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FIXED POINTS AND COMMON FIXED POINTS OF MAPPINGS ON CAT(0) SPACES

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Abstract. In this paper, we will consider the problem of existence of common fixed points for two mappings T and S on a CAT(0) space X. We will suppose that T and S belong to the class of mappings satisfying a generalization of Suzuki's condition (C). Our result improves a number of very recent results of A. Abkar, M. Eslamian in [2] and, as well as, those of B. Nanjaras et al. in [3].
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1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples are pre-Hilbert spaces, R-trees, the complex Hilbert ball with a hyperbolic metric.

In 2008, Suzuki [4] introduced a condition which is weaker than nonexpansiveness and stronger than quasinonexpansiveness. Suzuki's condition, which was named by him the condition (C), reads as follows: a mapping T on a subset K of a Banach space X is said to satisfy the condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in K.$$

In [4], Suzuki proved some fixed point and convergence theorems for such mappings.

Motivated by this result, Garcia-Falset et al. in [5] introduced two kinds of generalizations for the condition (C) and studied both the existence of fixed points and their asymptotic behavior. Very recently, some authors used a modified Suzuki condition for multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying this condition in Banach spaces [6, 7].

In this paper, we will consider the problem of existence of common fixed points for two mappings T and S on a CAT(0) space X. We will suppose that T and S belong to the class of mappings satisfying some conditions, which are generalizations of Suzuki's condition (C). Our result improves a number of very recent results of A. Abkar, M. Eslamian in [2] and, as well as, those of B. Nanjaras et al. in [3].

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subseteq R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$.
A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:
"Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})."$$

Here we recall some useful lemma which will be used next.

Lemma 2.1. *([8]) Let (X, d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z of the above lemma.

Lemma 2.2. *([8, Lemma 2.4]) Let (X, d) be a CAT(0) space. Then*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z),$$

for $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.3. *([8, Lemma 2.5]) Let (X, d) be a CAT(0) space. Then*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, , z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2,$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

In particular by Lemma 2.3 we have

$$d(z, \frac{1}{2}x \oplus \frac{1}{2}y)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2, \text{ for all } x, y, z \in X,$$

which is called the (CN) inequality of Bruhat-Tits, as it was shown in [9]. In fact (cf. [10], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN)

inequality.
Let $\{x_n\}$ be a bounded sequence in X and K be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\},$$

where

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x),$$

and the set

$$A = A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to K. Similarly, set A is called the *asymptotic center* of $\{x_n\}$ relative to K.
In the CAT(0) space, the asymptotic center $A = A(K, \{x_n\})$ of $\{x_n\}$ consists of exactly one point whenever K is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space X said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\triangle$-convergent to x and given $y \in X$ with $x \neq y$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

So every CAT(0) space X satisfies the Opial property.

Lemma 2.4. *(*[11]*) Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 2.5. *(*[12]*) If K is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of is in K.*

Definition 2.6. *(*[4]*) Let T be a mapping on a subset K of a CAT(0) space (X, d). Then T said to satisfy* condition (C) *if*

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in K$.

Definition 2.7. *(*[2]*) Let T be a mapping on a subset K of a CAT(0) space X and $\mu \geq 1$. T is said to satisfy condition (E_μ) if*

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y), \quad x, y \in K.$$

We say that T satisfies condition (E) whenever T satisfies the condition (E_μ) for some $\mu \geq 1$.

Definition 2.8. *(*[2]*) Let T be a mapping on a subset K of a CAT(0) space X and $\lambda \in (0, 1)$. T is said to satisfy condition (C_λ) if*

$$\lambda d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y), \quad x, y \in K.$$

Notice that if $0 < \lambda_1 < \lambda_2 < 1$ then the condition (C_{λ_1}) implies the condition (C_{λ_2}). The following example shows that the class of mappings satisfying the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$ is broader than the class of mappings satisfying the condition (C).

Definition 2.9. *A mapping T on a subset K of a CAT(0) space X is called quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $d(T(x), z) \leq d(x, z)$ for all $x \in K$ and $z \in Fix(T)$.*

Lemma 2.10. *(*[2, Lemma 2.9]*) Assume that a mapping T satisfies the condition* (E) *and has a fixed point. Then T is a quasi-nonexpansive mapping.*

The converse of the above implication is, in general, not true.

The following lemma is a consequence of Proposition 2 proved by Goebel and Kirk [13].

