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ULAM-HYERS STABILITY OF OPERATORIAL INCLUSIONS IN COMPLETE GAUGE SPACES

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Abstract. Using the weakly Picard operator technique, we will present some Ulam-Hyers stability results for operatorial inclusions using the weakly Picard operator technique. We study the Ulam-Hyers stability of some inclusions, where the multivalued operator satisfies some contraction conditions. We also give an application to integral inclusion.
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1. Introduction

In 1959, G. Marinescu [15] extended the Banach Contraction Principle to locally convex spaces, while I. Colojoară [7] and N. Gheorghiu [12] do the same thing to gauge spaces and R. J. Knill [14] to uniform spaces. In 1971, Cain and Nashed [4] extended the notion of contraction to Hausdorff locally convex linear spaces. They showed that on sequentially complete subset Banach Contraction Principle is still valid. V.G. Angelov [1] introduced in 1987 the notion of generalized φ-contractive single-valued map in gauge spaces, meanwhile the concept for multivalued operators was given in 1998 (see V.G. Angelov [2]). In 2000, M. Frigon [10] introduced the notion of singlevalued generalized contraction in gauge spaces and proved that every generalized contraction on a complete gauge space has a unique fixed point.

Definition 1.1. Let X be any set. A map $p : X \times X \to \mathbb{R}_+$ is called a pseudometric (a gauge) in X whenever

(1) $p(x, y) \geq 0$, for all $x, y \in X$;
(2) If $x = y$, then $p(x, y) = 0$;
(3) $p(x, y) = p(y, x)$, for all $x, y \in X$;
(4) $p(x, z) \leq p(x, y) + p(y, z)$, for all $x, y, z \in X$.

Definition 1.2. A family $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ of pseudometrics on X (or a gauge structure on X), where A is a directed set, is said to be separating if for each pair of points $x, y \in X$, with $x \neq y$, there is a $p_\alpha \in \mathcal{P}$ such that $p_\alpha(x, y) \neq 0$.

A pair $(X, \mathcal{P})$ of a nonempty set X and a separating gauge structure $\mathcal{P}$ on X is called a gauge space.

It is well known (see Dugundji [8], pages 198-204) that any family $\mathcal{P}$ of pseudometrics on a set X induces on X a uniform structure $\mathcal{U}$ and conversely, any uniform structure $\mathcal{U}$ on X is induced by a family of pseudometrics on X. In addition, we have that $\mathcal{U}$ is separating (or Hausdorff) if and only if $\mathcal{P}$ is separating. Thus we may identify the gauge spaces and the Hausdorff uniform spaces.

Definition 1.3. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in X is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_n, x_{n+p}) \leq \varepsilon$ for all $n \geq N$ and $p \in \mathbb{N}$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is called convergent if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_0, x_n) \leq \varepsilon$ for all $n \geq N$.

Definition 1.4. A gauge space is called complete if any Cauchy sequence is convergent.

A subset of X is said to be sequentially closed if it contains the limit of any convergent sequence of its elements.

For further details see J. Dugundji [8], A. Granas, J. Dugundji [13], M. Frigon [11], I. A. Rus, A. Petruşel, G. Petruşel [24], T.P. Petru [17].

Let $(X, \mathcal{P})$ be a gauge space and consider the following families of subsets of X:

$$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\},\ P_b(X) := \{Y \in P(X) |\ Y \text{ is bounded}\},$$

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\},\ P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\}.$$

If $(X, \mathcal{P})$ is a gauge space, then the gap functional in $P((X, \mathcal{P}))$ for every $\alpha \in A$ is defined as

$$D_\alpha : P((X, \mathcal{P})) \times P((X, \mathcal{P})) \to \mathbb{R}_+,\ D_\alpha(B, C) = \inf\{p_\alpha(b, c) \mid b \in B,\ c \in C\}.$$

In particular, if $x_0 \in X$ then $D_\alpha(x_0, B) := D_\alpha(\{x_0\}, B)$.

