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FIXED POINTS FOR φ-CONTRACTIONS IN E-BANACH SPACES

IOAN-RADU PETRE

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Kogălniceanu No. 1, 400084, Cluj-Napoca, Romania
E-mail: ioan.petre@ubbcluj.ro

Abstract. In this work we will present some extensions of Banach-Caccioppoli fixed point principle for classes of φ-contractions in E-metric spaces. Also, we extend Krasnoselskii's fixed point theorem to the case of E-Banach spaces and we give an application to a Fredholm-Volterra type differential equation where one of the integral operators satisfies a φ-contraction condition.
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1. Introduction

We will recall first some notations, auxiliary concepts and results, see e.g., [2], [10] and [8].

A set E equipped with a partial order "$\leq$" is called a partially ordered set. In a partially ordered set $(E, \leq)$ the notation $x < y$ means $x \leq y$ and $x \neq y$. An order interval $[x, y]$ is the set $\{z \in E : x \leq z \leq y\}$. Notice that if $x \nleq y$, then $[x, y] = \emptyset$.

A partially ordered set $(E, \leq)$ is a lattice if each pair of elements $x, y \in E$ has a supremum and an infimum. A real linear space E with an order relation "$\leq$" on E which is compatible with the algebraic structure of E, in the sense that satisfies properties:

(1) $x \leq y$ implies $x + z \leq y + z$, for each $z \in E, x, y \in E$ and
(2) $x \leq y$ implies $tx \leq ty$, for each $t > 0, x, y \in E$

is called an ordered linear space. An ordered linear space E for which $(E, \leq)$ is a lattice is called a Riesz space or linear lattice. Many familiar spaces are Riesz spaces, as some examples show.

Example 1.1. ([2]) The space $\mathbb{R}^n$ with norm defined by

$$||x|| = \left(\sum_{i=1}^{n} x_i\right)^{\frac{1}{2}}$$

and with the usual ordering relation, where $x = (x_1, \ldots, x_n) \leq y = (y_1, \ldots, y_n)$ whenever $x_i \leq y_i$, for each $i = 1, \ldots, n$ is a Riesz space. The infimum and supremum of two vectors x and y are given by

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}) \text{ and}$$
$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).$$

Example 1.2. ([2]) Both the vector space $C(X)$ of all continuous real functions (with X a compact set) and the vector space $C_b(X)$ of all bounded continuous real functions on the topological space X, with norms defined by

$$||f||_\infty = \sup\{|f(x)| : x \in X\}$$

and with the ordering relation defined pointwise, i.e. $f \leq g$ whenever $f(x) \leq g(x)$, for each $x \in X$ are Riesz spaces. The lattice operations of the real functions f and g are given by

$$(f \vee g)(x) = \max(f(x), g(x)) \text{ and}$$
$$(f \wedge g)(x) = \min(f(x), g(x)).$$

Example 1.3. ([2]) The vector space $L_p(\mu), 0 \leq p \leq \infty$, with norm defined by

$$||f||_p = \begin{cases} \left(\int |f|^p d\mu\right)^{\frac{1}{p}}, 0 \leq p < \infty \\ ess \sup |f|, p = \infty \end{cases}$$

and with the almost everywhere pointwise ordering relation, i.e. $f \leq g$ in $L_p(\mu)$ whenever $f(x) \leq g(x)$, for μ-almost every x is a Riesz space. The lattice operations are given by

$$(f \vee g)(x) = \max(f(x), g(x)) \text{ and}$$
$$(f \wedge g)(x) = \min(f(x), g(x)).$$

The notation $x_n \downarrow x$ means that x_n is a decreasing sequence and $\inf\{x_n\} = x$. If $(x_n), (y_n) \subset E$, then some basic properties of decreasing sequences are:

$x_n \downarrow x$ and $y_m \downarrow y$ implies $x_n + y_m \downarrow x + y$;

$x_n \downarrow x$ implies $\lambda x_n \downarrow \lambda x$, for $\lambda > 0$ and $\lambda x_n \uparrow \lambda x$, for $\lambda < 0$;

$x_n \downarrow x$ and $y_m \downarrow y$ implies $x_n \vee y_m \downarrow x \vee y$ and $x_n \wedge y_m \downarrow x \wedge y$.

