$$

We know that $\mathcal{P}_{FPP}(c_0) \subset \mathcal{P}_0(c_0)$ [2] although it is an open problem whether $\mathcal{P}_{FPP}(c_0)$ could be nonempty. On the other hand, $\mathcal{P}_0(c_0) \setminus \mathcal{P}_{FPP}(c_0) \neq \emptyset$. Indeed, let us consider the following equivalent norm on $c_0$:

$$|x|_{\infty} := \sup_n |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n}$$

if $x = (t_n)_{n} \in c_0$. It can be proved that $(c_0, |\cdot|_{\infty})$ fails to have both, an asymptotically isometric copy of $c_0$ [2], and the FPP [5].

We recall James’ Lemma:
**Lemma 2.2.** [11] Let $(X, ||·||)$ be a Banach space which contains a subspace isomorphic to $c_0$. Then for any positive number $\delta > 0$ there is a sequence $(y_n)$ in $X$ such that

$$(1 - \delta) \sup_n |t_n| < \left\| \sum_n t_n y_n \right\| \leq \sup_n |t_n|$$

for all sequences $(t_n) \in c_0$.

In [1] T. Domínguez-Benavides proved that $P(c_0) \setminus P_{FPP}(c_0)$ is dense in $P(c_0)$. The main object of this manuscript is to prove that $P_0(c_0) \setminus P_{FPP}(c_0)$ is also dense in $P(c_0)$, that is, that for every equivalent norm $p$ in $c_0$ we are able to find another equivalent norm $q$ as close as $p$ as we like in the $\rho$ metric such that $q$ fails to have both, the FPP and to have an asymptotically isometric copy of $c_0$. Notice that $P_0(c_0) \setminus P_{FPP}(c_0)$ is a set of equivalent norms of $c_0$ which is strictly contained in $P(c_0) \setminus P_{FPP}(c_0)$.

3. Main result

**Theorem 3.1.** The set $P_0(c_0) \setminus P_{FPP}(c_0)$ is dense in $P(c_0)$.

Proof. Let $p$ be any norm in $P(c_0)$ and $\epsilon > 0$. Take a constant $b$ such that $p(x) \leq b\|x\|_{\infty}$ for all $x \in c_0$ and $\delta$ a positive number such that

$$b \frac{\delta}{1 - \delta} < \frac{\epsilon}{2}$$

Using Lemma 2.2 we obtain a sequence $(y_n)$ in $c_0$ such that

$$(1 - \delta) \sup_n |t_n| \leq p \left( \sum_n t_n y_n \right) \leq \sup_n |t_n|$$

for all $(t_n) \in c_0$.

Moreover, following James’ proof (see [11], Lemma 2.2) we can achieve that the sequence $(y_n)_n$ is finitely supported and that supp$(y_n) \cap$ supp$(y_m) = \emptyset$ if $n \neq m$, where by supp$(y)$ we denote the support of the vector $y$. Moreover, from the proof of Lemma 2.2 in [11] we can also deduce that $\|y_n\|_{\infty} = \|y_m\|_{\infty}$ if $n, m \in \mathbb{N}$.

Let us define $Y := \text{span}(y_n)$ which is a subspace of $c_0$. We define in the subspace $Y$ the equivalent norm $|y| := \sup_n |t_n|$ whenever $y = \sum_n t_n y_n$. According to the above inequalities we have

$$(1 - \delta) |y| \leq p(y) \leq |y|$$

for all $y \in Y$.

Since $Y$ is a closed subspace of $c_0$, applying Lemma 2.2 in [1], we can extend the norm $|·|$ to the whole Banach space $c_0$ keeping the same equivalent constants, that is, there exists a norm $q$ in $c_0$ such that $q(y) = |y|$ whenever $y \in Y$ and

$$(1 - \delta) q(x) \leq p(x) \leq q(x),$$

for all $x \in c_0$. 

