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ITERATIONS AND FIXED POINTS FOR THE BERNSTEIN MAX-PRODUCT OPERATOR

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Abstract. In this paper we study the sequence of successive approximations, the fixed points and the Ishikawa iterates for the Bernstein max-product operator.
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1. Introduction

For a function $f : [0, 1] \to \mathbb{R}_+$ (here $x \in \mathbb{R}_+$ means $x \geq 0$), the Bernstein max-product approximation operator was for the first time defined (and formally studied) in [7], pp. 325-326, by the formula

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)},$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $\bigvee_{k=0}^{n} p_{n,k}(x) = \max_{k=\{0,\dots,n\}}\{p_{n,k}(x)\}$.

Notice that the Bernstein max-product operator is obtained from the linear Bernstein polynomial written in the form

$$B_n(f)(x) = \frac{\sum_{k=0}^{n} p_{n,k}(x) f(k/n)}{\sum_{k=0}^{n} p_{n,k}(x)}$$

and replacing here the "sum" operator by the "maximum" operator.

Surprisingly, with respect to the classical Bernstein polynomials, in the whole class of continuous functions on $[0, 1]$, the max-product Bernstein operators do not loose the approximation properties. Moreover, they present the advantage that for large classes of functions improve the order of approximation to the Jackson-type order.

In more details, it was proved in [3], [4] that $B_n^{(M)}$ is a nonlinear (more exactly sublinear on the space of positive functions) operator, well-defined for all $x \in \mathbb{R}$, and a piecewise rational function on $\mathbb{R}$. Also, in [3] it was proved that $B_n^{(M)}$ possesses some interesting direct approximation results and shape preserving properties. For example, while in general the order of uniform approximation was found to be $\omega_1(f; 1/\sqrt{n})_{[0,1]}$, however, for some subclasses of functions including for example the class of concave functions and also a subclass of the convex functions, the order of approximation is essentially better, namely is $\omega_1(f; 1/n)_{[0,1]}$. In addition, in [3] it was proved that $B_n^{(M)}(f)$ is continuous for any positive function f, preserves the monotonicity and the quasiconvexity of f.

For strictly positive functions, improved direct approximation results by the Bernstein max-product operator we have obtained in [5].

For the classical Bernstein polynomials $B_n(f)(x)$, in the paper of Rus [11] the well-known Kelisky–Rivlin's result in [10] stating that for all $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$ it holds $\lim_{m\to\infty} B_n^m(f)(x) = f(0) + [f(1) - f(0)]x = B_1(f)(x)$ (here $B_n^m(f)$ denotes the mth iterate of the sequence of successive approximations), is proved in a very simple and elegant manner, by using the Banach fixed point theorem. Note here that $B_1(f)(x) = f(0) + [f(1) - f(0)]x$ is a fixed point for the operator B_n.

Also, if $m = m_n$ depends on n and if $\lim_{n\to\infty} \frac{m_n}{n} = 0$, then it is known that (see e.g. [10]) $lim_{n\to\infty} B_n^{m_n}(f)(x) = f(x)$ uniformly in $[0,1]$.

Similar studies for the iterates of other kinds of Bernstein-type operators were obtained via fixed point theory in e.g. Agratini [1], Rus [12] and Agratini-Rus [2].

The main aim of the present paper is to make a similar study for the iterates of the Bernstein max-product operator $B_n^{(M)}$. It is worth noting that due to the fact that $B_n^{(M)}$ is not a contraction (is only a non-expansive operator), the methods used in the case of Bernstein polynomials cannot be used for the Bernstein max-product operators, so that new methods are required.

The plan of the paper goes as follows.

Although the Bernstein max-product operator is not a contraction, as an analogue of the above mentioned Kelisky–Rivlin's results for the Bernstein polynomial, in Section 2 of the present paper firstly we prove by a direct method that for any fixed $n \in \mathbb{N}$ and $f : [0,1] \to [0, +\infty)$, the sequence of successive approximations of the nonlinear operator $B_n^{(M)}$, denoted by $a_n^m(f)(x) = [B_n^{(M)}]^m(f)(x)$, still uniformly converges for $m \to \infty$ to a fixed point of $B_n^{(M)}$. Also, the limits of the double sequence $(a_n^m(f))_{m,n\in\mathbb{N}}$ for other interdependences between m and n are calculated.

In the same section, important subsets of the set of fixed points of the operator $B_n^{(M)}$ are concretely determined.

Finally, in Section 3 we study the convergence of so-called Ishikawa iterates for the operator $B_n^{(M)}$.

2. The sequence of successive approximations and fixed points for $B_n^{(M)}$

For the proof of the convergence of the sequence of successive approximations of $B_n^{(M)}$, we need the following three auxiliary results.

The first result obtained one refers to the fact that unlike the classical Bernstein (linear) operator $B_n(f)$ which is a contraction, the max-product Bernstein (nonlinear) operator $B_n^{(M)}(f)$ is only a nonexpansive operator. This means that the Banach fixed point theorem cannot be applied in this case.

Theorem 2.1. *For any $n \in \mathbb{N}$, the max-product Bernstein operator $B_n^{(M)} : C_+[0,1] \to C_+[0,1]$ is nonexpansive, that is*

$$\|B_n^{(M)}(f) - B_n^{(M)}(g)\| \leq \|f - g\|, \text{ for all } f, g \in C_+[0,1],$$

where $C_+[0,1] = \{f : [0,1] \to \mathbb{R}_+; f$ is continuous on $[0,1]\}$, $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$ and $\|\cdot\|$ denote the uniform norm in $C_+[0,1]$.