Lemma 2.11. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space X and let $\{\alpha_n\} \subseteq [0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)x_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(y_n, x_n) = 0$.*

Lemma 2.12. *(*[14]*) Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n,$$

where $\lambda_n \in (0,1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\frac{b_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.13. *(*[1, Lemma 1]*) Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$,*

$$a_{n+1} \leq a_n + b_n \quad \forall n \geq N_0.$$

(a) *If $\sum_{n=1}^{\infty} b_n < \infty$, then, $\lim a_n$ exists.*
(b) *If $\sum_{n=1}^{\infty} b_n < \infty$, and $\{a_n\}$ has a subsequence converging to zero, then, $\lim a_n = 0$.*

3. Main results

We generalize first Lemma 2.11 and, then, we will prove a common fixed point for two mappings which satisfy the conditions (E) and (C_λ). Our result improves a number of very recent results of A. Abkar, M. Eslamian [2] and B. Nanjaras et al. [3].

Lemma 3.1. *Let $\{y_n\}, \{z_n\}, \{u_n\}$ and $\{v_n\}$ be bounded sequences in a complete CAT(0) space X and let $\{\alpha_n\} \subseteq (0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Let $\{x_n\}$ be a sequence in X defined by*

$$x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)z_n$$

and suppose

$$\begin{aligned} d(y_{n+1}, u_n) &\leq d(x_{n+1}, z_n) \\ d(v_n, z_{n+1}) &\leq d(x_{n+1}, y_n) \\ d(u_n, v_n) &\leq c_n, \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{c_n\}$ is a sequence in $\mathbb{R}^+$. Then the sequence $\{x_n\}$ is bounded and

- *if $\sum_{n=1}^{\infty} c_n < +\infty$, then $\lim_{n\to\infty} d(y_n, z_n)$ exists.*

- *if* $d(y_n, z_n) + \frac{c_n}{\alpha_n} \to 0$ *as* $n \to \infty$, *then* $\lim_{n\to\infty} d(y_n, z_n) = 0$.

Proof. Since there exist $a \in X$ and $r > 0$ such that $d(y_n, a) \le r$ and $d(z_n, a) \le r$, we have

$$\begin{aligned} d(x_{n+1}, a) &\le d(x_{n+1}, y_n) + d(a, y_n) \\ &= (1 - \alpha_n)d(y_n, z_n) + d(a, y_n) \\ &\le (1 - \alpha_n)(d(y_n, a) + d(a, z_n)) + d(a, y_n) \\ &\le 2hr + r, \end{aligned}$$

where $h = \limsup_n \alpha_n < 1$. This proves $\{x_n\}$ is bounded.
To prove the convergence of $a_n := d(y_n, z_n)$ notice that

$$\begin{aligned} a_{n+1} &= d(y_{n+1}, z_{n+1}) \\ &\le d(y_{n+1}, u_n) + d(u_n, v_n) + d(v_n, z_{n+1}) \\ &\le \alpha_n a_n + c_n + (1 - \alpha_n)a_n \\ &= a_n + c_n. \end{aligned}$$

Now, by Lemma 2.13, $\lim a_n$ exists.
To prove that $a_n := d(y_n, z_n) \to 0$ we observe

$$\begin{aligned} a_{n+1} &= d(y_{n+1}, z_{n+1}) \\ &\le d(y_{n+1}, u_n) + d(u_n, v_n) + d(v_n, z_{n+1}) \\ &\le \alpha_n a_n + c_n + (1 - \alpha_n)a_n \end{aligned}$$

Now, by Lemma 2.12, $a_n \to 0$. □

Example 3.2. *Let* $X := \mathbb{R}$ *with the usual metric* $d(x, y) = |x - y|$. *Put*

$$c_n = \frac{1}{n^2},\ \alpha_n = \frac{1}{n+1},\ z_n := k,\ y_n := k+1,\ u_n = v_n = k + \frac{n}{n+1},$$

where k *is real number. So* $x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n)z_n = k + \frac{1}{n+1}$ *and all conditions of Lemma 3.1 hold for first case, while* $\lim_{n\to\infty} d(y_n, z_n) = 1$ *exists but* $d(y_n, z_n) \not\to 0$ *as* $n \to \infty$.
For second part, put

$$z_n = \frac{1}{\sqrt{n}},\ y_n = z_n + \frac{1}{n},\ u_n = v_n = z_{n+1} + \frac{1}{n+1}.$$

Thus $x_{n+1} = z_n + \frac{1}{n(n+1)}$ *and all conditions of Lemma 3.1 hold for second case, and since*

$$d(y_n, z_n) + \frac{c_n}{\alpha_n} = \frac{1}{n} + \frac{n+1}{n^2} \to 0,$$

therefore $d(y_n, z_n) \to 0$ *as* $n \to \infty$.

