Fixed Point Theory, 13(2012), No. 2, 659-668
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ULAM-HYERS-RASSIAS STABILITY FOR SET INTEGRAL EQUATIONS

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Abstract. The purpose of this paper is to study different kinds of Ulam stabilities for set integral equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.
Key Words and Phrases: Ulam-Rassias stability, Ulam-Hyers-Rassias stability, fixed point equation, set integral equation.
2010 Mathematics Subject Classification: 45M10, 47H10, 47H04.

1. Introduction and preliminaries

In this paper we will study the following aspects concerning a set integral equation: Ulam stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

We will present first some notions and symbols used in this paper.

Definition 1.1. (I.A. Rus [9]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if it satisfies:

(i) φ is monotone increasing;
(ii) $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0, for all $t > 0$.

Definition 1.2. (I.A. Rus [9]) A comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be:

(i) a strict comparison function if it satisfies $t - \varphi(t) \to \infty$, for $t \to \infty$;
(ii) a strong function if it satisfies $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for all $t > 0$.

Remark 1.3. *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function then $\varphi(0) = 0$ and $\varphi(t) < t$, for all $t > 0$.*

Example 1.4. (I.A. Rus [9]) The functions $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_1(t) = at$ (where $a \in]0, 1[$) and $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_2(t) = \frac{t}{1+t}$ are strict comparison functions.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function. We denote:

$$\varphi_\eta := sup\{t \in \mathbb{R}_+ | \ \ t - \varphi(t) \leq \eta\}.$$

Definition 1.5. (I.A. Rus [10]) Let (X, d) be a metric space. A mapping $A : X \to X$ is a φ-contraction if φ is a comparison function and

$$d(A(x), A(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

We present now the concept of generalized Ulam-Hyers stability.

Let (X, d) be a metric space, $A : X \to X$ be an operator. Consider the following fixed point equation:

$$x = A(x), \ x \in X \tag{1.1}$$

and for $\varepsilon > 0$ the following inequation

$$d(y, A(y)) \leq \varepsilon. \tag{1.2}$$

Definition 1.6. (I.A. Rus [10]) The equation (1.1) is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of (1.2), there exists a solution $x^* \in X$ of (1.1) such that:

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

In the case that $\psi(t) := ct$, (for some $c > 0$), for all $t \in \mathbb{R}_+$, then the equation (1.1) is said to be Ulam-Hyers stable.

Theorem 1.7. *Let (X, d) be a complete metric space and $A : X \to X$ be φ-contraction. If the function $\psi := 1_{\mathbb{R}_+} - \varphi$ is strictly increasing and surjective, then the fixed point equation*

$$x = A(x), \quad x \in X$$

is generalized Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$ and $y^* \in X$ with the property $d(y^*, A(y^*)) \leq \varepsilon$. By Matkowski-Rus theorem (see [9]) we know that there exists a unique fixed point $x_A^* \in X$ for A, i.e., $x_A^* \in A(x_A^*)$. Then

$$\begin{aligned} d(x, x_A^*) &\leq d(x, A(x)) + d(A(x), x_A^*) \\ &\leq d(x, A(x)) + \varphi(d(x, x_A^*)). \end{aligned}$$

If follows that: $d(x, x_A^*) \leq \psi^{-1}(d(x, A(x))), \text{ for all } x \in X.$

In particular, for $x := y^*$ we have that

$$d(y^*, x_A^*) \leq \psi^{-1}(d(y^*, A(y^*))) \leq \psi^{-1}(\varepsilon).$$

Then the fixed point equation is generalized Ulam-Hyers stable (with the function ψ^{-1}). □

Remark 1.8. *If, in the above result, φ is a strict comparison function, then we have the conclusion*

$$d(y^*, x_A^*) \leq \varphi_\varepsilon.$$

2. Ulam-Hyers stability

Let us denote $P_{cp,cv}(\mathbb{R}^n) := \{X \in P(\mathbb{R}^n) | X \ \ is \ \ compact \ \ and \ \ convex\}$ and let $\mathbf{B}_r := \{X \in P_{cp,cv}(\mathbb{R}^n) | diam(X) \leq r\}$, where $r > 0$.

The set $\mathbf{B}_r$ is convex and endowed with Pompeiu-Hausdorff metric H is complete. Let $F : [a,b] \times [a,b] \times \mathbf{B}_r \to \mathbf{B}_{r/2}$, and a set $A \in \mathbf{B}_{r/2}$.

