ULAM-HYERS-RASSIAS STABILITY
FOR SET INTEGRAL EQUATIONS

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Key Words and Phrases: Ulam-Rassias stability, Ulam-Hyers-Rassias stability, fixed point equation, set integral equation.

2010 Mathematics Subject Classification: 45M10, 47H10, 47H04.

1. INTRODUCTION AND PRELIMINARIES

In this paper we will study the following aspects concerning a set integral equation: Ulam stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

We will present first some notions and symbols used in this paper.

Definition 1.1. (I.A. Rus [9]) A function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a comparison function if it satisfies:

(i) \( \varphi \) is monotone increasing;
(ii) \( \varphi^n(t) \) converges to 0, for all \( t > 0 \).

Definition 1.2. (I.A. Rus [9]) A comparison function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be:

(i) a strict comparison function if it satisfies \( t - \varphi(t) \to \infty \), for \( t \to \infty \);
(ii) a strong function if it satisfies \( \sum_{n=1}^{\infty} \varphi^n(t) < \infty \), for all \( t > 0 \).

Remark 1.3. If \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a comparison function then \( \varphi(0) = 0 \) and \( \varphi(t) < t \), for all \( t > 0 \).

Example 1.4. (I.A. Rus [9]) The functions \( \varphi_1 : \mathbb{R}^+ \to \mathbb{R}^+ \), \( \varphi_1(t) = at \) (where \( a \in [0,1] \)) and \( \varphi_2 : \mathbb{R}^+ \to \mathbb{R}^+ \), \( \varphi_2(t) = \frac{t}{1+t} \) are strict comparison functions.

Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a strict comparison function. We denote:

\[ \varphi_\eta := \sup\{t \in \mathbb{R}^+ | t - \varphi(t) \leq \eta\}. \]
Definition 1.5. (I.A. Rus [10]) Let \((X, d)\) be a metric space. A mapping \(A : X \to X\) is a \(\varphi\)-contraction if \(\varphi\) is a comparison function and
\[
d(A(x), A(y)) \leq \varphi(d(x, y)), \quad \text{for all } x, y \in X.
\]

We present now the concept of generalized Ulam-Hyers stability. Let \((X, d)\) be a metric space, \(A : X \to X\) be an operator. Consider the following fixed point equation:
\[
x = A(x), \quad x \in X \tag{1.1}
\]
and for \(\varepsilon > 0\) the following inequation
\[
d(y, A(y)) \leq \varepsilon. \tag{1.2}
\]

Definition 1.6. (I.A. Rus [10]) The equation (1.1) is called generalized Ulam-Hyers stable if there exists \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) increasing, continuous in 0 with \(\psi(0) = 0\), such that for each \(\varepsilon > 0\) and for each solution \(y^* \in X\) of (1.2), there exists a solution \(x^* \in X\) of (1.1) such that:
\[
d(y^*, x^*) \leq \psi(\varepsilon).
\]
In the case that \(\psi(t) := ct\) (for some \(c > 0\)), for all \(t \in \mathbb{R}_+\), then the equation (1.1) is said to be Ulam-Hyers stable.

Theorem 1.7. Let \((X, d)\) be a complete metric space and \(A : X \to X\) be \(\varphi\)-contraction. If the function \(\psi := 1_{\mathbb{R}_+} - \varphi\) is strictly increasing and surjective, then the fixed point equation
\[
x = A(x), \quad x \in X
\]
is generalized Ulam-Hyers stable.

Proof. Let \(\varepsilon > 0\) and \(y^* \in X\) with the property \(d(y^*, A(y^*)) \leq \varepsilon\). By Matkowski-Rus theorem (see [9]) we know that there exists a unique fixed point \(x^*_A \in X\) for \(A\), i.e., \(x^*_A \in A(x^*_A)\). Then
\[
d(x, x^*_A) \leq d(x, A(x)) + d(A(x), x^*_A)
\]
\[
\leq d(x, A(x)) + \varphi(d(x, x^*_A)).
\]
If follows that:
\[
d(x, x^*_A) \leq \psi^{-1}(d(x, A(x))), \quad \text{for all } x \in X.
\]
In particular, for \(x := y^*\) we have that
\[
d(y^*, x^*_A) \leq \psi^{-1}(d(y^*, A(y^*))) \leq \psi^{-1}(\varepsilon).
\]
Then the fixed point equation is generalized Ulam-Hyers stable (with the function \(\psi^{-1}\)). \(\square\)

Remark 1.8. If, in the above result, \(\varphi\) is a strict comparison function, then we have the conclusion
\[
d(y^*, x^*_A) \leq \varphi \varepsilon.
\]
2. Ulam-Hyers stability

Let us denote $P_{cp,cv}(\mathbb{R}^n) := \{X \in P(\mathbb{R}^n) \mid X \text{ is compact and convex}\}$ and let $B_r := \{X \in P_{cp,cv}(\mathbb{R}^n) \mid \text{diam}(X) \leq r\}$, where $r > 0$.