Proof. We easily get

$$|B_n^{(M)}(f)(x) - B_n^{(M)}(g)(x)| \leq \frac{\bigvee_{k=0}^{n} |p_{n,k}(x)f(k/n) - p_{n,k}(x)g(k/n)|}{\bigvee_{k=0}^{n} p_{n,k}(x)} \leq \|f - g\|,$$

which proves the theorem. □

Remarks. 1) In general, the inequality in Theorem 2.1 is not strict, that is there exists $f, g \in C_+[0,1]$, such that $\|B_n^{(M)}(f) - B_n^{(M)}(g)\| = \|f - g\|$. Indeed, let us choose, for example, f nonincreasing on $[0,1]$ and $g = 0$ on $[0,1]$. By Corollary 5.6 in [3], it follows that $B_n^{(M)}(f)$ is also nonincreasing on $[0,1]$, which implies that $\|f\| = f(0)$, $\|B_n^{(M)}(f)\| = B_n^{(M)}(f)(0)$ and by the obvious relationship $B_n^{(M)}(f)(0) = f(0)$, it implies $\|B_n^{(M)}(f) - B_n^{(M)}(g)\| = \|B_n^{(M)}(f)\| = f(0) = \|f\| = \|f - g\|$.

2) Note that Lemma 2.5 in [6] shows that for any bounded $f : [0,1] \to \mathbb{R}_+$ and $n \in \mathbb{N}$, $B_n^{(M)}(f) \in Lip_L\, 1$, with $L = Cn^2\|f\|$, $C > 0$ being a constant independent of f and n, where

$$Lip_L\, 1 = \{f : [0,1] \to \mathbb{R}; |f(x) - f(y)| \leq L|x - y|, \text{ for all } x, y \in [0,1]\}.$$

In the next result we obtain an explicit value for C in the above Remark 2.

Theorem 2.2. *For all $f \in C_+[0,1]$ and $h \geq 0$ we have*

$$\omega_1(B_n^{(M)}(f); h) \leq 6\pi e^2 n^2 \|f\| h,$$

where $\omega_1(f; h) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x - y| \leq h\}$ denotes the modulus of continuity.

Proof. Analysing the proof of Lemma 2.5 in [6], we get $\omega_1(B_n^{(M)}(f); h) \leq \frac{1}{c_1^2} n^2 \|f\| h$, where it is easy to observe that the constant $c_1 > 0$ (independent of x and n) comes from Lemma 2.4 in [6] as satisfying the inequality $\bigvee_{k=0}^{n} p_{n,k}(x) \geq \frac{c_1}{\sqrt{n}}$, for all $x \in [0,1]$ and $n \in \mathbb{N}$.

Analysing now the proof of Lemma 2.4 in [6], it easily follows that $c_1 = c_2 \cdot \frac{1}{e}$, where $c_2 > 0$ is now the constant that appear in the statement of Lemma 2.3 in [6] as satisfying

$$\min\left\{p_{n,j}(\frac{j}{n+1}), p_{n,j}(\frac{j+1}{n+1})\right\} \geq \frac{c_2}{\sqrt{n}},$$

for all $n \in \mathbb{N}$, and $j \in \{0, 1, ..., n\}$, where $c_2 > 0$ is an absolute constant independent of n and j.

In continuation, analysing the proof of Lemma 2.3 in [6] and denoting $A_n = \frac{(2^n n!)^2}{(2n)!} \cdot \frac{1}{\sqrt{2n+1}}$, since $\lim_{n\to\infty} A_n = \sqrt{\frac{\pi}{2}}$ and because it is easy to prove that $(A_n)_n$ is increasing, we get

$$\frac{2}{\sqrt{3}} < A_n < \sqrt{\frac{\pi}{2}}, \text{ for all } n \in \mathbb{N}.$$

This immediately implies

$$\frac{(2n)!}{4^n (n!)^2} > \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{\sqrt{n}}, \text{ for all } n \in \mathbb{N}.$$

Therefore, following the lines in the proof of Lemma 2.3 in [6], case (i), we immediately obtain

$$p_n\left(\frac{j}{n+1}\right) > \frac{1}{\sqrt{e}} \cdot \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{\sqrt{n}} = \frac{\sqrt{2}}{\sqrt{3\pi e}} \cdot \frac{1}{\sqrt{n}}.$$

Similarly, following the lines in the proof of Lemma 2.3 in [6], case (ii), we get

$$p_{n,n_1}\left(\frac{n_1+1}{n+1}\right) = \frac{(2n_1)!}{4^{n_1}(n_1)^2} \cdot \frac{2n_1+1}{2n_1+2} > \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{\sqrt{n}} \cdot \frac{1}{2} = \frac{1}{\sqrt{6\pi}} \cdot \frac{1}{\sqrt{n}}.$$

Combining the cases (i) and (ii) in the proof of Lemma 2.3 in [6], since $\frac{\sqrt{2}}{\sqrt{3\pi e}} > \frac{1}{\sqrt{6\pi}}$, it follows that the constant c_2 in the statement of Lemma 2.3 in [6] can be chosen as $c_2 = \frac{1}{\sqrt{6\pi}}$.

In conclusion, going back with the values of the constants, we obtain $c_1 = \frac{1}{\sqrt{6\pi}} \cdot \frac{1}{e}$ and $\frac{1}{c_1^2} = 6\pi e^2$, which finish the proof. □

Also, we present:

Lemma 2.3. *For any $f \in C_+[0,1]$ and $n \in \mathbb{N}$ we have*

$$B_n^{(M)}[B_n^{(M)}(f)](x) \geq B_n^{(M)}(f)(x), \text{ for all } x \in [0,1].$$

Proof. Let us choose arbitrary $j \in \{0, 1, ..., n\}$. By relation (4.17) in [3], one has

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x),\ x \in [j/(n+1), (j+1)/(n+1)], \tag{1}$$

where

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(k/n)$$

for all $k \in \{0, 1, ..., n\}$. Relation (1) implies $B_n^{(M)}(f)(x) \geq f_{k,n,j}(x)$ for all $x \in [j/(n+1), (j+1)/(n+1)]$ and $k \in \{0, 1, ..., n\}$. In particular, for $x = j/n \in [j/(n+1), (j+1)/(n+1)]$ and $k = j$, we get $B_n^{(M)}(f)(j/n) \geq f_{j,n,j}(j/n) = f(j/n)$, $j \in \{0, 1, ..., n\}$. Therefore, taking into account the relationship of definition for $B_n^{(M)}(f)(x)$ in Introduction, we immediately get the statement of the lemma. □

We are now in position to prove the first main result of this section.