Consider the set integral equation

$$X(t) = A + \int_a^b F(t,s,X(s))ds. \tag{2.1}$$

By a solution of the equation (2.1) we understand a continuous function $X : [a,b] \to \mathbf{B}_r$, which satisfies (2.1) for every $t \in [a,b]$.

Definition 2.1. Let $F : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ and $A \in P_{cp,cv}(\mathbb{R}^n)$. The integral equation (2.1) is generalized Ulam-Hyers stable if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $Y^* \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$ of the inequation

$$H(Y(t), A + \int_a^b F(t,s,Y(s))ds) \leq \varepsilon, \quad t \in [a,b]$$

there exists a solution X^* of the equation (2.1) such that

$$||X^* - Y^*||_{C([a,b],P_{cp,cv}(\mathbb{R}^n))} \leq \psi(\varepsilon).$$

If $\psi(t) = ct$, for each $t \in \mathbb{R}_+$ $(c > 0)$ then the equation (2.1) is said to be Ulam-Hyers stable.

Theorem 2.2. *Let $F : [a,b] \times [a,b] \times \mathbf{B}_r \to \mathbf{B}_{r/2}$ be continuous and suppose there exist a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $p : [a,b] \times [a,b] \to \mathbb{R}_+$ such that:*

$$H(F(t,s,X), F(t,s,Y)) \leq p(t,s)\varphi(H(X,Y))$$

for every $t, s \in [a,b]$, $X, Y \in \mathbf{B}_r$, where $\max\limits_{t\in[a,b]} \int_a^b p(t,s) \leq 1$. Then:

- (a) *For each $A \in \mathbf{B}_{r/2}$ the integral equation (2.1) has a unique solution $X(\cdot, A) : [a,b] \to \mathbf{B}_r$ with depends continuously on A (see* [14]*).*
- (b) *If, in addition, we assume that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = t - \varphi(t)$ is strict increasing and surjective, then the set integral equation (2.1) is generalized Ulam-Hyers stable.*

Proof. For (a) the proof is given in [14]. By (a) we have that the operator $U : (\mathcal{C}([a,b], \mathbf{B}_r), ||\cdot||_{\mathcal{C}([a,b],\mathbf{B}_r)}) \to (\mathcal{C}([a,b], \mathbf{B}_r), ||\cdot||_{\mathcal{C}([a,b],\mathbf{B}_r)})$, given by $UX(t) = A + \int_a^b F(t,s,X(s))ds$, where $X \in \mathcal{C}([a,b], \mathbf{B}_r)$ and $t \in [a,b]$ is a φ- contraction, i.e., $||UX - UY||_{\mathcal{C}([a,b],\mathbf{B}_r)} \leq \varphi(||X - Y||_{\mathcal{C}([a,b],\mathbf{B}_r)})$, for all $X, Y \in \mathcal{C}([a,b], \mathbf{B}_r)$.

Notice that the fixed point equation $X = UX$ is equivalent with the equation (2.1) and X^* denotes the unique solution of this equation.

Let $\varepsilon > 0$ and $Y^* \in \mathcal{C}([a,b], \mathbf{B}_r)$ with the property: $||Y^* - UY^*||_{\mathcal{C}([a,b],\mathbf{B}_r)} \leq \varepsilon$.

For $X \in \mathcal{C}([a,b],\mathbf{B}_r)$, we have:

$$\begin{aligned}||X - X^*||_{\mathcal{C}([a,b],\mathbf{B}_r)} &\le ||X - UX||_{\mathcal{C}([a,b],\mathbf{B}_r)} + ||UX - X^*||_{\mathcal{C}([a,b],\mathbf{B}_r)}\\ &= ||X - UX||_{\mathcal{C}([a,b],\mathbf{B}_r)} + ||UX - UX^*||_{\mathcal{C}([a,b],\mathbf{B}_r)}\\ &\le ||X - UX||_{\mathcal{C}([a,b],\mathbf{B}_r)} + \varphi(||X - X^*||_{\mathcal{C}([a,b],\mathbf{B}_r)}).\end{aligned}$$

There, we have that $\psi(||X - X^*||_{\mathcal{C}([a,b],\mathbf{B}_r)}) \le ||X - UX||_{\mathcal{C}([a,b],\mathbf{B}_r)}$ and then $||X - X^*||_{\mathcal{C}([a,b],\mathbf{B}_r)} \le \psi^{-1}(||X - UX||_{\mathcal{C}([a,b],\mathbf{B}_r)})$, for all $X \in \mathcal{C}([a,b],\mathbf{B}_r)$.