The set $B_r$ is convex and endowed with Pompeiu-Hausdorff metric $H$. Let $F : [a, b] \times [a, b] \times B_r \to B_{r/2}$, and a set $A \in B_{r/2}$.

Consider the set integral equation

$$X(t) = A + \int_a^b F(t, s, X(s))ds.$$  \hfill (2.1)

By a solution of the equation (2.1) we understand a continuous function $X : [a, b] \to B_r$, which satisfies (2.1) for every $t \in [a, b]$.

**Definition 2.1.** Let $F : [a, b] \times [a, b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ and $A \in P_{cp,cv}(\mathbb{R}^n)$. The integral equation (2.1) is generalized Ulam-Hyers stable if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in $0$ with $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $Y^* \in C([a, b], P_{cp,cv}(\mathbb{R}^n))$ of the inequation

$$H(Y(t), A + \int_a^b F(t, s, Y(s))ds) \leq \varepsilon, \quad t \in [a, b]$$

there exists a solution $X^*$ of the equation (2.1) such that

$$||X^* - Y^*||_{C([a, b], P_{cp,cv}(\mathbb{R}^n))} \leq \psi(\varepsilon).$$

If $\psi(t) = ct$, for each $t \in \mathbb{R}_+ \ (c > 0)$ then the equation (2.1) is said to be Ulam-Hyers stable.

**Theorem 2.2.** Let $F : [a, b] \times [a, b] \times B_r \to B_{r/2}$ be continuous and suppose there exist a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $p : [a, b] \times [a, b] \to \mathbb{R}_+$ such that:

$$H(F(t, s, X), F(t, s, Y)) \leq p(t, s)\varphi(H(X, Y))$$

for every $t, s \in [a, b]$, $X, Y \in B_r$, where $\max_{t \in [a, b]} \int_a^b p(t, s) \leq 1$. Then:

(a) For each $A \in B_{r/2}$ the integral equation (2.1) has a unique solution $X(\cdot, A) : [a, b] \to B_r$ with depends continuously on $A$ (see [14]).

(b) If, in addition, we assume that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = t - \varphi(t)$ is strict increasing and surjective, then the set integral equation (2.1) is generalized Ulam-Hyers stable.

**Proof.** For (a) the proof is given in [14]. By (a) we have that the operator

$$UX(t) = A + \int_a^b F(t, s, X(s))ds,$$

where $X \in C([a, b], B_r)$ and $t \in [a, b]$ is a $\varphi$-contraction, i.e., $||UX - UY||_{C([a, b], B_r)} \leq \varphi(||X - Y||_{C([a, b], B_r)})$, for all $X, Y \in C([a, b], B_r)$.

Notice that the fixed point equation $X = UX$ is equivalent with the equation (2.1) and $X^*$ denotes the unique solution of this equation.

Let $\varepsilon > 0$ and $Y^* \in C([a, b], B_r)$ with the property: $||Y^* - UX^*||_{C([a, b], B_r)} \leq \varepsilon$. 

For $X \in \mathcal{C}([a, b], \mathbb{B}_r)$, we have:

$$||X - X^*||_{\mathcal{C}([a, b], \mathbb{B}_r)} \leq ||X - UX||_{\mathcal{C}([a, b], \mathbb{B}_r)} + ||UX - X^*||_{\mathcal{C}([a, b], \mathbb{B}_r)}$$

$$\leq ||X - UX||_{\mathcal{C}([a, b], \mathbb{B}_r)} + \varphi(||X - X^*||_{\mathcal{C}([a, b], \mathbb{B}_r)}).$$

There, we have that $\psi(||X - X^*||_{\mathcal{C}([a, b], \mathbb{B}_r)}) \leq ||X - UX||_{\mathcal{C}([a, b], \mathbb{B}_r)}$ and then $||X - X^*||_{\mathcal{C}([a, b], \mathbb{B}_r)} \leq \psi^{-1}(||X - UX||_{\mathcal{C}([a, b], \mathbb{B}_r)})$, for all $X \in \mathcal{C}([a, b], \mathbb{B}_r)$.