Theorem 2.4. *For a fixed* $f \in C_+[0,1]$*, let us consider the iterative sequence of successive approximations* $a_m^{(n)}(f)(x) = [B_n^{(M)}]^m(f)(x)$*,* $m, n \in \mathbb{N}$*,* $x \in [0,1]$*. Here* $[B_n^{(M)}]^2(f)(x) = B_n^{(M)}[B_n^{(M)}(f)](x)$ *and so on.*

(i) For any fixed $n \in \mathbb{N}$*, there exists* $f_n : [0,1] \to \mathbb{R}_+$*, such that* $f_n \in C_+[0,1]$*,* $f_n \in Lip_L\, 1$ *with* $L = 6\pi e^2 n^2 \|f\|$*,* $f_n(0) = f(0)$*,* $f_n(1) = f(1)$*,*

$$\lim_{m \to +\infty} a_m^{(n)}(f) = f_n, \ \ \textit{uniformly in } [0,1],$$

$B_n^{(M)}(f_n)(x) = f_n(x)$ *for all* $x \in [0,1]$ *(that is* f_n *is a fixed point for the operator* $B_n^{(M)}$*) and*

$$B_n^{(M)}(f)(x) = a_1^{(n)}(f)(x) \leq a_m^{(n)}(f)(x) \leq a_{m+1}^{(n)}(f)(x) \leq f_n(x) \leq \|f\|,$$

for all $x \in [0,1]$*,* $m \in \mathbb{N}$*;*

(ii) For all $m, n \in \mathbb{N}$ *and* $x \in [0,1]$*, we have the estimate*

$$|[B_n^{(M)}]^m(f)(x) - f(x)| \leq 12 \cdot \omega_1 \left(f; \frac{m}{\sqrt{n+1}}\right),$$

where $\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; |x - y| \leq \delta\}$*;*

(iii) For any fixed $m \in \mathbb{N}$ *we have* $\lim_{n\to\infty} a_m^{(n)}(f)(x) = f(x)$*, uniformly in* $[0,1]$*;*

(iv) Let $m = m_n$ *depending on* n *such that* $\lim_{n\to\infty} \frac{m_n}{\sqrt{n}} = 0$*. Then we have* $\lim_{n\to\infty} a_{m_n}^{(n)}(f)(x) = f(x)$*, uniformly in* $[0,1]$*;*

(v) Suppose, in addition, that $f \in Lip_L\, 1$ *and that it is strictly positive on* $[0,1]$*. Then, for all* $m, n \in \mathbb{N}$ *we have the estimate*

$$\|[B_n^{(M)}]^m(f) - f\| \leq \frac{m}{n} \cdot L \left(\frac{L}{m_f} + 4\right),$$

where $m_f = \inf\{f(x); x \in [0,1]\} > 0$*;*

(vi) Suppose that $f \in Lip_L\, 1$ *and that it is strictly positive on* $[0,1]$*. Let* $m = m_n$ *depending on* n *such that* $\lim_{n\to\infty} \frac{m_n}{n} = 0$*. Then uniformly on* $[0,1]$ *we have* $\lim_{n\to\infty} a_{m_n}^{(n)}(f)(x) = f(x)$*.*

(vii) Suppose that $f \in C_+[0,1]$ *is such that for any* $n \in \mathbb{N}$*, the function* $B_n^{(M)}(f)$ *is a fixed point for the operator* $B_n^{(M)}$*. Then, for any sequence of natural numbers,* $(m_n)_{n\in\mathbb{N}}$*, the sequence of iterates* $a_{m_n}^{(n)}(f) = [B_n^{(M)}]^{m_n}(f)$ *converges uniformly on* $[0,1]$ *to* f*, as* $n \to \infty$*.*

Proof. (i) From the above Lemma 2.3, easily follow the inequalities

$$0 \leq B_n^{(M)}(f)(x) = a_1^{(n)}(f)(x) \leq ... \leq a_m^{(n)}(f)(x) \leq a_{m+1}^{(n)}(f)(x) \leq ... \leq \|f\|,$$

for all $m, n \in \mathbb{N}$. The last inequality follows from the obvious inequality $0 \leq B_n^{(M)}(f)(x) \leq \|f\|$.

Fixing $n \in \mathbb{N}$ and $x \in [0,1]$, the sequence of positive numbers $(a_m^{(n)}(f)(x))_{m\in\mathbb{N}}$ is bounded and monotonically nondecreasing, which implies, for $m \to +\infty$, its convergence to a limit, denote it by $f_n(x)$. Since $B_n^{(M)}(f)(x) \leq \|f\|$, we easily obtain $a_m^{(n)}(f)(x) \leq \|f\|$, for all m, that is the sequence $(a_m^{(n)}(f))_{m\in\mathbb{N}}$ is uniformly bounded. Passing to limit with $m \to +\infty$ we get $f_n(x) \leq \|f\|$ for all $x \in [0,1]$, $n \in \mathbb{N}$.

Also, since it is easy to check that $B_n^{(M)}(f)(0) = f(0)$ and $B_n^{(M)}(f)(1) = f(1)$, it is immediate that $a_m^{(n)}(f)(0) = f(0)$ and $a_m^{(n)}(f)(1) = f(1)$ for all $m \in \mathbb{N}$, which therefore implies that $f_n(0) = f(0)$, $f_n(1) = f(1)$.