Lemma 3.3. *Let* $\{y_n\}$ *and* $\{z_n\}$ *be bounded sequences in a complete CAT(0) space* X *and let* $\{\alpha_n\} \subseteq (0, 1)$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\limsup_n \alpha_n < 1$. *Suppose* $\{x_n\}$ *be a sequence that defined by* $x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n)z_n$ *and*

$$d(y_{n+1}, z_{n+1}) \le d(x_{n+1}, z_n), \tag{3.1}$$

or

$$d(y_{n+1}, z_{n+1}) \leq d(x_{n+1}, y_n) \tag{3.2}$$

for all $n \in \mathbb{N}$*. Then the sequence* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} d(y_n, z_n) = 0$*.*

Proof. By the relation (3.1), we have

$$a_{n+1} = d(y_{n+1}, z_{n+1}) \leq d(x_{n+1}, z_n) = \alpha_n d(y_n, z_n) = \alpha_n a_n.$$

Therefore

$$0 \leq a_{n+1} \leq \alpha_n a_n \leq a_n, \tag{3.3}$$

namely $\{a_n\}$ is bounded bellow and decreasing sequence which make it to be convergence to l, i.e. $a_n \to l$ as $n \to \infty$. It is clear that $l \geq 0$. Now by (3.3) we have

$$l = \limsup_{n\to\infty} a_{n+1} \leq \limsup_{n\to\infty}(\alpha_n a_n) = \limsup_{n\to\infty}(\alpha_n) l$$

and since $\limsup_{n\to\infty}(\alpha_n) < 1$ we get that $l = 0$.

Now if the relation (3.2) takes place, we have

$$a_{n+1} = d(y_{n+1}, z_{n+1}) \leq d(x_{n+1}, y_n) = (1-\alpha_n) d(y_n, z_n) = (1-\alpha_n) a_n.$$

Therefore

$$0 \leq a_{n+1} \leq (1-\alpha_n) a_n \leq a_n, \tag{3.4}$$

so $a_n \to l$ as $n \to \infty$ for some l. Now by (3.4) we have

$$l = \liminf_{n\to\infty} a_{n+1} \leq \liminf_{n\to\infty}((1-\alpha_n) a_n) = (1-h) l,$$

and since $h = \limsup_{n\to\infty}(\alpha_n) < 1$ we get again that $l = 0$. □

Example 3.4. *Let* $X; = \mathbb{R}$ *with the usual metric* $d(x, y) = |x - y|$*. Put*

$$\alpha_n = \frac{1}{n+1}, \; z_n := \frac{1}{n!}, \; y_n := \frac{k}{n!},$$

where $1 \neq k \in \mathbb{R}$*. So* $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n) z_n = \frac{k+n}{(n+1)!}$ *and all the conditions of Lemma 3.3 with relation (3.1) hold, and we have*

$$\lim_{n\to\infty} d(y_n, z_n) = \lim_{n\to\infty} \frac{|k-1|}{(n+1)!} = 0.$$

For relation (3.2) from Lemma 3.3, put

$$\alpha_n = \frac{1}{n+1}, \; z_n := \frac{1}{\sqrt{n}}, \; y_n := z_n + \frac{1}{n},$$

so $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n) z_n = z_n + \frac{1}{n(n+1)}$ *and all the conditions of Lemma 3.3 with relation (3.2) hold, and we have again* $\lim_{n\to\infty} d(y_n, z_n) = \lim_{n\to\infty} \frac{1}{n} = 0$*.*

Our first main result is the following.

Theorem 3.5. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Suppose that $T, S : K \to K$ satisfy the condition (C_λ) for some $\lambda \in (0,1)$ and $TS = ST$. Let $x_1 \in K$ and define*

$$x_{n+1} = \alpha_n Tx_n \oplus (1-\alpha_n)Sx_n, \text{ for } n \geq 1.$$

Let $\{\alpha_n\} \subseteq [\lambda, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$ and suppose that

$$\alpha_n d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n),$$
$$\alpha_n d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n).$$

Then $\lim_{n\to\infty} d(Tx_n, Sx_n)$ exists.