By putting $X := Y^*$, we get

$$||Y^* - X^*||_{\mathcal{C}([a, b], \mathbb{B}_r)} \leq \psi^{-1}(||Y^* - UX^*||_{\mathcal{C}([a, b], \mathbb{B}_r)}) \leq \psi^{-1}(\varepsilon).$$

Then the fixed point equation $X = UX$, is generalized Ulam-Hyers stable with function $\psi^{-1}$. Then, the integral equation (2.1) is generalized Ulam-Hyers stable. \qed

3. ULAM-HYERS-RASSIAS STABILITY

We consider the following integral equations in the space of multivalued operators:

$$X(t) = \int_{a}^{b} K(t, s, X(s))ds + X_0(t), \quad t \in [a, b] \quad (3.1)$$

$$X(t) = \int_{a}^{t} K(t, s, X(s))ds + X_0(t), \quad t \in [a, b], \quad (3.2)$$

where $K : [a, b] \times [a, b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ is a continuous operator and $X_0 \in \mathcal{C}([a, b], P_{cp,cv}(\mathbb{R}^n))$.

A solution of these integral equations in the space of the multivalued operators means a continuous operator $X : [a, b] \to P_{cp,cv}(\mathbb{R}^n)$ which satisfies (3.1) respectively (3.2), for each $t \in [a, b]$.

An auxiliary result is the following.

**Lemma 3.1.** (I.A. Rus [12]) Let $h \in \mathcal{C}([a, b], \mathbb{R}_+)$ and $\beta > 0$ with $\beta(b - a) < 1$. If $u \in \mathcal{C}([a, b], \mathbb{R}_+)$ satisfies

$$u(t) \leq h(t) + \beta \int_{a}^{b} u(s)ds, \text{ for all } t \in [a, b],$$

then

$$u(t) \leq h(t) + \beta(1 - \beta(b - a))^{-1} \int_{a}^{b} h(s)ds, \text{ for all } t \in [a, b].$$

**Theorem 3.2.** Consider the integral equation (3.1). Let $K : [a, b] \times [a, b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ be a multivalued operator. Suppose that:

(i) $K$ is continuous on $[a, b] \times [a, b] \times P_{cp,cv}(\mathbb{R}^n)$ and $X_0 \in \mathcal{C}([a, b], P_{cp,cv}(\mathbb{R}^n))$;
(ii) $K(t, s, \cdot)$ is Lipschitz, i.e. there exists $L_K \geq 0$ such that:

$$H(K(t, s, A), K(t, s, B)) \leq L_K H(A, B),$$

for all $A, B \in P_{cp,cv}(\mathbb{R}^n)$ and for all $t, s \in [a, b]$;

(iii) $L_K(b-a) < 1$;

(iv) $\varphi \in C([a, b], (0, +\infty))$.

Then:

(a) the integral equation (3.1) has a unique solution denoted with $X^*$;

(b) the integral equation (3.1) is generalized Ulam-Hyers-Rassias stable, i.e., if $X \in C([a, b], P_{cp,cv}(\mathbb{R}^n))$ has the property

$$H(X(t), \int_a^b K(t, s, X(s))ds) \leq \varphi(t), \text{ for all } t \in [a, b],$$

then there exists $c_\varphi > 0$ such that

$$H(X(t), X^*(t)) \leq c_\varphi \cdot \varphi(t), \text{ for all } t \in [a, b].$$

Proof. For the proof of (a) we refer to [15]. By (a) we have that $\Gamma : C([a, b], P_{cp,cv}(\mathbb{R}^n)) \to C([a, b], P_{cp,cv}(\mathbb{R}^n))$ the operator, given by $\Gamma X(t) = \int_a^b K(t, s, X(s))ds + X_0$, for all $t \in [a, b]$ is a contraction.