Now, from $\|B_n(f)\| \le \|f\|$ and applying successively Theorem 2.2, we easily obtain that $a_m^{(n)}(f) = [B_n^{(M)}]^m(f) \in Lip_L 1$, for all $m \in \mathbb{N}$. Therefore, the sequence (of functions of successive approximation) $(a_m^{(n)}(f))_{m\in\mathbb{N}}$ clearly is equicontinuous, which combined with the fact that the sequence is uniformly bounded, by the Arzela-Ascoli theorem implies that it contains a subsequence $(a_{m_k}^{(n)}(f))_{k\in\mathbb{N}}$, uniformly convergent. Because the whole sequence is pointwise convergent to $f_n(x)$, we get that $lim_{k\to\infty} a_{m_k}^{(n)}(f) = f_n$ uniformly in $[0,1]$ and as a consequence, it immediately follows that $f_n \in C_+[0,1]$, in fact moreover, that $f_n \in Lip_L\, 1$ with $L = 6\pi e^2 n^2 \|f\|$.

Applying now the well-known Dini's theorem to the pointwise convergent monotone sequence of continuous functions $(a_m^{(n)}(f))_{m\in\mathbb{N}}$, it follows that in fact we have $lim_{m\to\infty} a_m^{(n)}(f) = f_n$ uniformly in $[0,1]$.

Also, the monotonicity of the sequence $(a_m^{(n)})_{m\in\mathbb{N}}$ implies $a_m^{(n)}(f)(x) \le f_n(x)$ for all $x \in [0,1]$, $m, n \in \mathbb{N}$.

Finally, since $a_{m+1}^{(n)}(f) = B_n^{(M)}[a_m^{(n)}(f)]$ and $lim_{m\to\infty} a_{m+1}^{(n)}(f) = f_n$ uniformly in $[0,1]$, taking also into account that by Theorem 2.1, $B_n^{(M)}$ is nonexpansive, for any fixed n it follows that for all $m \in \mathbb{N}$ we have

$$\|B_n^{(M)}(f_n) - f_n\| \le \|B_n^{(M)}(f_n) - a_{m+1}^{(n)}(f)\| + \|a_{m+1}^{(n)}(f) - f_n\|$$

$$\le \|f_n - a_m^{(n)}(f)\| + \|a_{m+1}^{(n)}(f) - f_n\|.$$

Passing here with $m \to \infty$, we get $\|B_n^{(M)}(f_n) - f_n\| = 0$, that is $B_n^{(M)}(f_n)(x) - f_n(x) = 0$, for all $x \in [0,1]$.

(ii) For any fixed $m \in \mathbb{N}$ and $n \in \mathbb{N}$ variable, it is easy to see that the sequence $([B_n^{(M)}]^m(f))_{n\in\mathbb{N}}$ satisfies the Corollary 2.4 in [3], that is for all $\delta > 0$ we get

$$|[B_n^{(M)}]^m(f)(x) - f(x)| \le \left[1 + \frac{1}{\delta}[B_n^{(M)}]^m(\varphi_x)(x)\right] \omega_1(f;\delta), x \in [0,1],$$

where $\varphi_x(t) = |t - x|$, for all $t \in [0,1]$.

In what follows we prove by mathematical induction that $[B_n^{(M)}]^m(\varphi_x)(x) \le 6 \cdot \frac{m}{\sqrt{n+1}}$, for all $m, n \in \mathbb{N}$, $x \in [0,1]$, which replaced in the above estimate and by choosing then $\delta = 6 \cdot \frac{m}{\sqrt{n+1}}$, will immediately imply

$$|[B_n^{(M)}]^m(f)(x) - f(x)| \le 12 \cdot \omega_1\left(f; \frac{m}{\sqrt{n+1}}\right).$$

Indeed, denoting

$$m_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-j},$$

by [3], relationship (4.17), we can write

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} m_{k,n,j}(x) f\left(\frac{k}{n}\right), \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

This immediately implies

$$\begin{aligned}[B_n^{(M)}]^2(f)(x) &= \bigvee_{k=0}^{n} m_{k,n,j}(x) B_n^{(M)}(f)(k/n) \\ &= \bigvee_{k=0}^{n} m_{k,n,j}(x) \left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n) f(i/n)\right].\end{aligned}$$

Replacing here $f(t) = |t - x| = \varphi_x(t)$ with x fixed, and taking into account the inequality

$$\left|\frac{i}{n} - x\right| \leq \left|\frac{i}{n} - \frac{k}{n}\right| + \left|\frac{k}{n} - x\right|,$$

for all $x \in [j/(n+1), (j+1)/(n+1)]$ we get

$$\begin{aligned}[B_n^{(M)}]^2(\varphi_x)(x) &= \bigvee_{k=0}^{n} m_{k,n,j}(x) \left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n) \left|\frac{i}{n} - x\right|\right] \\ &\leq \bigvee_{k=0}^{n} m_{k,n,j}(x) \left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n) \left|\frac{k}{n} - \frac{i}{n}\right|\right] \\ &+ \bigvee_{k=0}^{n} m_{k,n,j}(x) \left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n) \left|\frac{k}{n} - x\right|\right] \\ &= \bigvee_{k=0}^{n} m_{k,n,j}(x) \left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n) \left|\frac{k}{n} - \frac{i}{n}\right|\right] \\ &+ \bigvee_{k=0}^{n} m_{k,n,j}(x) \left|\frac{k}{n} - x\right| \left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n)\right] \\ &\leq 6 \cdot \frac{1}{\sqrt{n+1}} + 6 \cdot \frac{1}{\sqrt{n+1}} = 6 \cdot \frac{2}{\sqrt{n+1}}.\end{aligned}$$