By putting $X := Y^*$, we get

$$||Y^* - X^*||_{\mathcal{C}([a,b],\mathbf{B}_r)} \le \psi^{-1}(||Y^* - UY^*||_{\mathcal{C}([a,b],\mathbf{B}_r)}) \le \psi^{-1}(\varepsilon).$$

Then the fixed point equation $X = UX$, is generalized Ulam-Hyers stable with function ψ^{-1}. Then, the integral equation (2.1) is generalized Ulam-Hyers stable. □

3. Ulam-Hyers-Rassias stability

We consider the following integral equations in the space of multivalued operators:

$$X(t) = \int_a^b K(t,s,X(s))ds + X_0(t), \quad t \in [a,b] \tag{3.1}$$

$$X(t) = \int_a^t K(t,s,X(s))ds + X_0(t), \quad t \in [a,b], \tag{3.2}$$

where $K : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ is a continuous operator and $X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$.

A solution of these integral equations in the space of the multivalued operators means a continuous operator $X : [a,b] \to P_{cp,cv}(\mathbb{R}^n)$ which satisfies (3.1) respectively (3.2), for each $t \in [a,b]$.

An auxiliary result is the following.

Lemma 3.1. (I.A. Rus [12]) *Let $h \in C([a,b],\mathbb{R}_+)$ and $\beta > 0$ with $\beta(b-a) < 1$. If $u \in C([a,b],\mathbb{R}_+)$ satisfies*

$$u(t) \le h(t) + \beta \int_a^b u(s)ds, \ \textit{for all } t \in [a,b],$$

then

$$u(t) \le h(t) + \beta(1 - \beta(b-a))^{-1} \int_a^b h(s)ds, \ \textit{for all } t \in [a,b].$$

Theorem 3.2. *Consider the integral equation (3.1). Let*

$$K : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$$

be a multivalued operator. Suppose that:

(i) *K is continuous on $[a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n)$ and $X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$;*

(ii) $K(t,s,\cdot)$ *is Lipschitz, i.e. there exists* $L_K \geq 0$ *such that:*

$$H(K(t,s,A),K(t,s,B)) \leq L_K H(A,B),$$

for all $A,B \in P_{cp,cv}(\mathbb{R}^n)$ *and for all* $t,s \in [a,b]$*;*

(iii) $L_K(b-a) < 1$.

(iv) $\varphi \in C([a,b],(0,+\infty))$.

Then:

(a) *the integral equation (3.1) has a unique solution denoted with* X^**;*

(b) *the integral equation (3.1) it is generalized Ulam-Hyers-Rassias stable, i.e., if* $X \in C([a,b],P_{cp,cv}(\mathbb{R}^n))$ *has the property*

$$H(X(t),\int_a^b K(t,s,X(s))ds) \leq \varphi(t), \text{ for all } t \in [a,b],$$

then there exists $c_\varphi > 0$ *such that*

$$H(X(t),X^*(t)) \leq c_\varphi \cdot \varphi(t), \text{ for all } t \in [a,b].$$

Proof. For the proof of (a) we refer to [15] . By (a) we have that $\Gamma : C([a,b],P_{cp,cv}(\mathbb{R}^n)) \to C([a,b],P_{cp,cv}(\mathbb{R}^n))$ the operator, given by $\Gamma X(t) = \int_a^b K(t,s,X(s))ds + X_0$, for all $t \in [a,b]$ is a contraction.

