FIXED POINTS AND COMMON FIXED POINTS
OF MAPPINGS ON CAT(0) SPACES

MEHDI ASADI

*Department of Mathematics, Zanjan Branch
Islamic Azad University
Zanjan, Iran
E-mail: masadi@azu.ac.ir
Fax: +98-241-4220030

Abstract. In this paper, we will consider the problem of existence of common fixed points for two mappings $T$ and $S$ on a CAT(0) space $X$. We will suppose that $T$ and $S$ belong to the class of mappings satisfying a generalization of Suzuki's condition ($C$). Our result improves a number of very recent results of A. Abkar, M. Eslamian in [2] and, as well as, those of B. Nanjaras et al. in [3].

Key Words and Phrases: CAT(0) spaces, fixed points, common fixed point.

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1. Introduction

A metric space $X$ is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in $X$ is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples are pre-Hilbert spaces, R-trees, the complex Hilbert ball with a hyperbolic metric.

In 2008, Suzuki [4] introduced a condition which is weaker than nonexpansiveness and stronger than quasinonexpansiveness. Suzuki’s condition, which was named by him the condition ($C$), reads as follows: a mapping $T$ on a subset $K$ of a Banach space $X$ is said to satisfy the condition ($C$) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in K.$$ 

In [4], Suzuki proved some fixed point and convergence theorems for such mappings.

Motivated by this result, Garcia-Falset et al. in [5] introduced two kinds of generalizations for the condition ($C$) and studied both the existence of fixed points and their asymptotic behavior. Very recently, some authors used a modified Suzuki condition for multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying this condition in Banach spaces [6, 7].

In this paper, we will consider the problem of existence of common fixed points for two mappings $T$ and $S$ on a CAT(0) space $X$. We will suppose that $T$ and $S$ belong to the class of mappings satisfying some conditions, which are generalizations
of Suzuki’s condition (C). Our result improves a number of very recent results of A. Abkar, M. Eslamian in [2] and, as well as, those of B. Nanjaras et al. in [3].

2. Preliminaries

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subseteq \mathbb{R}$ to $X$ such that $c(0) = x, c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\triangle(x_1, x_2, x_3) := \triangle(x_1, x_2, x_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x_i, x_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

“Let $\triangle$ be a geodesic triangle in $X$ and let $\triangle$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

Here we recall some useful lemma which will be used next.

**Lemma 2.1.** ([8]) Let $(X, d)$ be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1-t)d(x, y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point $z$ of the above lemma.

**Lemma 2.2.** ([8, Lemma 2.4]) Let $(X, d)$ be a CAT(0) space. Then

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z),$$

for $x, y, z \in X$ and $t \in [0, 1]$.

**Lemma 2.3.** ([8, Lemma 2.5]) Let $(X, d)$ be a CAT(0) space. Then

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2,$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

In particular by Lemma 2.3 we have

$$d(z, \frac{1}{2}x \oplus \frac{1}{2}y)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2,$$

for all $x, y, z \in X$, which is called the (CN) inequality of Bruhat-Tits, as it was shown in [9]. In fact (cf. [10], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN)
Let $\{x_n\}$ be a bounded sequence in $X$ and $K$ be a nonempty bounded subset of $X$. We associate this sequence with the number
\[ r = r(K, \{x_n\}) = \inf \{ r(x, \{x_n\}) : x \in K \}, \]
where
\[ r(x, \{x_n\}) = \limsup_{n \to \infty} d(x_n, x), \]
and the set
\[ A = A(K, \{x_n\}) = \{ x \in K : r(x, \{x_n\}) = r \}. \]

The number $r$ is known as the asymptotic radius of $\{x_n\}$ relative to $K$. Similarly, set $A$ is called the asymptotic center of $\{x_n\}$ relative to $K$.