For the last estimate we used the inequalities which follow from the relationship (4.6) in the proof of Theorem 4.1 in [3]

$$m_{k,n,j}(x) \left|\frac{k}{n} - x\right| \leq \frac{6}{\sqrt{n+1}}, \quad m_{i,n,k}(k/n) \left|\frac{k}{n} - \frac{i}{n}\right| \leq \frac{6}{\sqrt{n+1}}$$

and the inequalities obtained from Lemma 3.2 in [3]

$$m_{k,n,j}(x) \leq 1, \quad m_{i,n,k}(k/n) \leq 1.$$

Similarly, taking into account that for all $x \in [j/(n+1), (j+1)/(n+1)]$ we can write

$$[B_n^{(M)}]^3(f)(x)$$

$$= \bigvee_{k=0}^{n} m_{k,n,j}(x) \left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n) \left[\bigvee_{l=0}^{n} m_{l,n,i}(i/n) f(l/n) \right] \right],$$

replacing here $f(t) = |t - x| = \varphi_x(t)$, taking into account the inequality

$$\left| \frac{l}{n} - x \right| \leq \left| \frac{l}{n} - \frac{i}{n} \right| + \left| \frac{i}{n} - \frac{k}{n} \right| + \left| \frac{k}{n} - x \right|,$$

and reasoning exactly as in the case of $[B_n^{(M)}]^2$, we easily obtain

$$[B_n^{(M)}]^3(\varphi_x)(x) \leq 6 \cdot \frac{3}{\sqrt{n+1}}, \; x \in [j/(n+1), (j+1)/(n+1)],$$

valid for all $j = 0, 1, ..., n$. Therefore, the above inequality is in fact valid for all $x \in [0, 1]$.

Reasoning now by mathematical induction, we get the desired estimate in the statement for arbitrary $m \in \mathbb{N}$.

(iii) It is immediate by passing to limit with $n \to \infty$ in the inequality from the above point (ii).

(iv) It is immediate by replacing m with m_n in the estimate in (ii), by passing to limit with $n \to \infty$ and taking into account that $\lim_{n\to\infty} \frac{m_n}{\sqrt{n+1}} = 0$.

(v) We obviously can write

$$\|[B_n^{(M)}]^m(f) - f\| \leq \sum_{j=1}^{m} \|[B_n^{(M)}]^j(f) - [B_n^{(M)}]^{j-1}(f)\|,$$

where by convention $[B_n^{(M)}]^0(f)(x) = f(x)$.

But by applying successively Theorem 2.1, we easily get that

$$\|[B_n^{(M)}]^j(f) - [B_n^{(M)}]^{j-1}(f)\| \leq \|[B_n^{(M)}]^{j-1}(f) - [B_n^{(M)}]^{j-2}(f)\|$$
$$\leq ... \leq \|[B_n^{(M)}](f) - (f)\| \leq \omega_1 \left(f; \frac{1}{n}\right) \cdot \left[\frac{n \cdot \omega_1(f; 1/n)}{m_f} + 4 \right],$$

where for the last estimate above we used Theorem 4.6 in [5], valid for strictly positive functions only.

Now, taking into account that $f \in Lip_L 1$, from the above estimate we get

$$\|[B_n^{(M)}]^j(f) - [B_n^{(M)}]^{j-1}(f)\| \leq \frac{1}{n} \left[L \left(\frac{L}{m_f} + 4 \right) \right],$$

for all $j = 1, ..., m$, which finally implies

$$\|[B_n^{(M)}]^m(f) - f\| \leq \frac{m}{n} \left[L \left(\frac{L}{m_f} + 4 \right) \right].$$

(vi) It is immediate by taking $m = m_n$ and passing to limit in the estimate from the above point (v).

(vii) By hypothesis, we have $B_n^{(M)}[B_n^{(M)}(f)] = B_n^{(M)}(f)$, for all $n \in \mathbb{N}$, and therefore it easily follows that $[B_n^{(M)}]^{m_n}(f) = B_n^{(M)}(f)$, for all $n \in \mathbb{N}$. Consequently, by Theorem 4.1 in [3], we obtain

$$\|[B_n^{(M)}]^{m_n}(f)(x) - f(x)| = |B_n^{(M)}(f)(x) - f(x)| \leq 12 \cdot \omega_1(f; 1/\sqrt{n+1}),$$

and passing to limit with $n \to \infty$, we immediately get the desired conclusion. $\square$

Remarks. 1) In the class of Lipschitz, strictly positive functions, Theorem 2.4, (vi), is more general than Theorem 2.4, (iv). Indeed, while $lim_{n\to\infty} \frac{m_n}{\sqrt{n}} = 0$ implies $lim_{n\to\infty} \frac{m_n}{n} = 0$, the converse is not true. Note that the case of Theorem 2.4, (vi), is similar with what happens in the case of the iterates of Bernstein polynomials.

2) As a consequence of the well-known Trotter's approximation result in the theory of the semigroups of linear operators (see e.g. [9]), it is known that in the case of Bernstein polynomials $B_n(f)(x)$, if f is twice differentiable and $\lim_{n\to\infty} \frac{m_n}{n} = t > 0$, then $\lim_{n\to\infty} B_n^{m_n}(f)(x) = e^{tA(x)}$, where $A(x) = \frac{x(1-x)f''(x)}{2}$, for all $x \in [0,1]$.

It remains as an interesting open question what happens with the iterates $[B_n^{(M)}]^{m_n}(f)$, when $\lim_{n\to\infty} \frac{m_n}{n} = t > 0$. Let us first observe that by Theorem 2.4, (vii), if f satisfies the hypothesis there, then $[B_n^{(M)}]^{m_n}(f)$ uniformly converges to f on $[0,1]$. It is worth mentioning that by the next Theorems 2.5 and 2.6, we put in evidence large classes of functions f satisfying the hypothesis in Theorem 2.4, (vii). Therefore, the above mentioned open problem for the Bernstein max-product operator, gets a sense only if f does not satisfy the hypothesis in Theorem 2.4, (vii). Also, notice here that the Bernstein max-product operator $[B_n^{(M)}]^{m_n}$ is not linear.