Using now Theorem 4.1 in [9], for every \( \lambda > 0 \), the norm \( q + \lambda \cdot \| \cdot \|_\infty \) fails to have an asymptotically isometric copy of \( c_0 \), where \( \| \cdot \|_\infty \) was defined in Section 2. Let us prove that \((c_0, q + \lambda \cdot \| \cdot \|_\infty)\) also fails to have the FPP for every \( \lambda > 0 \):

For \( x = \sum_n t_ne_n \in c_0 \), we set

\[
\eta(x) := \sum_n \frac{|t_n|}{2^n}.
\]

With this notation \( \|x\|_\infty = \|x\|_\infty + \eta(x) \) for all \( x \in c_0 \). Notice that \( \eta(ax + by) = |a|\eta(x) + |b|\eta(y) \) if \( x, y \) are two vectors in \( c_0 \) with disjoint supports and \( a, b \in \mathbb{R} \). Since the sequence \((y_n)\) is bounded and disjointly supported, we can check that \( \lim_n \eta(y_n) = 0 \). Hence, without loss of generality, we can extract a subsequence \((y_n)_k \subset (y_n)_n\) such that \( \eta(y_{n(k+1)}) \leq \eta(y_{n_k}) \) for all \( k \in \mathbb{N} \).

Define the set

\[
C := \{ \sum_{k=1}^{\infty} t_k y_{n_k} : 0 \leq t_k \leq 1 \},
\]

which is a closed convex bounded subset of \( c_0 \) and it is contained in \( Y \). Let us consider the mapping \( T : C \to C \) given by

\[
T(\sum_{k=1}^{\infty} t_k y_{n_k}) = y_1 + \sum_{k=1}^{\infty} t_n y_{n(k+1)}
\]

It is easy to check that \( T \) is fixed point free since \((y_{n_k})_k\) is a basic sequence. Let us prove that \( T \) is nonexpansive for the norm \( q + \lambda \cdot \| \cdot \|_\infty \).

Firstly, \( T \) is nonexpansive for the \( q \) norm: set \( x = \sum_{k=1}^{\infty} t_k y_{n_k}, \ y = \sum_{k=1}^{\infty} s_k y_{n_k} \) two vectors in \( C \).

\[
q(Tx - Ty) = |Tx - Ty| = \sup_k |t_k - s_k| = |x - y| = q(x - y).
\]

On the other hand, \( T \) is also nonexpansive for the \( \| \cdot \|_\infty \) norm. Indeed:

\[
|Tx - Ty|_\infty = \left| \sum_{k=1}^{\infty} (t_k - s_k) y_{n(k+1)} \right|_\infty
\]

\[
= \left| \sum_{k=1}^{\infty} (t_k - s_k) y_{n(k+1)} \right|_\infty + \eta \left( \sum_{k=1}^{\infty} (t_k - s_k) y_{n(k+1)} \right)
\]

\[
= \sup_k |t_k - s_k| \| y_{n(k+1)} \|_\infty + \sum_{k=1}^{\infty} |t_k - s_k| \eta(y_{n(k+1)})
\]

\[
\leq \sup_k |t_k - s_k| \| y_{n_k} \|_\infty + \sum_{k=1}^{\infty} |t_k - s_k| \eta(y_{n_k})
\]

\[
= \left| \sum_{k=1}^{\infty} (t_k - s_k) y_{n_k} \right|_\infty + \eta \left( \sum_{k=1}^{\infty} (t_k - s_k) y_{n_k} \right)
\]
\[
\left| \sum_{k=1}^{\infty} (t_k - s_k) y_{nk} \right| = |x - y|_\infty.
\]

The above shows that \( T \) is nonexpansive for the norm \( q + \lambda \cdot | \cdot |_\infty \) for every \( \lambda > 0 \).

Consider \( 0 < \lambda < \epsilon/4 \) and take \( x \in c_0 \) with \( \|x\|_\infty \leq 1 \):

\[
|p(x) - q(x) - \lambda|x|_\infty| \leq |p(x) - q(x)| + \lambda|x|_\infty \\
\leq \frac{1}{b} |p(x) - p(x) + \lambda|x|_\infty| \\
\leq \frac{1}{b} b \|x\|_\infty + 2 \lambda \|x\|_\infty
\]

which implies that \( \rho(p, q + \lambda \cdot | \cdot |_\infty) < \epsilon \) and this finishes the proof.

**Remark 3.2.** A classical way to measure the distance between two isomorphic Banach spaces is to consider the Banach-Mazur distance defined by

\[
d(X, Y) = \inf\{\|T\| \|T^{-1}\| : T : X \to Y \text{ isomorphism } \}.
\]

From the proof of Theorem 3.1 and by using the identity operator between \( c_0 \) endowed with two equivalent norms, we can derive that for every renorming of \( c_0 \) we can find an isomorphic Banach space without the FPP, failing to have an asymptotically isometric copy of \( c_0 \) and whose Banach-Mazur distance is as close to one as we like.

**References**

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