In the CAT(0) space, the asymptotic center $A = A(K, \{x_n\})$ of $\{x_n\}$ consists of exactly one point whenever $K$ is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space $X$ said to be $\Delta$-convergent to $x \in X$ if $x$ is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\Delta$-convergent to $x$ and given $y \in X$ with $x \neq y$,
\[ \limsup_{n \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, y). \]
So every CAT(0) space $X$ satisfies the Opial property.

**Lemma 2.4.** ([11]) Every bounded sequence in a complete CAT(0) space has a $\Delta$-convergent subsequence.

**Lemma 2.5.** ([12]) If $K$ is a closed convex subset of a complete CAT(0) space and $\{x_n\}$ is a bounded sequence in $K$, then the asymptotic center of is in $K$.

**Definition 2.6.** ([4]) Let $T$ be a mapping on a subset $K$ of a CAT(0) space $(X, d)$. Then $T$ said to satisfy condition (C) if
\[ \frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y), \]
for all $x, y \in K$.

**Definition 2.7.** ([2]) Let $T$ be a mapping on a subset $K$ of a CAT(0) space $X$ and $\mu \geq 1$. $T$ is said to satisfy condition $(E_\mu)$ if
\[ d(x, Ty) \leq \mu d(x, Tx) + d(x, y), \quad x, y \in K. \]

We say that $T$ satisfies condition (E) whenever $T$ satisfies the condition $(E_\mu)$ for some $\mu \geq 1$.

**Definition 2.8.** ([2]) Let $T$ be a mapping on a subset $K$ of a CAT(0) space $X$ and $\lambda \in (0, 1)$. $T$ is said to satisfy condition $(C_\lambda)$ if
\[ \lambda d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y), \quad x, y \in K. \]

Notice that if $0 < \lambda_1 < \lambda_2 < 1$ then the condition $(C_{\lambda_1})$ implies the condition $(C_{\lambda_2})$. The following example shows that the class of mappings satisfying the conditions (E) and $(C_\lambda)$ for some $\lambda \in (0, 1)$ is broader than the class of mappings satisfying the condition (C).
Definition 2.9. A mapping $T$ on a subset $K$ of a CAT(0) space $X$ is called quasi-nonexpansive if $\text{Fix}(T) \neq \emptyset$ and $d(T(x), z) \leq d(x, z)$ for all $x \in K$ and $z \in \text{Fix}(T)$.

Lemma 2.10. (\cite[Lemma 2.9]{1}) Assume that a mapping $T$ satisfies the condition $(E)$ and has a fixed point. Then $T$ is a quasi-nonexpansive mapping.

The converse of the above implication is, in general, not true.

The following lemma is a consequence of Proposition 2 proved by Goebel and Kirk \cite{13}.

Lemma 2.11. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space $X$ and let $\{\alpha_n\} \subseteq [0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n) x_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} d(y_n, x_n) = 0$.

Lemma 2.12. (\cite{14}) Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences satisfying the following inequality:

$$a_{n+1} \leq (1 - \lambda_n) a_n + b_n,$$

where $\lambda_n \in (0, 1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\frac{b_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.13. (\cite[Lemma 1]{1}) Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$,

$$a_{n+1} \leq a_n + b_n \quad \forall n \geq N_0.$$

(a) If $\sum_{n=1}^{\infty} b_n < \infty$, then, $\lim a_n$ exists.

(b) If $\sum_{n=1}^{\infty} b_n < \infty$, and $\{a_n\}$ has a subsequence converging to zero, then, $\lim a_n = 0$.

3. Main results

We generalize first Lemma 2.11 and, then, we will prove a common fixed point for two mappings which satisfy the conditions $(E)$ and $(C_\lambda)$. Our result improves a number of very recent results of A. Abkar, M. Eslamian \cite{2} and B. Nanjaras et al. \cite{3}.