We will denote by H_α the generalized Pompeiu-Hausdorff functional on $P((X, \mathcal{P}))$, defined as

$$H_\alpha : P((X, \mathcal{P})) \times P((X, \mathcal{P})) \to \mathbb{R}_+ \cup \{+\infty\},$$
$$H_\alpha(B, C) = \max\{\sup_{b\in B} D_\alpha(b, C), \sup_{c\in C} D_\alpha(c, B)\}.$$

Let $(X, \mathcal{P})$ be a gauge space. If $F : (X, \mathcal{P}) \to P((X, \mathcal{P}))$ is a multivalued operator, then $x \in X$ is called fixed point for F if and only if $x \in F(x)$. The set

$$Fix(F) := \{x \in X |\ x \in F(x)\}$$

is called the fixed point set of F.

For a multivalued operator $F : (X, \mathcal{P}) \to P((X, \mathcal{P}))$ we will denote by

$$Graph(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

the graphic of F.

Notice that $f : (X, \mathcal{P}) \to (X, \mathcal{P})$ is a selection for $F : (X, \mathcal{P}) \to P((X, \mathcal{P}))$ if $f(x) \in F(x)$, for each $x \in X$.

For the following notions see I.A. Rus [21] and [20], I.A. Rus, A. Petruşel, A. Sîntămărian [25] and A. Petruşel [19].

Definition 1.5. Let $(X,\mathcal{P})$ be a gauge space and $F:(X,\mathcal{P})\to P_{cl}((X,\mathcal{P}))$ be a multivalued operator. By definition, F is a multivalued weakly Picard (briefly MWP) operator if for each $x\in X$ and each $y\in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0=x$, $x_1=y$;

(ii) $x_{n+1}\in F(x_n)$, for each $n\in\mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Remark 1.6. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the condition *(*i*)* and *(*ii*)*, in the Definition *1* is called a sequence of successive approximations of F starting from $(x,y)\in Graph(F)$.

If $F:(X,\mathcal{P})\to P((X,\mathcal{P}))$ is a MWP operator, then we define $F^{\infty}:Graph(F)\to P(FixF)$ by the formula $F^{\infty}(x,y):=\{\ z\in Fix(F)\mid$ there exists a sequence of successive approximations of F starting from (x,y) that converges to $z\ \}$.

Definition 1.7. Let $(X,\mathcal{P})$ be a gauge space and $F:(X,\mathcal{P})\to P((X,\mathcal{P}))$ be a MWP operator. Then, F is called a c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection f^{∞} of F^{∞} such that

$$p_\alpha(x,f^{\infty}(x,y))\le c\cdot p_\alpha(x,y),\ \text{for all } (x,y)\in Graph(F),\ \alpha\in A.$$

For the theory of multivalued weakly Picard operators see [19] and [25].

The purpose of this paper is to present some results concerning the Ulam-Hyers stability of some operatorial inclusions (such as the fixed point inclusion) by using the weakly Picard operator technique.

2. Ulam-Hyers stability for fixed point inclusions

Definition 2.1. Let $(X,\mathcal{P})$ be a gauge space and $F:(X,\mathcal{P})\to P((X,\mathcal{P}))$ be a multivalued operator. The fixed point inclusion

$$x\in F(x),\ x\in X \tag{2.1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi:=\{\psi_\alpha\}_{\alpha\in A}$ a family of mappings, where each $\psi_\alpha:\mathbb{R}_+\to\mathbb{R}_+$ is increasing, continuous in 0 and $\psi_\alpha(0)=0$ such that for each $\varepsilon_\alpha>0$ and for each solution $y^*\in X$ of the inequation

$$D_\alpha(y,F(y))\le\varepsilon_\alpha \tag{2.2}$$

there exists a solution x^* of the fixed point inclusion (2.1) such that

$$p_\alpha(y^*,x^*)\le\psi_\alpha(\varepsilon_\alpha),\ \text{for all } \alpha\in A.$$

If there exists $c=\{c_\alpha\}_{\alpha\in A}\in(0,\infty)^A$ such that $\psi_\alpha(t):=c_\alpha t$, for each $t\in\mathbb{R}_+$ and $\alpha\in A$, then the fixed point inclusion (2.1) is said to be Ulam-Hyers stable.