Then the fixed point equation $X = \Gamma X$ has a unique solution

$$X^* \in C([a,b],P_{cp,cv}(\mathbb{R}^n)).$$

We have:

$$H(\Gamma X(t),\Gamma X^*(t)) = H(\int_a^b K(t,s,X(s))ds + X_0(t),\int_a^b K(t,s,X^*(s))ds + X_0(t)) \leq$$
$$\leq \int_a^b H(K(t,s,X(s)),K(t,s,X^*(s)))ds \leq L_K \int_a^b H(X(s),X^*(s))ds.$$

For $X \in C([a,b],P_{cp,cv}(\mathbb{R}^n))$, we have:

$$\begin{aligned} H(X(t),X^*(t)) &\leq H(X(t),\Gamma X(t)) + H(\Gamma X(t),X^*(t)) \\ &= H(X(t),\Gamma X(t)) + H(\Gamma X(t),\Gamma X^*(t)) \\ &\leq \varphi(t) + L_K \int_a^b H(X(s),X^*(s))ds. \end{aligned}$$

By Lemma 3.1 we have:

$$\begin{aligned} H(X(t),X^*(t)) &\leq \varphi(t) + L_K(1-L_K(b-a))^{-1}\int_a^b \varphi(s)ds \\ &= \varphi(t)[1+L_K(1-L_K(b-a))^{-1}\frac{\int_a^b \varphi(s)ds}{\varphi(t)}]. \end{aligned}$$

By the Mean Integral Theorem, there exists $\alpha \in (a,b)$ such that

$$\varphi(t)[1+L_K(1-L_K(b-a))^{-1}\frac{\varphi(\alpha)}{\varphi(t)}] \leq \varphi(t)[1+L_K(1-L_K(b-a))^{-1}\frac{M_\varphi}{m_\varphi}] = c_\varphi \cdot \varphi(t).$$

Then, the integral equation (3.1) is generalized Ulam-Hyers-Rassias stable. □

Lemma 3.3. (I.A.Rus [12]) *Let J be an interval in $\mathbb{R}$, $t_0 \in J$ and $h, k, u \in C(J, \mathbb{R}_+)$. If*

$$u(t) \leq h(t) + \left| \int_{t_0}^{t} k(s)u(s)ds \right|, \text{ for all } t \in J,$$

then

$$u(t) \leq h(t) + \left| \int_{t_0}^{t} h(s)k(s)e^{\left| \int_s^t k(\sigma)d\sigma \right|} ds \right|, \text{ for all } t \in J.$$

Theorem 3.4. *Consider the integral equation (3.2). Let $K : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ be a multivalued operator and $X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$. Suppose that:*

(i) *K is continuous on $[a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n)$;*
(ii) *$K(t,s,\cdot)$ is Lipschitz, i.e. there exists $L_K \geq 0$ such that*

$$H(K(t,s,A), K(t,s,B)) \leq L_K H(A,B),$$

for all $A, B \in P_{cp,cv}(\mathbb{R}^n)$ and $t, s \in [a,b]$;

(iii) *there exists $\varphi \in C([a,b], (0,+\infty))$ and $\eta_\varphi > 0$ such that $\int_a^t \varphi(s)ds \leq \eta_\varphi \cdot \varphi(t)$ for all $t \in [a,b]$.*

Then:

(a) *the integral equation (3.2) has a unique solution denoted with X^*;*
(b) *the integral equation (3.2) it is generalized Ulam-Hyers-Rassias stable, i.e., if $X \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$ has the property*

$$H(X(t), \int_a^t K(t,s,X(s))ds) \leq \varphi(t), \text{ for all } t \in [a,b],$$

then there exists $c_\varphi > 0$ such that

$$H(X(t), X^*(t)) \leq c_\varphi \cdot \varphi(t), \text{ for all } t \in [a,b].$$

Proof. For the proof of (a) we refer to [15] . By (a) we have that $\Gamma : C([a,b], P_{cp,cv}(\mathbb{R}^n)) \to C([a,b], P_{cp,cv}(\mathbb{R}^n))$, given by the operator $\Gamma X(t) = \int_a^t K(t,s,X(s))ds + X_0(t), \quad t \in [a,b]$ is a contraction.

Consider the fixed point equation $X = \Gamma X$ and let X^* be the unique solution of this equation. We have:

$$\begin{aligned} H(\Gamma X(t), \Gamma X^*(t)) &\leq H(\int_a^t K(t,s,X(s))ds, \int_a^t K(t,s,X^*(s))ds) \\ &\leq \int_a^t H(K(t,s,X(s)), K(t,s,X^*(s)))ds \\ &\leq L_K \int_a^t H(X(s), X^*(s))ds. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} H(X(t), X^*(t)) &\leq H(X(t), \Gamma(X)(t)) + H(\Gamma(X)(t), X^*(t)) \\ &= H(X(t), \Gamma(X)(t)) + H(\Gamma(X)(t), \Gamma(X^*)(t)) \\ &\leq \varphi(t) + L_K \int_a^t H(X(s), X^*(s))ds. \end{aligned}$$