A Riesz space E is Archimedean if $\frac{1}{n}x \downarrow 0$ holds for every $x \in E_+$ (see [2]), where

$$E_+ = \{x \in E : x \geq 0\} \text{ is the positive cone of } E.$$

A Riesz space E is order complete or Dedekind complete if every nonempty subset of E which is bounded from above has a supremum (equivalently, every nonempty subset of E which is bounded from below has an infimum), see [2]. Any order complete Riesz space is Archimedean. The converse is false.

If E is a Riesz space, we will denote by

$$|x| := x \vee (-x) \text{ the absolute value of } x \in E.$$

We present now some useful definitions for our main results and the concept of vector metric space given in [2], [8] and [10].

Definition 1.4. Let E be a Riesz space. A sequence (b_n) in E is called order-convergent (or o-convergent) to b (we write $b_n \xrightarrow{o} b$), if there exists a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$, for all $n \in \mathbb{N}$.

Definition 1.5. Let E, F be two Riesz spaces and $f : E \to F$. The function f is order continuous (or o-continuous) if $b_n \xrightarrow{o} b$ in E implies $f(b_n) \xrightarrow{o} f(b)$ in F.

Definition 1.6. Let E be a Riesz space. A sequence (b_n) in E is called order-Cauchy (or o-Cauchy), if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$, for all $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$.

Definition 1.7. A Riesz space E is called o-complete if every o-Cauchy sequence is o-convergent.

Definition 1.8. Let X be a nonempty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a vector metric or E-metric if it satisfies the following properties:

(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Also, the triple (X, d, E) is said to be a vector metric space or an E-metric space.

It is obvious that E-metric spaces generalize the notion of metric spaces and for arbitrary elements x, y, z, w of an E-metric space, the following properties holds:

(i) $0 \leq d(x, y)$;
(ii) $d(x, y) = d(y, x)$;
(iii) $|d(x, z) - d(y, z)| \leq d(x, y)$;
(iv) $|d(x, z) - d(y, w)| \leq |d(x, y) - d(z, w)|$.

Some examples of E-metric spaces, from [2], [10] and [8], are given now.

Example 1.9. A Riesz space E is an E-metric space with $d : E \times E \to E$ defined by

$$d(x, y) = |x - y|.$$

This E-metric is called to be the absolute valued metric on E.

Example 1.10. It is well known the $\mathbb{R}^2$ is a Riesz space with coordinatwise ordering defined by

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2, \text{ for } (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$$

or with lexicographical ordering defined by

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2$$

and endowed with the vector metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$d((x_1, y_1), (x_2, y_2)) = (\alpha |x_1 - y_1|, \beta |x_2 - y_2|), \text{ where } \alpha, \beta \in \mathbb{R}_+$$

becomes an E-metric space.

Other necessary notions (see [2], [10], [8]) are the following.

Definition 1.11. Let (X, d, E) be an E-metric space. A sequence (x_n) in X E-converges to some $x \in E$, written $x_n \xrightarrow{d,E} x$, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$, for all $n \in \mathbb{N}$.

Definition 1.12. Let X, Y be two E-metric spaces and $f : X \to Y$. The function f is E-continuous if $x_n \xrightarrow{d,E} x$ in X implies $f(x_n) \xrightarrow{d,E} f(x)$ in Y.

Definition 1.13. Let (X, d, E) be an E-metric space. A sequence (x_n) in X is called to be E-Cauchy, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$, for all $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$.

Definition 1.14. An E-metric space X is called E-complete if each E-Cauchy sequence in X E-converges to a limit in X.

Definition 1.15. Let (X, d, E) be an E-metric space. We say that a subset $Y \subset X$ is E-closed if $(x_n) \subset Y$ and $x_n \xrightarrow{d,E} x$ implies $x \in Y$.

Lemma 1.16. *Let (X, d, E) be an E-metric space. If $x_n \xrightarrow{d,E} x$ then the following properties hold:*

(1) The limit x is unique;
(2) Any subsequence of (x_n) E-converges to x;
(3) If $y_n \xrightarrow{d,E} y$ then $d(x_n, y_n) \xrightarrow{o} d(x, y)$.