Definition 2.2. (Espinola-Petruşel [9]) Let $(X,\mathcal{P})$ be a gauge space and $F:(X,\mathcal{P})\to P((X,\mathcal{P}))$ be a multivalued operator. Then F is an admissible $c_\alpha-$MWP operator (briefly $c_\alpha-$ AMWP operator) if and only if $c_\alpha\in(0,\infty)$, for each $\alpha\in A$ and the following conditions are satisfied:

1) there exists a selection f^∞ of F^∞ such that

$$p_\alpha(x, f^\infty(x,y)) \leq c_\alpha p_\alpha(x,y), \text{ for all } (x,y) \in Graph(F) \text{ and for all } \alpha \in A.$$

2) for every $x \in X$ and every $q = \{q_\alpha\} \in (1,\infty)^A$, there exists $y \in F(x)$ such that

$$p_\alpha(x,y) \leq q_\alpha D_\alpha(x, F(x)), \text{ for all } \alpha \in A.$$

Theorem 2.3. Let $(X, \mathcal{P})$ be a gauge space and $F : (X,\mathcal{P}) \to P((X,\mathcal{P}))$ be a c_α-AMWP operator. Then, the fixed point inclusion (2.1) is Ulam-Hyers stable.

Proof. Let $\varepsilon_\alpha > 0$ and $y^* \in F^\infty(x,y)$ be a solution of (2.2), i.e., $D_\alpha(y^*, F(y^*)) \leq \varepsilon_\alpha$, for all $\alpha \in A$. From condition 2) of Definition 2 we have that for every $q = \{q_\alpha\} \in (1,\infty)^A$ there exists $u^* \in F(y^*)$ such that $p_\alpha(y^*, u^*) \leq q_\alpha D_\alpha(y^*, F(y^*))$. Then from condition 1) from Definition 2 for $(u^*, y^*) \in Graph(F)$ we consider $x^* := f^\infty(y^*, u^*)$ which satisfies the following relations:

(i) x^* is a solution of the fixed point inclusion (2.1).

(ii) $p_\alpha(y^*, x^*) = p_\alpha(y^*, f^\infty(y^*, u^*)) \leq c_\alpha p_\alpha(y^*, u^*)$
$\leq c_\alpha q_\alpha D_\alpha(y^*, F(y^*)) \leq c_\alpha q_\alpha \varepsilon_\alpha.$

Thus the fixed point inclusion is Ulam-Hyers stable.

O'Regan, Petruşel and Petru proved in [16] the existence of a fixed point for a Ćirić-type multivalued operator. We will show now that the fixed point inclusion (2.1) is Ulam-Hyers stable provided the multivalued operator F satisfies a Ćirić-type contraction condition. Our tool will be the abstract result given in Theorem 2.

Theorem 2.4. Let $(X, \mathcal{P})$ be a complete gauge space and $F : (X,\mathcal{P}) \to P((X,\mathcal{P})$ be a multivalued operator with closed graph. We suppose that:

(i) there exists $\{a_\alpha\}_{\alpha\in A} \in (0,1)^A$ such that, for every $\alpha \in A$, the following implication holds: for each $x, y \in X$ we have:

$$H_\alpha(F(x), F(y)) \leq a_\alpha \cdot M_\alpha^F(x,y),$$

where

$$M_\alpha^F(x,y) := \max\{p_\alpha(x,y), D_\alpha(x,F(x)), D_\alpha(y,F(y)), \frac{1}{2}[D_\alpha(x,F(y)) + D_\alpha(y,F(x))]\}.$$

(ii) for every $x, y \in X$, every $u \in F(x)$ and every $q = \{q_\alpha\}_{\alpha\in A} \in (1,\infty)^A$ there exists $v \in F(y)$ such that $p_\alpha(u,v) \leq q_\alpha \cdot H_\alpha(F(x), F(y))$, for every $\alpha \in A$.

Then the fixed point inclusion (2.1) is Ulam-Hyers stable.

Proof. We have to show that F is a c_α-AMWP operator.