By Lemma 3.3 we have:

$$\begin{aligned} H(X(t), X^*(t)) &\leq \varphi(t) + L_K e^{L_K(b-a)} \int_a^t \varphi(s)ds \\ &\leq [1 + \eta_\varphi L_K e^{L_K(b-a)}]\varphi(t) = c_\varphi \varphi(t). \end{aligned}$$

Then, the integral equation (3.2) is generalized Ulam-Hyers-Rassias stable. □

Another stability result is the following.

Theorem 3.5. *Consider the following equation*

$$X(t) = \int_{t-\tau}^t F(s, X(s))ds, \text{ where } \tau > 1, \quad t, s \in [-\tau, T]. \tag{3.3}$$

Suppose that:

(i) $F : [-\tau, T] \times P_{cp,cv}(\mathbb{R}_+) \to P_{cp,cv}(\mathbb{R}_+)$, *is continuous;*

(ii) *there exists* $k \in L^1[-\tau, T]$ *such that* $H(F(s, A), F(s, B) \leq k(s)H(A, B)$, *for all* $A, B \in P_{cp,cv}(\mathbb{R}_+)$ *and* $s \in [-\tau, T]$;

(iii) $\varphi \in C([-\tau, T], P_{cp,cv}(\mathbb{R}_+))$;

(iv) *there exists* $\lambda_\varphi > 0$ *such that:* $\int_{t-\tau}^t \varphi(s)ds \leq \lambda_\varphi \cdot \varphi(t)$, *for each* $t \in [-\tau, T]$.

Then, the integral equation (3.3) is generalized Ulam-Hyers-Rassias stable with respect to φ, *i.e., there exists* $c_{F,\varphi} > 0$ *such that for each solution* $Y \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+))$ *of the inequation*

$$H(Y(t), \int_{t-\tau}^t F(s, Y(s))ds) \leq \varphi(t), \text{ for all } t \in [-\tau, T]$$

with the property

$$Y(0) = \int_{-\tau}^0 F(s, Y(s))ds,$$

there exists a solution $X^* \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+))$ *of the equation (3.3) such that:*

$$H(Y(t), X^*(t)) \leq c_{F,\varphi} \cdot \varphi(t), \text{ for all } t \in [0, T].$$

Proof. Let $Y \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+))$ be a solution of the inequality

$$H(Y(t), \int_{t-\tau}^t F(s, Y(s))ds \leq \varphi(t), \text{ for all } t \in [-\tau, T].$$

Let $X^* \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+))$ be the unique solution of the Cauchy problem:

$$\begin{cases} X(t) = \int_{t-\tau}^{t} F(s, X(s))ds, \text{ where } \tau > 1, & t, s \in [-\tau, T] \\ & X(0) = Y(0) \end{cases}$$

We have

$$H(Y(t), X^*(t)) \leq H(Y(t), \int_{t-\tau}^{t} F(s, Y(s))ds) + H(\int_{t-\tau}^{t} F(s, Y(s))ds, X^*(t))$$

$$\leq H(Y(t), \int_{t-\tau}^{t} F(s, Y(s))ds) + H(\int_{t-\tau}^{t} F(s, Y(s))ds, \int_{t-\tau}^{t} F(s, X^*(s))ds)$$

$$\leq \lambda_\varphi \cdot \varphi(t) + \int_{t-\tau}^{t} H(F(s, Y(s)), F(s, X^*(s)))ds$$

$$\leq \lambda_\varphi \cdot \varphi(t) + \int_{t-\tau}^{t} k(s) H(Y(s), X^*(s))ds.$$

By Gronwall Lemma we have:

$$H(Y(s), X^*(s)) \leq \lambda_\varphi \cdot \varphi(t) \cdot e^{\int_{t-\tau}^{t} k(s)ds}$$

$$\leq [\lambda_\varphi \cdot l^{\int_{t-\tau}^{+\infty} k(s)ds}] \cdot \varphi(t) = c_{F,\varphi} \cdot \varphi(t)$$

Then, the integral equation (3.3) is generalized Ulam-Hyers-Rassias stable. □

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Received: May 23, 2010; Accepted: February 3, 2011.