Definition 1.17. Let (X, d, E) be an E-metric space. If $A \subset X$ is a nonempty set, then the symbol

$$\delta(A) = \sup \{d(x, y) : x, y \in A\}$$

is called the E-diameter of X if $\sup \{d(x, y) : x, y \in A\}$ is in E. Furthermore, if there exists an $a > 0$ in E such that $d(x, y) \leq a$, for $x, y \in A$, then A is called an E-bounded set.

Remark 1.18. If $E = \mathbb{R}$, the concepts of E-convergence and metric convergence are the same, respectively the concepts of E-Cauchy sequence and Cauchy sequence are the same. If $X = E$ and d is the absolute valued vector metric on X, then the concepts of E-convergence and o-convergence are the same.

The purpose of this paper is to give some extensions of the Contraction principle to the case of E-metric spaces. More precisely we will realize the study of the fixed point theory for (local and global) nonlinear contractions with an o-comparison function in E-metric spaces. Our results generalize some theorems given in [8], [15], [27], [28], [29].

2. Main Results

In the first part we give some existence and uniqueness results for nonlinear φ-contractions in E-metric spaces, using the lattice structure and the order relation of the Riesz space E.

Definition 2.1. Let (X, d, E) be an E-metric space and $\varphi : E_+ \to E_+$ be an increasing operator such that $\varphi(t) < t$ and $\varphi^n(t) \xrightarrow{o} 0$, for any $t > 0$. We say that the operator $T : X \to X$ is a nonlinear φ-contraction, if and only if

$$d(Tx, Ty) \leq \varphi[d(x, y)], \text{ for any } x, y \in X.$$

Definition 2.2. By definition, an operator $\varphi : E_+ \to E_+$ which satisfies the above properties is called an o-comparison function.

In what follows, by Theorems 2.3, 2.4, 2.8 and 2.11 we will obtain some extensions of the Contraction principle to the case of E-metric spaces and we will realize the study of the fixed point theory for nonlinear contractions with an o-comparison function in the entire space, in a closed ball of the space, respectively in a certain subset of the E-metric space.

Theorem 2.3. *Let (X, d, E) be an E-complete metric space and let $T : X \to X$ be a nonlinear φ-contraction. Then:*

(i) there exists a unique fixed point $z \in X$ for T and for any $x \in X$, $T^n(x) \xrightarrow{d,E} z$;
(ii) $d(z, T^n(x)) \leq \varphi^n[d(z, x)]$, for any $n \in \mathbb{N}$.

Proof. Let $x \in X$ be arbitrarily. Inductively, we have

$$d\left(T^n(x), T^{n+p}(x)\right) \leq \varphi^n[d(x, T^p(x))], \text{ for any } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

Since $\varphi^n(d(x, T^p(x))) \xrightarrow{o} 0$ as $n \to \infty$, we have that there exists (η_n) in E such that $\eta_n \downarrow 0$ and $\varphi^n(d(x, T^p(x))) \leq \eta_n$, for any $n \in \mathbb{N}$. Thus,

$$d\left(T^n(x), T^{n+p}(x)\right) \leq \eta_n, \text{ for any } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

Letting $n \to \infty$, we obtain that the sequence $(T^n(x))$ is E-Cauchy in X. By the E-completeness of X, it follows that there exists $z \in X$ such that for any $x \in X$, $T^n(x) \xrightarrow{d,E} z$. Thus, there exists a sequence (ε_n) in E such that $\varepsilon_n \downarrow 0$ and $d(T^n(x), z) \leq \varepsilon_n$, for any $n \in \mathbb{N}$. We have

$$\begin{aligned} d(z, Tz) &\leq d\left(z, T^{n+1}(x)\right) + d\left(T^{n+1}(x), Tz\right) \\ &\leq \varepsilon_{n+1} + \varphi[d(T^n(x), z)] \\ &\leq \varepsilon_{n+1} + \varphi(\varepsilon_n) \leq 2\varepsilon_n \downarrow 0, \text{ when } n \to \infty. \end{aligned}$$

Thus, z is a fixed point of T in X. For the uniqueness, we suppose that $y \in X$ is another fixed point of T with $y \neq z$. Then

$$d(y, z) = d(Ty, Tz) \leq \varphi[d(y, z)].$$

Thus, by the properties of φ, we get that $d(y, z) = 0$ and so, $y = z$.