Let $x_0 \in X$ and $x_1 \in F(x_0)$ be arbitrary. For every $q = \{q_\alpha\}_{\alpha\in A} \in (1,\infty)^A$, by (ii), there exists $x_2 \in F(x_1)$ such that

$$p_\alpha(x_1, x_2) \leq q_\alpha H_\alpha(F(x_0), F(x_1)), \text{ for each } \alpha \in A.$$

Then:

$$\begin{aligned} p_\alpha(x_1,x_2) &\leq q_\alpha H_\alpha(F(x_0), F(x_1)) \\ &\leq q_\alpha a_\alpha M_\alpha^F(x_0, x_1) \\ &= q_\alpha a_\alpha \max\{p_\alpha(x_0,x_1), D_\alpha(x_0,F(x_0)), D_\alpha(x_1,F(x_1)), \frac{1}{2}D_\alpha(x_0,F(x_1))\}. \end{aligned}$$

We introduce the following notation:

$$\Gamma := \max\{p_\alpha(x_0,x_1), D_\alpha(x_0,F(x_0)), D_\alpha(x_1,F(x_1)), \frac{1}{2}D_\alpha(x_0,F(x_1))\}$$

and will choose $q=\{q_\alpha\}_{\alpha\in A}\in(1,\infty)^A$ such that $1<q_\alpha<\frac{1}{a_\alpha}$, for each $\alpha\in A$.

If $\Gamma=p_\alpha(x_0,x_1)$ then $p_\alpha(x_1,x_2)\le q_\alpha a_\alpha p_\alpha(x_0,x_1)$.

If $\Gamma=D_\alpha(x_0,F(x_0))$ then since $D_\alpha(x_0,F(x_0))\le p_\alpha(x_0,x_1)$ we have

$$p_\alpha(x_1,x_2)\le q_\alpha a_\alpha p_\alpha(x_0,x_1).$$

If $\Gamma = D_\alpha(x_1,F(x_1))$ then $p_\alpha(x_1,x_2)\le q_\alpha a_\alpha D_\alpha(x_1,F(x_1))\le q_\alpha a_\alpha p_\alpha(x_1,x_2)$, which is a contradiction since $1<q_\alpha<\frac{1}{a_\alpha}$, for each $\alpha\in A$.

If $\Gamma=\frac{1}{2}D_\alpha(x_0,F(x_1))$ then

$$\begin{aligned} p_\alpha(x_1,x_2) &\le q_\alpha a_\alpha\frac{1}{2}D_\alpha(x_0,F(x_1))\le\frac{q_\alpha a_\alpha}{2}p_\alpha(x_0,x_2)\le\\ &\le\frac{q_\alpha a_\alpha}{2}[p_\alpha(x_0,x_1)+p_\alpha(x_1,x_2)].\end{aligned}$$

Hence, we obtain that

$$p_\alpha(x_1,x_2)\le\frac{q_\alpha a_\alpha}{2-q_\alpha a_\alpha}p_\alpha(x_0,x_1).$$

Then

$$\begin{aligned}\Gamma &= \frac{1}{2}D_\alpha(x_0,F(x_1))\le\frac{1}{2}p_\alpha(x_0,x_1)\le\frac{1}{2}[p_\alpha(x_0,x_1)+p_\alpha(x_1,x_2)]\le\\ &\le\frac{1}{2}[1+\frac{q_\alpha a_\alpha}{2-q_\alpha a_\alpha}]p_\alpha(x_0,x_1)=\frac{1}{2-q_\alpha a_\alpha}p_\alpha(x_0,x_1)<p_\alpha(x_0,x_1),\end{aligned}$$

which is a contradiction with the definition of Γ.

Thus in all cases we have that

$$p_\alpha(x_1,x_2)\le q_\alpha a_\alpha p_\alpha(x_0,x_1).$$

By induction, we will obtain a sequence $(x_n)_{n\in\mathbb{N}}$ of succesive approximations for F starting from x_0, satisfying the following assertion:

$$p_\alpha(x_n,x_{n+1})\le(q_\alpha a_\alpha)^n p_\alpha(x_0,x_1),\ \text{for every } n\in\mathbb{N}^* \text{ and } \alpha\in A.$$