Proof. Put $a_n := d(Tx_n, Sx_n), u_n := TSx_n$ and $v_n = STx_n$. It follows that

$$\lambda d(x_n, Tx_n) \leq \alpha_n d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n) = \alpha_n a_n$$
$$\lambda d(x_n, Sx_n) \leq \alpha_n d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n) = (1-\alpha_n)a_n$$

By condition (C_λ), we have

$$d(Tx_{n+1}, TSx_n) \leq \alpha_n a_n$$
$$d(Sx_{n+1}, STx_n) \leq (1-\alpha_n)a_n$$
$$d(u_n, v_n) = d(STx_n, TSx_n) = 0.$$

Now according to Lemma (3.1) we get that $\lim_{n\to\infty} d(Tx_n, Sx_n)$ exists. □

Remark 3.6. *If $a_n := d(y_n, z_n)$ and $d(u_n, v_n) \leq c_n$, then, in the conditions of Lemma 3.1, we have*

$$\begin{aligned} d(u_n, v_n) &\leq d(u_n, y_{n+1}) + d(y_{n+1}, z_{n+1}) + d(z_{n+1}, v_n) \\ &\leq \alpha_n a_n + a_{n+1} + (1-\alpha_n)a_n \\ &= a_n + a_{n+1}. \end{aligned}$$

Corollary 3.7. *(*[3, Lemma 3.6]*) Let K be a nonempty bounded and convex subset of a complete CAT(0) space X and suppose $T : K \to K$ satisfies condition (C). Define a sequence $\{x_n\}$ by $x_1 \in K$ and*

$$x_{n+1} = \alpha_n Tx_n \oplus (1-\alpha_n)x_n, \text{ for all } n \geq 1,$$

where $\{\alpha_n\} \subset [\frac{1}{2}, 1)$ is such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$.
Then $\lim_{n\to\infty} d(Tx_n, x_n)$ exists.

Proof. It is enough that we take $Sx = x$ and $\lambda = \frac{1}{2}$. Then $\alpha_n d(x_n, Tx_n) = d(x_{n+1}, x_n)$ and all the conditions from Theorem 3.5 hold. □

Our second main result is the following.

Theorem 3.8. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Suppose $T, S : K \to K$ satisfy the conditions (E) and (C_λ) for some $\lambda \in (0,1)$, $TS = ST$ and the relations*

$$\lambda d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n), \quad \lambda d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n), \tag{3.5}$$

hold for some sequence $\{x_n\}$ in K.
Then $\lim_{n\to\infty} d(Tx_n, Sx_n)$ exists and if, additionally $\lim_{n\to\infty} d(Tx_n, Sx_n) = 0$, then T and S have a common fixed point in K.

Proof. Let $x_1 \in K$ and define $x_{n+1} = \lambda Tx_n \oplus (1-\lambda)Sx_n$. By relation (3.5) and Theorem 3.5, we obtain that $\lim_{n\to\infty} d(Tx_n, Sx_n)$ exists. Notice that

$$d(x_{n+1}, Sx_n) = \lambda d(Tx_n, Sx_n), \quad d(x_{n+1}, Tx_n) = (1-\lambda)d(Tx_n, Sx_n). \tag{3.6}$$

When $\lim_{n\to\infty} d(Tx_n, Sx_n) = 0$ we can show that

$$\limsup_{n\to\infty} d(Tx_n, x_n) = \limsup_{n\to\infty} d(Sx_n, x_n) = 0.$$

Since

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, Sx_n) + d(Sx_n, Tx_n) \\ d(x_n, Sx_n) &\leq d(x_n, Tx_n) + d(Tx_n, Sx_n), \end{aligned}$$

we obtain

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, Tx_n) &\leq \limsup_{n\to\infty} d(x_n, Sx_n) \\ \limsup_{n\to\infty} d(x_n, Sx_n) &\leq \limsup_{n\to\infty} d(x_n, Tx_n). \end{aligned}$$

Thus, from (3.6) we get that

$$\limsup_{n\to\infty} d(Tx_n, x_n) = \limsup_{n\to\infty} d(Sx_n, x_n) = 0.$$

Let $A(\{x_n\}) = \{x_0\}$. By Lemma 2.5 we have $x_0 \in K$. Since T and S satisfy the condition (E) we have

$$\begin{aligned} d(x_n, Tx_0) &\leq \mu_1 d(x_n, Tx_n) + d(x_n, x_0) \\ d(x_n, Sx_0) &\leq \mu_2 d(x_n, Sx_n) + d(x_n, x_0), \end{aligned}$$

for some $\mu_1, \mu_2 \geq 1$. Hence by taking limit superior on both sides in above inequalities, we obtain

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, Tx_0) &\leq \limsup_{n\to\infty} d(x_n, x_0) \\ \limsup_{n\to\infty} d(x_n, Sx_0) &\leq \limsup_{n\to\infty} d(x_n, x_0). \end{aligned}$$