For any $n \in \mathbb{N}$, the error estimate for the fixed point is given by

$$\begin{aligned} d(z, T^n(x)) &= d(T^n(z), T^n(x)) \\ &\leq \varphi^n[d(z, x)], \end{aligned}$$

which completes the proof. □

Another result of this type can be obtained in the closed ball

$$\bar{B}(x_0, r) := \{x \in X \mid d(x, x_0) \leq r\},$$

where $x_0 \in X$ and $r \in E_+$ with $r > 0$.

Theorem 2.4. *Let (X, d, E) be an E-complete metric space, $x_0 \in X, r \in E_+$, let $T : \bar{B}(x_0, r) \to X$ be an operator and there exists an increasing operator $\varphi : [0, r] \to [0, r] \subset E_+$ such that $\varphi^n(t) \xrightarrow{o} 0$ for any $t \in (0, r]$, with the property $d(Tx, Ty) \leq \varphi[d(x, y)]$ and $d(x, y) \leq r$, for any $x, y \in \bar{B}(x_0, r)$. We assume that $d(x_0, Tx_0) \leq r - \varphi(r)$.*
Then:

- *(i) T has a unique fixed point $z \in \bar{B}(x_0, r)$ and for any $x \in \bar{B}(x_0, r)$ we have $T^n(x) \xrightarrow{d,E} z$;*
- *(ii) $d(z, T^n(x)) \leq \varphi^n(r)$, for any $n \in \mathbb{N}$.*

Proof. We will show that $T\left(\bar{B}(x_0, r)\right) \subset \bar{B}(x_0, r)$.
Let $x \in \bar{B}(x_0, r)$ and by the estimation

$$\begin{aligned} d(x_0, Tx) &\leq d(x_0, Tx_0) + d(Tx_0, Tx) \\ &\leq r - \varphi(r) + \varphi[d(x_0, x)] \\ &\leq r - \varphi(r) + \varphi(r) = r, \end{aligned}$$

we get that $Tx \in \bar{B}(x_0, r)$. Thus, $T : \bar{B}(x_0, r) \to \bar{B}(x_0, r)$ and since X is E-complete, by Theorem 2.3, we get that there exists a unique fixed point z for T in $\bar{B}(x_0, r)$ and for any $x \in \bar{B}(x_0, r)$, $T^n(x) \xrightarrow{d,E} z$.

For any $n \in \mathbb{N}$, the error estimate for the fixed point is given by

$$\begin{aligned} d(z, T^n(y_0)) &= d(T^n(z), T^n(y_0)) \\ &\leq \varphi^n[d(z, x)] \leq \varphi^n(r), \end{aligned}$$

which completes the proof. □

Example 2.5. Let $I = [0, a]$, $a > 0$ be an interval of the real axis. Suppose that $T \in C\left(I^2 \times B, B\right)$, $g \in C(I, B)$ and we consider the Fredholm type integral equation

$$x(t) = \int_I T(t, s, x(s))\, ds + g(t), \tag{1}$$

in $C(I, B)$, i.e., in the space of all continuous functions defined on I, with values in a Banach space B, with the uniform convergence $\longrightarrow$ and with the metric defined by

$$d(x, y) = ||x(t) - y(t)||, \text{ for any } x, y \in C(I, B),$$

where $||\cdot|| : B \to \mathbb{R}_+$ is the norm of B. In $C(I, \mathbb{R}_+)$ we choose the usual partial order and the usual operations (addition and multiplication) and for the convergence relation $\downarrow$ we consider the pointwise convergence of decreasing sequences in $C(I, \mathbb{R}_+)$. Then, in this case, we can easily observe that the Riesz space E is $C(I, \mathbb{R}_+)$ and the abstract space X is $C(I, B)$.