Then the fixed point equation $X = \Gamma X$ has a unique solution

$$X^* \in C([a, b], P_{cp,cv}(\mathbb{R}^n)).$$

We have:

$$H(\Gamma X(t), \Gamma X^*(t)) = H(\int_a^b K(t, s, X(s))ds + X_0(t), \int_a^b K(t, s, X^*(s))ds + X_0(t)) \leq$$

$$\leq \int_a^b H(K(t, s, X(s)), K(t, s, X^*(s)))ds \leq L_K \int_a^b H(X(s), X^*(s))ds.$$

For $X \in C([a, b], P_{cp,cv}(\mathbb{R}^n))$, we have:

$$H(X(t), X^*(t)) \leq H(X(t), \Gamma X(t)) + H(\Gamma X(t), X^*(t))$$

$$= H(X(t), \Gamma X(t)) + H(\Gamma X(t), \Gamma X^*(t))$$

$$\leq \varphi(t) + L_K \int_a^b H(X(s), X^*(s))ds.$$

By Lemma 3.1 we have:

$$H(X(t), X^*(t)) \leq \varphi(t) + L_K(1 - L_K(b-a))^{-1} \int_a^b \varphi(s)ds$$

$$= \varphi(t)[1 + L_K(1 - L_K(b-a))^{-1} \int_a^b \frac{\varphi(s)ds}{\varphi(t)}].$$

By the Mean Integral Theorem, there exists $\alpha \in (a, b)$ such that

$$\varphi(t)[1 + L_K(1 - L_K(b-a))^{-1} \frac{\varphi(\alpha)}{\varphi(t)}] \leq \varphi(t)[1 + L_K(1 - L_K(b-a))^{-1} \frac{M_\varphi}{m_\varphi}] = c_\varphi \cdot \varphi(t).$$
Then, the integral equation (3.1) is generalized Ulam-Hyers-Rassias stable. □

**Lemma 3.3.** (I.A.Rus [12]) Let J be an interval in \( \mathbb{R} \), \( t_0 \in J \) and \( h,k,u \in C(J, \mathbb{R}_+) \). If

\[
    u(t) \leq h(t) + \left| \int_{t_0}^{t} k(s)u(s)ds \right|, \quad \text{for all } t \in J,
\]

then

\[
    u(t) \leq h(t) + \left| \int_{t_0}^{t} h(s)k(s)e^{\int_{t}^{s} k(\sigma)ds}ds \right|, \quad \text{for all } t \in J.
\]

**Theorem 3.4.** Consider the integral equation (3.2). Let \( K : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \rightarrow P_{cp,cv}(\mathbb{R}^n) \) be a multivalued operator and \( X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n)) \).

Suppose that:

(i) \( K \) is continuous on \([a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n)\);

(ii) \( K(t,s,\cdot) \) is Lipschitz, i.e. there exists \( L_K \geq 0 \) such that

\[
    H(K(t,s,A),K(t,s,B)) \leq L_K H(A,B),
\]

for all \( A,B \in P_{cp,cv}(\mathbb{R}^n) \) and \( t,s \in [a,b] \);

(iii) there exists \( \varphi \in C([a,b],[0,\infty)) \) and \( \eta_{\varphi} > 0 \) such that \( \int_{a}^{t} \varphi(s)ds \leq \eta_{\varphi} \cdot \varphi(t) \) for all \( t \in [a,b] \).

Then:

(a) the integral equation (3.2) has a unique solution denoted with \( X^* \);

(b) the integral equation (3.2) it is generalized Ulam-Hyers-Rassias stable, i.e., if \( X \in C([a,b], P_{cp,cv}(\mathbb{R}^n)) \) has the property

\[
    H(X(t), \int_{a}^{t} K(t,s,X(s))ds) \leq \varphi(t), \quad \text{for all } t \in [a,b],
\]

then there exists \( c_{\varphi} > 0 \) such that

\[
    H(X(t), X^*(t)) \leq c_{\varphi} \cdot \varphi(t), \quad \text{for all } t \in [a,b].
\]

**Proof.** For the proof of (a) we refer to [15]. By (a) we have that

\( \Gamma : C([a,b], P_{cp,cv}(\mathbb{R}^n)) \rightarrow C([a,b], P_{cp,cv}(\mathbb{R}^n)) \), given by the operator

\( \Gamma X(t) = \int_{a}^{t} K(t,s,X(s))ds + X_0(t), \quad t \in [a,b] \) is a contraction.