By the uniqueness of the asymptotic center, we obtain $Tx_0 = Sx_0 = x_0$.□

Corollary 3.9. *(*[2, Theorem 3.2]*) Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Suppose $T : K \to K$ satisfies the conditions (E) and (C_λ) for some $\lambda \in [0, 1)$. If $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, then T has a fixed point in K.*

Proof. Put $Sx = x$ so $\lambda d(x_n, Tx_n) = d(x_{n+1}, x_n)$ for all $n \geq 1$. By Theorem 3.8 we have $Tx = x$ for some $x \in K$. □

Our last main result is the following.

Theorem 3.10. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X and let $T, S : K \to K$ be two mappings which satisfy the conditions (E) and (C_λ) and $TS = ST$. Consider $x_1 \in K$ and define*

$$x_{n+1} = \alpha_n Tx_n \oplus (1-\alpha_n)Sx_n, \text{ for all } n \geq 1,$$

where (for some $\lambda \in (0,1)$*)* $\{\alpha_n\} \subseteq [\lambda, 1)$ *is such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\limsup_n \alpha_n < 1$*. Suppose that the following relations hold*

$$\alpha_n d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n),$$
$$\alpha_n d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n).$$

If $\lim_{n\to\infty} d(Tx_n, Sx_n) = 0$*, then* $\lim_{n\to\infty} d(x_n, p)$ *exists, for each* $p \in Fix(T) \cap Fix(S)$*.*

Proof. By Theorem 3.8, $Fix(T) \cap Fix(S) \neq \emptyset$. Given $p \in Fix(T) \cap Fix(S)$, by Lemma 2.2 and Lemma 2.10 we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n Tx_n \oplus (1-\alpha_n)Sx_n, p) \\ &\leq \alpha_n d(Tx_n, p) + (1-\alpha_n) d(Sx_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus $d(x_{n+1}, p) \leq d(x_n, p)$. So the sequence $\{d(x_n, p)\}$ which is bounded bellow and decreasing, which completes the proof. □

Corollary 3.11. *(*[3, Lemma 5.1]*) Let* K *be a nonempty bounded closed convex subset of a complete CAT(0) space* X *and let* $T : K \to K$ *be a mapping satisfying condition* (C)*. Define a sequence* $\{x_n\}$ *by* $x_1 \in K$ *and* $x_{n+1} = \alpha_n Tx_n \oplus (1-\alpha_n)x_n$ *where* $\{\alpha_n\} \subseteq [\frac{1}{2}, 1)$*. Then* $\lim_{n\to\infty} d(x_n, p)$ *exists, for each* $p \in Fix(T)$*.*

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References

[1] A. Rafiq, *Implicit fixed point iterations*, Rostock, Math. Kolloq., **62**(2007), 21-39.
[2] A. Abkar, M. Eslamian, *Common fixed point results in CAT(0) spaces,* Nonlinear Anal., **74**(2011), 18351840.
[3] B. Nanjaras, B. Panyanaka, W. Phuengrattana, *Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces*, Nonlinear Anal. Hybrid Systems, **4**(2010), 25-31.
[4] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340**(2008), 1088-1095.
[5] J. Garcia-Falset, E. Llorens-Fuster, T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Math. Anal. Appl., **375**(2011), 185195.
[6] A. Abkar, M. Eslamian, *A fixed point theorem for generalized nonexpansive multivalued mappings*, Fixed Point Theory, **12**(2011), 241-246.
[7] A. Abkar, M. Eslamian, *Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces*, Fixed Point Theory Appl., 2010(2010), Article ID 457935, 10 pp.
[8] S. Dhompongsa, B. Panyanak, *On* $\triangle$*-convergence theorems in CAT(0) spaces*, Computer Math. Appl., **56**(2008), 2572-2579.
[9] F. Bruhat, J. Tits, *Groupes ŕeductifs sur un corps local. I. Données radicielles valuées*, Inst. Hautes Études Sci. Publ. Math., **41**(1972), 5-251.
[10] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, Heidelberg, 1999.
[11] W.A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68**(2008), 36893696.

[12] S. Dhompongsa, W.A. Kirk, B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal., **8**(2007), 3545.
[13] K. Goebel, W.A. Kirk, *Iteration processes for nonexpansive mapping*, Contemp. Math., **21**(1983), 115123.
[14] F.P. Vasilev, *Numerical Methods for Solving Extremal Problems*, Moscow, 1988.

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