3) If f is a fixed point of $B_n^{(M)}$, i.e. $f(x) = B_n^{(M)}(f)(x)$ for all $x \in [0,1]$, we easily get $a_m^{(n)}(f)(x) = B_n^{(M)}(f)(x)$, for all $m \in \mathbb{N}$, $x \in [0,1]$, therefore in this case it is trivial in Theorem 2.4, (i), that $f_n(x) = B_n^{(M)}(f)(x)$, for all $x \in [0,1]$.

4) According to Theorem 2.4, (i), for each fixed $n \in \mathbb{N}$ it is important to determine the set of the fixed points for $B_n^{(M)}$. In this sense, we present the following results.

Theorem 2.5. *(i) If $f : [0,1] \to [0,\infty)$ is nondecreasing and such that the function $g : (0,1] \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing, then for any $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is a fixed point for the operator $B_n^{(M)}$, that is $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0,1]$;*

(ii) If $f : [0,1] \to [0,\infty)$ is nonincreasing and such that the function $h : [0,1) \to [0,\infty), h(x) = \frac{f(x)}{1-x}$ is nondecreasing, then for any $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is a fixed point for the operator $B_n^{(M)}$, that is $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0,1]$.

Proof. (i) From the relations (4.46) and (4.47) in the proof of Corollary 4.7 in [3], for all $x \in [j/(n+1),(j+1)/(n+1)]$ and $j \in \{0,1,...,n-1\}$ we can write

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}$$

and

$$B_n^{(M)}(f)(x) = f(1), \text{ for } x \in [n/(n+1),1],$$

where

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(k/n).$$

Taking above $x = j/n$, by simple calculation we obtain

$$B_n^{(M)}(f)(j/n) = \max\{f(j/n), f[(j+1)/n] \cdot j/(j+1)\},$$

which by the property of the auxiliary function g in hypothesis, implies $f(j/n) \geq \frac{j}{j+1} f[(j+1)/n]$, which replaced in the above equality gives $B_n^{(M)}(f)(j/n) = f(j/n)$.

But it is clear that if for $f \in C_+[0,1]$ we have $B_n^{(M)}(f)(j/n) = f(j/n)$ for all $j \in \{0, 1, ..., n\}$, then $g = B_n^{(M)}(f)$ is a fixed point for $B_n^{(M)}$, which implies the desired conclusion.

(ii) From the relations (4.49) and (4.50) in the proof of Corollary 4.7 in [3], for all $x \in [j/(n+1), (j+1)/(n+1)]$ and $j \in \{1, ..., n\}$ we can write

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x)\},$$

and

$$B_n^{(M)}(f)(x) = f(0), \text{ for } x \in [0, 1/(n+1)].$$

Taking above $x = j/n$, by simple calculation we obtain

$$B_n^{(M)}(f)(j/n) = \max\{f[(j-1)/n] \cdot (n-j)/(n-j+1), f(j/n)\},$$

which by the property of the auxiliary function g in hypothesis, implies $f(j/n) \geq \frac{n-j}{n-j+1} f[(j-1)/n]$, which replaced in the above equality gives $B_n^{(M)}(f)(j/n) = f(j/n)$.

Therefore, we again get the desired conclusion. □

Remarks. 1) According to Remark 4.8 in [3], if $f : [0,1] \to [0,\infty)$ is a convex, nondecreasing function satisfying $\frac{f(x)}{x} \geq f(1)$ for all $x \in [0,1]$, or if $f : [0,1] \to [0,\infty)$ is a convex, nonincreasing function satisfying $\frac{f(x)}{1-x} \geq f(0)$, then again f satisfies the hypothesis in Theorem 2.5, (i) and (ii), respectively, and consequently we get $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0,1]$.

2) Denote by $\mathcal{S}[0,1]$ the class of all functions f which satisfy the hypothesis in the statement of Theorem 2.5 (i), or of Theorem 2.5 (ii), or in the above Remark 1. Also, for any fixed arbitrary $n \in \mathbb{N}$, let us denote

$$\mathcal{T}_n^{(M)}[0,1] = B_n^{(M)}(\mathcal{S}[0,1])$$

$$= \{F \in C_+[0,1]; \exists f \in \mathcal{S}[0,1] \text{ such that } F(x) = B_n^{(M)}(f)(x), \forall x \in [0,1]\}.$$

Then if we denote by

$$\mathcal{F}_n^{(M)}[0,1] = \{F : [0,1] \to [0,+\infty); B_n^{(M)}(F)(x) = F(x), \text{ for all } x \in [0,1]\}$$

the set of all fixed points of the operator $B_n^{(M)} : C_+[0,1] \to C_+[0,1]$, the statement of Theorem 2.5 together with the above Remark 1 means that we have $\mathcal{T}_n^{(M)}[0,1] \subset \mathcal{F}_n^{(M)}[0,1]$.

3) By Lemma 4.6 in [3], any nondecreasing concave function satisfies the hypothesis of Theorem 2.5, (i), and any nonincreasing concave function satisfies the hypothesis of Theorem 2.5, (ii). Therefore, the class of all positive, monotone and concave functions on $[0,1]$ denoted by $MK_+[0,1]$, has the property $MK_+[0,1] \subset S[0,1]$, that is the function $H = B_n^{(M)}(f)$ satisfies $B_n^{(M)}(H)(x) = H(x)$, for all $x \in [0,1]$.

4) It is easy to consider concrete examples of functions in $S[0,1]$ (others than the constant functions which obviously are fixed points for $B_n^{(M)}$), like

$$x,\ e^x, 1+x^2, sin(x), cos(x), ln(1+x), e^{-x}, 1+x^3.$$

Indeed, it is easy to check that x, e^x and $1+x^2$ satisfy the first type of hypothesis in the above Remark 1, $sin(x)$, $cos(x)$ and $ln(1+x)$ belong to the class $MK_+[0,1]$ defined in the above Remark 3, while e^{-x} satisfy the second type of hypothesis in the above Remark 1. Therefore, for any f in this remark we have $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0,1]$ and $n \in \mathbb{N}$.