For each $n,m\in\mathbb{N}^*$ and for every $\alpha\in A$ we have

$$\begin{aligned} p_\alpha(x_n,x_{n+m}) &\le p_\alpha(x_n,x_{n+1})+\dots+p_\alpha(x_{n+m-1},x_{n+m})\le\\ &\le [1+\dots+(q_\alpha a_\alpha)^{m-1}]\cdot(q_\alpha a_\alpha)^n p_\alpha(x_0,x_1)=\\ &= \frac{1-(q_\alpha a_\alpha)^m}{1-q_\alpha a_\alpha}\cdot(q_\alpha a_\alpha)^n p_\alpha(x_0,x_1)\le\frac{(q_\alpha a_\alpha)^n}{1-q_\alpha a_\alpha}p_\alpha(x_0,x_1).\end{aligned}$$

Letting $n\to+\infty$ and taking into account the completeness of the gauge space, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Thus, the sequence is convergent and it converges to a fixed point of F since the multivalued operator $F:(X,\mathcal{P})\to P((X,\mathcal{P}))$ has closed graph.

Hence the multivalued operator F is a c_α-AMWP with $c_\alpha=\frac{1}{1-q_\alpha a_\alpha}$. Applying Theorem 2 we obtain the Ulam-Hyers stability of the fixed point inclusion (2.1).

3. Application

We give at the beginning of this section the notion of Ulam-Hyers stability for an integral inclusion. Let us consider the following integral inclusion:

$$x(t) \in \int_{-t}^{t} K(t,s,x(s))ds + g(t) \text{ a.e. } t \in [0,\infty), \tag{3.1}$$

where $\int_{-t}^{t} K(t,s,x(s))ds$ denotes the multivalued integral in Aumann' sense, see [3].

A solution of the integral inclusion (3.1) is a continuous function which satisfies the inclusion.

Definition 3.1. The integral inclusion (3.1) is Ulam-Hyers stable if and only if there exists $c = \{c_\alpha\}_{\alpha \in A} \in (0,\infty)^A$ such that for each $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in A} \in (0,\infty)^A$ and for any ε-solution y^* of (3.1) (i.e., any $y^* \in C(\mathbb{R},\mathbb{R}^m)$ which satisfies the inequality

$$|y^*(t) - \int_{-t}^{t} K(t,s,x(s))ds - g(t)| \le \varepsilon_\alpha, \text{ for each } t \ge 0) \tag{3.2}$$

there exists a solution x^* of the inclusion (3.1) such that

$$|y^*(t) - x^*(t)| \le c_\alpha \cdot \varepsilon_\alpha, \text{ for each } t \ge 0.$$

Then we have the following result.

Theorem 3.2. We suppose that

(i) $K : [0,\infty) \times [0,\infty) \times \mathbb{R}^m \to P_{cp}(\mathbb{R}^m)$ is jointly measurable for all $x \in C[0,\infty)$;
(ii) $g \in C(\mathbb{R},\mathbb{R}^m)$;
(iii) for almost every $(t,s) \in [0,\infty) \times [0,\infty)$ the multivalued operator $K(t,s,\cdot) : \mathbb{R}^m \to P(\mathbb{R}^m)$ is continuous;
(iv) there exists $L_K > 0$ such that for every $u, v \in \mathbb{R}^m$

$$H(K(t,s,u), K(t,s,v)) \le L_K \cdot \|u - v\|.$$

Then:

a) The integral inclusion (3.1) has at least one solution in $C(\mathbb{R},\mathbb{R}^m)$.
b) The integral inclusion (3.1) is Ulam-Hyers stable.

Proof. a) We consider the sequentially complete gauge space $\Big(C(\mathbb{R},\mathbb{R}^m), (d_n)_{n\in\mathbb{N}}\Big)$ where

$$d_n(x,y) = \sup_{t \in [-n,n]} \Big\{ \|x(t) - y(t)\| \cdot e^{-\tau\|t\|} \Big\}, \ \tau > 0, \ n \in \mathbb{N}^*,$$

and the multivalued operator

$$F(x)(t) = \int_{-t}^{t} K(t,s,x(s))ds + g(t).$$

Let $x_1, x_2 \in C([-n,n],\mathbb{R}^m)$ and $u_1 \in F(x_1)$. Then $u_1 \in \mathbb{R}^m$ and