Lemma 3.1. Let $\{y_n\}, \{z_n\}, \{u_n\}$ and $\{v_n\}$ be bounded sequences in a complete CAT(0) space $X$ and let $\{\alpha_n\} \subseteq (0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Let $\{x_n\}$ be a sequence in $X$ defined by

$$x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n) z_n$$

and suppose

$$d(y_{n+1}, u_n) \leq d(x_{n+1}, z_n)$$

$$d(v_n, z_{n+1}) \leq d(x_{n+1}, y_n)$$

$$d(u_n, v_n) \leq c_n,$$

for all $n \in \mathbb{N}$, where $\{c_n\}$ is a sequence in $\mathbb{R}^+$. Then the sequence $\{x_n\}$ is bounded and

- if $\sum_{n=1}^{\infty} c_n < +\infty$, then $\lim_{n \to \infty} d(y_n, z_n)$ exists.
Example 3.2. Let \( d(y_n, z_n) + \frac{c_n}{\alpha_n} \to 0 \) as \( n \to \infty \), then \( \lim_{n \to \infty} d(y_n, z_n) = 0 \).

Proof. Since there exist \( a \in X \) and \( r > 0 \) such that \( d(y_n, a) \leq r \) and \( d(z_n, a) \leq r \), we have

\[
d(x_{n+1}, a) \leq d(x_{n+1}, y_n) + d(a, y_n) = (1 - \alpha_n)d(y_n, z_n) + d(a, y_n)
\leq (1 - \alpha_n)(d(y_n, a) + d(a, z_n)) + d(a, y_n)
\leq 2hr + r,
\]

where \( h = \limsup_n \alpha_n < 1 \). This proves \( \{x_n\} \) is bounded.

To prove the convergence of \( a_n := d(y_n, z_n) \) notice that

\[
a_{n+1} = d(y_{n+1}, z_{n+1})
\leq d(y_{n+1}, u_n) + d(u_n, v_n) + d(v_n, z_{n+1})
\leq \alpha_n a_n + c_n + (1 - \alpha_n)a_n
= a_n + c_n.
\]

Now, by Lemma 2.13, \( \lim a_n \) exists.

To prove that \( a_n := d(y_n, z_n) \to 0 \) we observe

\[
a_{n+1} = d(y_{n+1}, z_{n+1})
\leq d(y_{n+1}, u_n) + d(u_n, v_n) + d(v_n, z_{n+1})
\leq \alpha_n a_n + c_n + (1 - \alpha_n)a_n
\]

Now, by Lemma 2.12, \( a_n \to 0 \). \( \square \)

Example 3.2. Let \( X := \mathbb{R} \) with the usual metric \( d(x, y) = |x - y| \). Put

\[
c_n = \frac{1}{n^2}, \quad \alpha_n = \frac{1}{n + 1}, \quad z_n := k, \quad y_n := k + 1, \quad u_n = v_n = k + \frac{n}{n+1},
\]

where \( k \) is real number. So \( x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n)z_n = k + \frac{1}{n+1} \) and all conditions of Lemma 3.1 hold for first case, while \( \lim_{n \to \infty} d(y_n, z_n) = 1 \) exists but \( d(y_n, z_n) \neq 0 \) as \( n \to \infty \).

For second part, put

\[
z_n = \frac{1}{\sqrt{n}}, \quad y_n = z_n + \frac{1}{n}, \quad u_n = v_n = z_{n+1} + \frac{1}{n + 1}.
\]

Thus \( x_{n+1} = z_n + \frac{1}{n(n+1)} \) and all conditions of Lemma 3.1 hold for second case, and since

\[
d(y_n, z_n) + \frac{c_n}{\alpha_n} = \frac{1}{n} + \frac{n + 1}{n^2} \to 0,
\]

therefore \( d(y_n, z_n) \to 0 \) as \( n \to \infty \).

Lemma 3.3. Let \( \{y_n\} \) and \( \{z_n\} \) be bounded sequences in a complete \( CAT(0) \) space \( X \) and let \( \{\alpha_n\} \subseteq (0, 1) \) such that \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and \( \limsup_n \alpha_n < 1 \). Suppose \( \{x_n\} \) be a sequence that defined by \( x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n)z_n \) and

\[
d(y_{n+1}, z_{n+1}) \leq d(x_{n+1}, z_n), \quad (3.1)
\]
or
\[ d(y_{n+1}, z_{n+1}) \leq d(x_{n+1}, y_n) \]  
(3.2)
for all \( n \in \mathbb{N} \). Then the sequence \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} d(y_n, z_n) = 0 \).