The results expressed by the above Remark 3 can be generalized to the whole class of concave functions, as follows.

Theorem 2.6. *If* $f : [0,1] \to [0,\infty)$ *is a continuous concave function then we have* $B_n^{(M)}\left[B_n^{(M)}(f)\right] = B_n^{(M)}(f)$ *for all* $n \in \mathbb{N}$.

Proof. By the proof of Corollary 4.6. in [3] we get

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\}$$

for all $x \in [j/(n+1),(j+1)/(n+1)]$ and $j \in \{1,...,n-1\}$,

$$B_n^{(M)}(f)(x) = \max\{f_{0,n,0}(x), f_{0,n,1}(x)\} \text{ for all } x \in [0,1/(n+1)]$$

and

$$B_n^{(M)}(f)(x) = \max\{f_{n,n,n-1}(x), f_{n,n,n}(x)\}, \text{ for all } x \in [n/(n+1),1].$$

Here recall that

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(k/n).$$

Since $j/n \in [j/(n+1),(j+1)/(n+1)]$, replacing $x = j/n$ in the above formulas for $B_n^{(M)}(f)(x)$, we easily obtain (see the reasonings in the proof of Theorem 2.5, (i) and (ii)) that $B_n^{(M)}(f)(j/n) = f(j/n)$ for all $j \in \{0,1,...,n\}$, which form the formula of definition of $B_n^{(M)}(f)(x)$ easily implies the desired conclusion. □

Remarks. 1) Theorems 2.5 and 2.6 put in evidence large classes of functions $f \in C_+[0,1]$, with the property that $B_n^{(M)}(f)$ is a fixed point for the operator $B_n^{(M)}$, for all $n \in \mathbb{N}$.

The following example of f is that of a function for which there exists $n \in \mathbb{N}$ (in fact an infinity of such of n) such that $B_n^{(M)}(f)$ is not anymore fixed point for the operator $B_n^{(M)}$. Indeed, let $f : [0,1] \to [0,\infty)$ be defined by $f(x) = 1/2 - x$ if $x \in [0,1/2]$ and $f(x) = x - 1/2$ if $x \in (1/2,1]$. For $n = 5$, by the formula of definition of $B_n^{(M)}(f)(x)$ in Introduction, we easily get

$$\begin{aligned} B_5^{(M)}(f)(0) &= B_5^{(M)}(f)(1) = 1/2, \\ B_5^{(M)}(f)(1/5) &= B_5^{(M)}(f)(4/5) = 2/5, \\ B_5^{(M)}(f)(2/5) &= B_5^{(M)}(f)(3/5) = 9/40, \end{aligned}$$

and

$$B_5^{(M)}(B_5^{(M)}(f))(2/5) = 3/10.$$

Therefore, it follows $B_5^{(M)}(B_5^{(M)}(f))(2/5) \neq B_5^{(M)}(f)(2/5)$, which clearly implies that $B_5^{(M)}(f)$ is not a fixed point for the operator $B_5^{(M)}$.

In fact, by using for example MATLAB, one can easily show that for many other values of n (sufficiently large), again we get the same conclusion.

2) Theorem 2.6 is also useful to show that the method in the case of Bernstein polynomials in [11] cannot be use here, because for any $a, b \in \mathbb{R}_+$, the operator $B_n^{(M)}$ cannot be a contraction on the subspace $U_{a,b} = \{f \in C_+[0,1]; f(0) = a\,,\ f(1) = b\}$.

In this sense, we can prove that for any natural number n, there exist two continuous functions $f, g : [0,1] \to [0,\infty)$ satisfying $f(0) = g(0) = a$, $f(1) = g(1) = b$ and such that $\left\| B_n^{(M)}(f) - B_n^{(M)}(g) \right\| = \|f - g\|$.

Indeed, let us define as $y = f(x)$ the equation of the straight line passing through the points $(0, a)$ and $(1, b)$ and let g be the function whose graph is the polygonal line passing through the points $(0, a)$, $(1/2, c)$ and $(1, b)$, where the value c can be any real number which satisfies $c > f(1/2)$. (Note that the graphs of both functions f and g form a triangle.)

By simple geometrical reasonings we get that $\|f - g\| = g(1/2) - f(1/2)$.

Firstly, we suppose that n is even. Since f and g are concave functions, by the proof of the above Theorem 2.6, we get $B_n^{(M)}(f)(j/n) = f(j/n)$ and similarly, $B_n^{(M)}(g)(j/n) = g(j/n)$ for all $j \in \{0, 1, ..., n\}$. Therefore, taking $j(n) = n/2$, we obtain that $B_n^{(M)}(f)(1/2) = f(1/2)$ and $B_n^{(M)}(g)(1/2) = g(1/2)$. In conclusion, we have

$$g(1/2) - f(1/2) = \|f - g\| \geq \left\| B_n^{(M)}(f) - B_n^{(M)}(g) \right\|$$

$$\geq \left| B_n^{(M)}(f)(1/2) - B_n^{(M)}(g)(1/2) \right| = g(1/2) - f(1/2),$$

which implies $\left\| B_n^{(M)}(f) - B_n^{(M)}(g) \right\| = \|f - g\|$, for any even natural number n.

The reasoning is similar in the case when n is and odd natural number, because it suffices to replace the pair $(1/2, c)$ in the definition of g with $(n_0/(2n_0 + 1), c)$ where $n = 2n_0 + 1$.

3. Ishikawa Iterations for $B_n^{(M)}$

The results in this section are based on the following two well-known results.

Theorem 3.1. (Ishikawa [8]) *Let C be a compact convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be nonexpansive. For $(\lambda_m)_{m\in\mathbb{N}}$ a sequence in $[0, b]$ with $b < 1$ and such that $\sum_{m=0}^{\infty} \lambda_m = +\infty$, let us define the iterates in X by*

$$x_{m+1} := (1 - \lambda_m)x_m + \lambda_m T(x_m).$$

Then for any starting point $x_0 \in C$, the sequence $(x_m)_{m\in\mathbb{N}}$ converges to a fixed point of T.