$$u_1(t) \in \int_{-t}^{t} K(t,s,x(s))ds + g(t).$$

Thus there exists $k_1(t,s) \in K(t,s,x(s))$ such that $u_1(t) = \int_{-t}^{t} k_1(t,s)ds + g(t)$. Since

$$H(K(t,s,x_1(s)), K(t,s,x_2(s))) \leq L_K \cdot \|x_1(s) - x_2(s)\|$$

follows that there exists $v \in K(t,s,x_2(s))$ such that

$$\|k_1(t,s) - v\| \leq L_K \cdot \|x_1(s) - x_2(s)\|.$$

Thus the multivalued operator G defined by

$$G(t) = K(t,s,x_2(s)) \cap \{v : \ \|k_1(t,s) - v\| \leq L_K \cdot \|x_1(s) - x_2(s)\|\}$$

has nonempty values and is measurable. By Kuratowski and Ryll Nardzewski's selection theorem there exists $k_2(t,s)$ a measurable selection for G. Then $k_2(t,s) \in K(t,s,x_2(s))$ and

$$\|k_1(t,s) - k_2(t,s)\| \leq L_K \cdot \|x_1(s) - x_2(s)\|.$$

Let $u_2(t) = \int_{-t}^{t} k_2(t,s)ds + g(t) \in F(x_2)$. Then for $t \in [-n,n], \ n \in \mathbb{N}^*$ we have

$$\begin{aligned}
\|u_1(t) - u_2(t)\| &\leq \int_{-\|t\|}^{\|t\|} \|k_1(t,s) - k_2(t,s)\|ds \\
&\leq \int_{-\|t\|}^{\|t\|} L_K \cdot \|x_1(s) - x_2(s)\| \ ds \\
&= \int_{-\|t\|}^{\|t\|} L_K \cdot \|x_1(s) - x_2(s)\| \cdot e^{-\tau\|s\|} \cdot e^{\tau\|s\|}ds \\
&\leq L_K \cdot d_n(x_1,x_2) \cdot \int_{-\|t\|}^{\|t\|} e^{\tau\|s\|}ds = L_K \cdot d_n(x_1,x_2) \cdot \frac{e^{\tau\|t\|}}{\tau}.
\end{aligned}$$

Thus $d_n(u_1,u_2) \leq \frac{L_K}{\tau} \cdot d_n(x_1,x_2)$. Choosing $\tau > L_K$ we have that $L_F := \frac{L_K}{\tau} < 1$. By the analogous relation obtained by interchanging the roles of x_1 and x_2 it follows that

$$H_n(F(x_1), F(x_2)) \leq L_F \cdot d_n(x_1,x_2).$$

In what follows we want to prove that the multivalued operator F is an admissible contraction. We have already obtained the first condition, so it remains to show that for every $x \in C([-n,n],\mathbb{R}^m)$ and every $\varepsilon \in (0,\infty)^{\mathbb{N}^*}$, there exists $y \in F(x)$ such that $d_n(x,y) \leq D_n(x,F(x)) + \varepsilon_n$, for every $n \in \mathbb{N}^*$.

Supposing the contrary, we have that there exists $\varepsilon \in (0, \infty)^{\mathbb{N}^*}$ and exists $x \in C([-n, n], \mathbb{R}^m)$ such that for all $y \in F(x)$ we have $d_n(x, y) > D_n(x, F(x)) + \varepsilon_n$. It follows that $D_n(x, F(x)) \geq D_n(x, F(x)) + \varepsilon_n$ and so $\varepsilon_n \leq 0$, for every $n \in \mathbb{N}^*$, which is a contradiction.

Thus the multivalued operator F is an admissible contraction. This implies that F is also a $c_n - AMWP$ operator with $c_n = (1 - L_F)^{-1}$, hence we have the existence of the solution.

b) Applying Theorem 2 the second conclusion follows too.

Remark 3.3. For other Ulam-Hyers stability theorems see [6] (for a classical approach) and [5], [18],[20], [22], [23] (by Picard and weakly Picard operator technique).

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