**Proof.** By the relation (3.1), we have
\[ a_{n+1} = d(y_{n+1}, z_{n+1}) \leq d(x_{n+1}, y_n) = \alpha_n d(y_n, z_n) = \alpha_n a_n. \]
Therefore
\[ 0 \leq a_{n+1} \leq \alpha_n a_n \leq a_n, \]  
(3.3)
namely \( \{a_n\} \) is bounded below and decreasing sequence which make it to be convergence to \( l \), i.e. \( a_n \to l \) as \( n \to \infty \). It is clear that \( l \geq 0 \). Now by (3.3) we have
\[ l = \limsup_{n \to \infty} a_{n+1} \leq \limsup_{n \to \infty} (\alpha_n a_n) = \limsup_{n \to \infty} (\alpha_n)l \]
and since \( \limsup_{n \to \infty} (\alpha_n) < 1 \) we get that \( l = 0 \).

Now if the relation (3.2) takes place, we have
\[ a_{n+1} = d(y_{n+1}, z_{n+1}) \leq d(x_{n+1}, y_n) = (1 - \alpha_n) d(y_n, z_n) = (1 - \alpha_n) a_n. \]
Therefore
\[ 0 \leq a_{n+1} \leq (1 - \alpha_n) a_n \leq a_n, \]  
(3.4)
so \( a_n \to l \) as \( n \to \infty \) for some \( l \). Now by (3.4) we have
\[ l = \liminf_{n \to \infty} a_{n+1} \leq \liminf_{n \to \infty} ((1 - \alpha_n) a_n) = (1 - h)l, \]
and since \( h = \limsup_{n \to \infty} (\alpha_n) < 1 \) we get again that \( l = 0 \). \( \square \)

**Example 3.4.** Let \( X = \mathbb{R} \) with the usual metric \( d(x, y) = |x - y| \). Put
\[ \alpha_n = \frac{1}{n+1}, \quad z_n := \frac{1}{n}, \quad y_n := \frac{k}{n}, \]
where \( 1 \neq k \in \mathbb{R} \). So \( x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n) z_n = \frac{k+n}{(n+1)!} \) and all the conditions of Lemma 3.3 with relation (3.1) hold, and we have
\[ \lim_{n \to \infty} d(y_n, z_n) = \lim_{n \to \infty} \frac{|k - 1|}{(n+1)!} = 0. \]

For relation (3.2) from Lemma 3.3, put
\[ \alpha_n = \frac{1}{n+1}, \quad z_n := \frac{1}{\sqrt{n}}, \quad y_n := z_n + \frac{1}{n}, \]
so \( x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n) z_n = z_n + \frac{1}{n(n+1)} \) and all the conditions of Lemma 3.3 with relation (3.2) hold, and we have again \( \lim_{n \to \infty} d(y_n, z_n) = \lim_{n \to \infty} \frac{1}{n} = 0 \).

Our first main result is the following.
Theorem 3.5. Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Suppose that T, S : K → K satisfy the condition (C\lambda) for some λ ∈ (0, 1) and TS = ST. Let x_1 ∈ K and define
\[ x_{n+1} = \alpha_nTx_n \oplus (1 - \alpha_n)Sx_n, \text{ for } n \geq 1. \]
Let \{α_n\} ⊆ [λ, 1) such that \(\sum_{n=1}^{\infty} α_n = \infty\) and lim sup \(n\to\infty\) α_n < 1 and suppose that
\[ α_n d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n), \]
\[ α_n d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n). \]
Then \(\lim_{n\to\infty} d(Tx_n, Sx_n)\) exists.