Theorem 3.2. (Ishikawa [8]) *Let C be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be nonexpansive. Let $(\lambda_m)_m$ be as in Theorem 3.1. Then for any starting point $x_0 \in C$, the following sequence, $(\|x_m - T(x_m)\|)_{m\in\mathbb{N}}$, converges to 0 (i.e. $(x_m)_n$ is a so-called approximate fixed-point sequence).*

Now, in order to can apply to our case the above Theorems 3.1 and 3.2, firstly we need to identify bounded closed convex and compact convex subsets in $C_+[0, 1]$. For example, it is easy to check that the subset

$$C_K^+[0,1] = \{f \in C_+[0,1]; \|f\| \le K\},$$

is bounded, closed and convex. Also, it is easy to check that the subset $C_{L,K} = C_K^+[0,1] \bigcap Lip_L\, 1$ is bounded, closed, convex and equicontinuous, which by the Arzela-Ascoli theorem implies that $C_{L,K}$ is a convex compact subset in $C_+[0,1]$ endowed with the uniform norm.

Another important hypothesis in the Theorems 3.1 and 3.2 is the invariance property of T. In our case, we need this invariance property for the Bernstein max-product operator. For this purpose, we will make use of the Theorem 2.2 in Section 2.

We have:

Theorem 3.3. *(i) If $f \in C_K^+[0,1]$ then for all $n \in \mathbb{N}$ we have $B_n^{(M)}(f) \in C_K^+[0,1]$;*

(ii) Let $K > 0$ and $L \ge 6\pi e^2 K$ be fixed constants and denote $C_{L,K} = C_K^+[0,1] \bigcap Lip_L\, 1$. Then, for all $n \in \mathbb{N}$ satisfying the inequality $n^2 \le \frac{L}{6\pi e^2 K}$, the invariance property $B_n^{(M)}(C_{L,K}) \subset C_{L,K}$ holds.

Proof. (i) Since $0 \le f(k/n) \le \|f\|$ for all $n \in \mathbb{N}$ and $k = 0, 1, ..., n$, it is immediate by the formula of definition of $B_n^{(M)}(f)(x)$, because we easily get $|B_n^{(M)}(f)(x)| \le \|f\|$, for all $x \in [0,1]$, which implies $\|B_n^{(M)}\| \le \|f\| \le K$, for all $n \in \mathbb{N}$.

(ii) Let $f \in C_{L,K}$. By (i) it follows that $\|B_n^{(M)}(f)\| \le K$ for all $n \in \mathbb{N}$ and by (i) it follows that $B_n^{(M)}(f) \in Lip_{6\pi e^2 n^2 \|f\|}\, 1 \subset Lip_{6\pi e^2 n^2 K}\, 1$, for all $n \in \mathbb{N}$. Then, by $n^2 \le \frac{L}{6\pi e^2 K}$ we get $B_n^{(M)}(f) \in Lip_{6\pi e^2 n^2 K}\, 1 \subset Lip_L\, 1$, which leads to the conclusion that $B_n^{(M)}(f) \in C_{L,K}$ for n satisfying $n^2 \le \frac{L}{6\pi e^2 K}$. □

As immediate consequences of the above considerations, we get the following two results.

Corollary 3.4. *Let $K > 0$ and $L \ge 6\pi e^2 K$ be fixed constants and $C_{L,K} = C_K^+[0,1] \bigcap Lip_L\, 1$. Also, let $(\lambda_m)_{m\in\mathbb{N}}$ be sequence in $[0,b]$ with $b < 1$ and such that $\sum_{m=0}^{\infty} \lambda_m = +\infty$. For any $n \in \mathbb{N}$ and $f_{n,1} \in C_{L,K}$ fixed, let us define the iterated sequence of functions*

$$f_{n,m+1}(x) = (1-\lambda_m) f_{n,m}(x) + \lambda_m \cdot B_n^{(M)}(f_{n,m})(x),\ m \in \mathbb{N},\ x \in [0,1].$$

Then, for any fixed $n \in \mathbb{N}$ satisfying the inequality $n^2 \le \frac{L}{6\pi e^2 K}$, the sequence of functions $(f_{n,m}(x))_{m\in\mathbb{N}}$ converges as $m \to \infty$ in the uniform norm, to a fixed point of the operator $B_n^{(M)}$.

Proof. Firstly, it is clear that $C_+[0,1]$ endowed with the uniform norm is a Banach space. By Theorem 2.1, by the comments between the statements of the Theorems 3.2 and 3.3 and by Theorem 3.3, (ii), the operator $B_n^{(M)} : C_{L,K} \to C_{L,K}$ is nonexpansive on the compact convex set $C_{L,K}$. Then the corollary is an immediate consequence of Theorem 3.1. □

Corollary 3.5. *Let $K > 0$ and $C_K^+[0,1] = \{f \in C_+[0,1]; \|f\| \le K\}$. Also, let $(\lambda_m)_m$ and the iterated sequence $(f_{n,m+1}(x))_{m\in\mathbb{N}}$ be defined as in the statement of Corollary 3.4. Then, for any $n \in \mathbb{N}$ and $f_{n,1} \in C_K^+[0,1]$ fixed, we have*

$$\lim_{m\to\infty} \|f_{n,m} - B_n^{(M)}(f_{n,m})\| = 0,$$

where $\|\cdot\|$ denotes the uniform norm.

Proof. By Theorem 2.1, by the comments between the statements of the Theorems 3.2 and 3.3 and by Theorem 3.3, (i), the operator $B_n^{(M)} : C_K^+[0,1] \to C_K^+[0,1]$ is nonexpansive on the bounded, closed and convex subset $C_K^+[0,1]$. Then the corollary is an immediate consequence of Theorem 3.2. □

Remark. The methods in this paper can be extended to other max-product operators of Bernstein-type.

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