Consider the fixed point equation \( X = \Gamma X \) and let \( X^* \) be the unique solution of this equation. We have:

\[
    H(\Gamma X(t), \Gamma X^*(t)) \leq H(\int_{a}^{t} K(t,s,X(s))ds, \int_{a}^{t} K(t,s,X^*(s))ds) \\
    \leq \int_{a}^{t} H(K(t,s,X(s)), K(t,s,X^*(s)))ds \\
    \leq L_K \int_{a}^{t} H(X(s), X^*(s))ds.
\]
Theorem 3.5. Suppose that:

\[ \phi \]

Then, the integral equation (3.3) is generalized Ulam-Hyers-Rassias stable with respect to the inequation

\[ \exists \text{ solution } X \]

Proof. Let \( Y \) be a solution of the equation (3.3) such that:

\[ H(Y(t), \int_{-\tau}^{t} F(s, Y(s))ds) \leq \varphi(t), \text{ for all } t \in [-\tau, T]. \]

Then, the integral equation (3.2) is generalized Ulam-Hyers-Rassias stable. \( \square \)

Another stability result is the following.

**Theorem 3.5.** Consider the following equation

\[ X(t) = \int_{t-\tau}^{t} F(s, X(s))ds, \text{ where } \tau > 1, \quad t, s \in [-\tau, T]. \] (3.3)

Suppose that:

(i) \( F : [-\tau, T] \times P_{cp,cv}(\mathbb{R}_+) \to P_{cp,cv}(\mathbb{R}_+) \), is continuous;

(ii) there exists \( k \in L^1[-\tau, T] \) such that

\[ H(F(s, A), F(s, B)) \leq k(s)H(A, B), \text{ for all } A, B \in P_{cp,cv}(\mathbb{R}_+) \text{ and } s \in [-\tau, T]; \]

(iii) \( \varphi \in C([-\tau, T], P_{cp,cv}(\mathbb{R}_+)) \);

(iv) there exists \( \lambda_\varphi > 0 \) such that:

\[ \int_{-\tau}^{t} \varphi(s)ds \leq \lambda_\varphi \varphi(t), \text{ for each } t \in [-\tau, T]. \]

Then, the integral equation (3.3) is generalized Ulam-Hyers-Rassias stable with respect to \( \varphi \), i.e., there exists \( c_{F,\varphi} > 0 \) such that for each solution \( Y \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+)) \) of the inequation

\[ H(Y(t), \int_{-\tau}^{t} F(s, Y(s))ds) \leq \varphi(t), \text{ for all } t \in [-\tau, T] \]

with the property

\[ Y(0) = \int_{-\tau}^{0} F(s, Y(s))ds, \]

there exists a solution \( X^* \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+)) \) of the equation (3.3) such that:

\[ H(Y(t), X^*(t)) \leq c_{F,\varphi} \varphi(t), \text{ for all } t \in [0, T]. \]

Proof. Let \( Y \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+)) \) be a solution of the inequality

\[ H(Y(t), \int_{-\tau}^{t} F(s, Y(s))ds) \leq \varphi(t), \text{ for all } t \in [-\tau, T]. \]

Let \( X^* \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+)) \) be the unique solution of the Cauchy problem:
\[
\left\{ \begin{array}{l}
X(t) = \int_{t-\tau}^{t} F(s, X(s))ds, \text{ where } \tau > 1, \quad t, s \in [-\tau, T] \\
X(0) = Y(0)
\end{array} \right.
\]

We have
\[
H(Y(t), X^*(t)) \leq H(Y(t), \int_{t-\tau}^{t} F(s, Y(s))ds) + H(\int_{t-\tau}^{t} F(s, Y(s))ds, X^*(t))
\]
\[
\leq H(Y(t), \int_{t-\tau}^{t} F(s, Y(s))ds) + H(\int_{t-\tau}^{t} F(s, Y(s))ds, \int_{t-\tau}^{t} F(s, X^*(s))ds)
\]
\[
\leq \lambda \varphi \cdot \varphi(t) + \int_{t-\tau}^{t} H(F(s, Y(s)), F(s, X^*(s)))ds
\]
\[
\leq \lambda \varphi \cdot \varphi(t) + \int_{t-\tau}^{t} k(s)H(Y(s), X^*(s))ds.
\]

By Gronwall Lemma we have:
\[
H(Y(s), X^*(s)) \leq \lambda \varphi \cdot \varphi(t) \cdot e^{\int_{t-\tau}^{t} k(s)ds}
\]
\[
\leq [\lambda \varphi \cdot \int_{t-\tau}^{t} k(s)ds] \cdot \varphi(t) = c_{F, \varphi} \cdot \varphi(t)
\]

Then, the integral equation (3.3) is generalized Ulam-Hyers-Rassias stable. □

**References**


Received: May 23, 2010; Accepted: February 3, 2011.