Proof. Put \(a_n := d(Tx_n, Sx_n), u_n := TSx_n\) and \(v_n = STx_n\). It follows that
\[ \lambda d(x_n, Tx_n) \leq α_n d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n) = α_n a_n \]
\[ \lambda d(x_n, Sx_n) \leq α_n d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n) = (1 - α_n)a_n \]
By condition (C\lambda), we have
\[ d(Tx_{n+1}, TSx_n) \leq α_n a_n \]
\[ d(Sx_{n+1}, STx_n) \leq (1 - α_n)a_n \]
\[ d(u_n, v_n) = d(STx_n, TSx_n) = 0. \]
Now according to Lemma (3.1) we get that \(\lim_{n\to\infty} d(Tx_n, Sx_n)\) exists.

Remark 3.6. If \(a_n := d(y_n, z_n)\) and \(d(u_n, v_n) \leq \epsilon_n\), then, in the conditions of Lemma 3.1, we have
\[ d(u_n, v_n) \leq d(u_n, y_{n+1}) + d(y_{n+1}, z_{n+1}) + d(z_{n+1}, v_n) \]
\[ \leq α_n a_n + a_{n+1} + (1 - α_n)a_n \]
\[ = a_n + a_{n+1}. \]

Corollary 3.7. ([3, Lemma 3.6]) Let K be a nonempty bounded and convex subset of a complete CAT(0) space X and suppose T : K → K satisfies condition (C). Define a sequence \{x_n\} by \(x_1 \in K\) and
\[ x_{n+1} = \alpha_nTx_n \oplus (1 - \alpha_n)x_n, \text{ for all } n \geq 1, \]
where \{α_n\} ⊆ [\frac{1}{2}, 1) is such that \(\sum_{n=1}^{\infty} α_n = \infty\) and lim sup \(n\to\infty\) α_n < 1.
Then \(\lim_{n\to\infty} d(Tx_n, x_n)\) exists.

Proof. It is enough that we take Sx = x and \(\lambda = \frac{1}{2}\). Then \(\alpha_n d(x_n, Tx_n) = d(x_{n+1}, x_n)\) and all the conditions from Theorem 3.5 hold.

Our second main result is the following.

Theorem 3.8. Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Suppose T, S : K → K satisfy the conditions (E) and (C\lambda) for some \(\lambda \in (0, 1)\), TS = ST and the relations
\[ \lambda d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n), \quad \lambda d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n), \tag{3.5} \]
hold for some sequence \{x_n\} in K.
Then \(\lim_{n\to\infty} d(Tx_n, Sx_n)\) exists and if, additionally \(\lim_{n\to\infty} d(Tx_n, Sx_n) = 0\), then T and S have a common fixed point in K.
Theorem 3.5, we obtain that \( \lim_{n \to \infty} d(Tx_n, Sx_n) \) exists. Notice that
\[
d(x_{n+1}, Sx_n) = \lambda d(Tx_n, Sx_n), \quad d(x_{n+1}, Tx_n) = (1 - \lambda)d(Tx_n, Sx_n). \tag{3.6}
\]

When \( \lim_{n \to \infty} d(Tx_n, Sx_n) = 0 \) we can show that
\[
\limsup_{n \to \infty} d(Tx_n, x_n) = \limsup_{n \to \infty} d(Sx_n, x_n) = 0.
\]

Since
\[
d(x_n, Tx_n) \leq d(x_n, Sx_n) + d(Sx_n, Tx_n)
d(x_n, Sx_n) \leq d(x_n, Tx_n) + d(Tx_n, Sx_n),
\]
we obtain
\[
\limsup_{n \to \infty} d(x_n, Tx_n) \leq \limsup_{n \to \infty} d(x_n, Sx_n)
\]
\[
\limsup_{n \to \infty} d(x_n, Sx_n) \leq \limsup_{n \to \infty} d(x_n, Tx_n).
\]
Thus, from (3.6) we get that
\[
\limsup_{n \to \infty} d(Tx_n, x_n) = \limsup_{n \to \infty} d(Sx_n, x_n) = 0.
\]

Let \( A(\{x_n\}) = \{x_0\} \). By Lemma 2.5 we have \( x_0 \in K \). Since \( T \) and \( S \) satisfy the condition \((E)\) we have
\[
d(x_n, Tx_0) \leq \mu_1 d(x_n, Tx_n) + d(x_n, x_0)
d(x_n, Sx_0) \leq \mu_2 d(x_n, Sx_n) + d(x_n, x_0),
\]
for some \( \mu_1, \mu_2 \geq 1 \). Hence by taking limit superior on both sides in above inequalities, we obtain
\[
\limsup_{n \to \infty} d(x_n, Tx_0) \leq \limsup_{n \to \infty} d(x_n, x_0)
\]
\[
\limsup_{n \to \infty} d(x_n, Sx_0) \leq \limsup_{n \to \infty} d(x_n, x_0).
\]
By the uniqueness of the asymptotic center, we obtain \( Tx_0 = Sx_0 = x_0 \). \( \square \)

Corollary 3.9. \((2, \text{Theorem 3.2})\) Let \( K \) be a nonempty closed convex bounded subset of a complete CAT(0) space \( X \). Suppose \( T : K \to K \) satisfies the conditions \((E)\) and \((C_\lambda)\) for some \( \lambda \in [0, 1) \). If \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \), then \( T \) has a fixed point in \( K \).

Proof. Put \( Sx = x \) so \( \lambda d(x_n, Tx_n) = d(x_{n+1}, x_n) \) for all \( n \geq 1 \). By Theorem 3.8 we have \( Tx = x \) for some \( x \in K \). \( \square \)

Our last main result is the following.

Theorem 3.10. Let \( K \) be a nonempty bounded closed convex subset of a complete CAT(0) space \( X \) and let \( T, S : K \to K \) be two mappings which satisfy the conditions \((E)\) and \((C_\lambda)\) and \( TS = ST \). Consider \( x_1 \in K \) and define
\[
x_{n+1} = \alpha_n Tx_n \oplus (1 - \alpha_n)Sx_n, \text{ for all } n \geq 1,
\]
where (for some \( \lambda \in (0,1) \)) \( \{\alpha_n\} \subseteq [\lambda, 1) \) is such that \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and \( \limsup_n \alpha_n < 1 \). Suppose that the following relations hold
\[
\alpha_n d(x_n, Tx_n) \leq d(x_{n+1}, Sx_n),
\]
\[
\alpha_n d(x_n, Sx_n) \leq d(x_{n+1}, Tx_n).
\]
If \( \lim_{n \to \infty} d(Tx_n, Sx_n) = 0 \), then \( \lim_{n \to \infty} d(x_n, p) \) exists, for each \( p \in Fix(T) \cap Fix(S) \).

\textbf{Proof.} By Theorem 3.8, \( Fix(T) \cap Fix(S) \neq \emptyset \). Given \( p \in Fix(T) \cap Fix(S) \), by Lemma 2.2 and Lemma 2.10 we have
\[
d(x_{n+1}, p) = d(\alpha_n Tx_n \oplus (1 - \alpha_n) Sx_n, p) \\
\leq \alpha_n d(Tx_n, p) + (1 - \alpha_n) d(Sx_n, p) \\
= d(x_n, p).
\]
Thus \( d(x_{n+1}, p) \leq d(x_n, p) \). So the sequence \( \{d(x_n, p)\} \) which is bounded below and decreasing, which completes the proof. \( \square \)

\textbf{Corollary 3.11.} ([3, Lemma 5.1]) Let \( K \) be a nonempty bounded closed convex subset of a complete CAT(0) space \( X \) and let \( T : K \to X \) be a mapping satisfying condition (C). Define a sequence \( \{x_n\} \) by \( x_1 \in K \) and \( x_{n+1} = \alpha_n Tx_n \oplus (1 - \alpha_n)x_n \) where \( \{\alpha_n\} \subseteq [\frac{1}{2}, 1) \). Then \( \lim_{n \to \infty} d(x_n, p) \) exists, for each \( p \in Fix(T) \).

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\textbf{References}


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