Moreover, if we assume that there exists a continuous function $\omega \in C\left(I^2, \mathbb{R}_+\right)$ with $\sup_{t\in I} \int_I \omega(t,s)\, ds \leq 1$, such that

$$||T(t,s,x) - T(t,s,y)|| \leq \omega(t,s)\, \varphi(||x-y||)\, , \text{ for each } t,s \in I,\ x,y \in B,$$

where $\varphi : E_+ \to E_+$ is an o-comparison operator.

Then, the integral equation (1) has a unique solution in $C(I,B)$.

We attach to the integral equation (1) the operator

$$A : C(I,B) \to C(I,B)\, ,$$

$$A(x)(t) := \int_I T(t,s,x(s))\, ds + g(t)\, , \text{ for any } t \in I.$$

Thus, A is well defined. We prove that A is a nonlinear φ-contraction. Let $x_1, x_2 \in C(I,B)$. We have

$$\begin{aligned} ||A(x_1)(t) - A(x_2)(t)|| &\leq \int_I ||T(t,s,x_1(s)) - T(t,s,x_2(s))||\, ds \\ &\leq \int_I \omega(t,s)\, \varphi(||x_1(s) - x_2(s)||)\, ds \\ &\leq \varphi(||x_1 - x_2||_\infty) \int_I \omega(t,s)\, ds. \end{aligned}$$

Passing to the norm $||\cdot||_\infty$, we get

$$||A(x_1) - A(x_2)||_\infty \leq \varphi(||x_1 - x_2||_\infty)\, , \text{ for any } x_1, x_2 \in C(I,B)\, .$$

By Theorem 2.3, it follows that the equation (1) has a unique solution in $C(I,B)$.

Remark 2.6. The interval I could be replaced by a compact subset of a topological space and similarly the proof runs (see Example 1.2).

Remark 2.7. Theorem 2.3 and Theorem 2.4 represent global and local extensions (to the case of E-metric spaces) of some well known fixed point principles, see J. Matkowski [17] and I.A. Rus [24].

If we choose $X = E$ and d an absolute valued metric on E, then we obtain fixed point theorems in the Riesz space E, see R. Cristescu [10]. Moreover, similarly can be proved other fixed point theorems in E-metric spaces for operators which satisfies nonlinear generalized φ-contraction conditions and the theory of such a theorem can be considered, see [19] and [26].

Following the ideas from [15], other results with equivalent conclusions with Theorems 2.3 and 2.4 can be obtained in the space

$$X(x_0, r) := \bigcup_{\lambda \in E_+} \bar{B}(x_0, \lambda r) = \bigcup_{\lambda \in E_+} \{x |\ x \in X, d(x, x_0) \leq \lambda r\}\, .$$

Theorem 2.8. *Let (X,d,E) be an E-complete metric space, $x_0 \in X, r \in E_+$, let $T : X(x_0,r) \to X$ be an operator and there exists an increasing operator $\varphi : E_+ \to E_+$ (not necessary o-continuous) such that $\varphi^n(t) \overset{o}{\longrightarrow} 0$ for any $t > 0$, with properties:*
1°) $\varphi(\lambda r) \leq \varphi(\lambda)\, r$, for $\lambda \in E_+$;
2°) $d(Tx,Ty) \leq \varphi[d(x,y)]$ and $d(x,y) \leq \lambda r$, for any $x,y \in X(x_0,r)$ and for $\lambda \in E_+$;

3°) $d(x_0, Tx_0) \leq \lambda_0 r$, *for* $\lambda_0 \in E_+$.
Then:

(i) there exists a unique fixed point z *for* T *in* $X(x_0, r)$ *and for any* $x \in X(x_0, r)$, $T^n(x) \xrightarrow{d,E} z$;

(ii) $d(z, T^n(x)) \leq \varphi^n[d(z, x)]$, *for any* $n \in \mathbb{N}$.

Proof. We have to prove that $X(x_0, r)$ is invariant with respect to T, i.e., $T(X(x_0, r)) \subset X(x_0, r)$. Let $x \in X(x_0, r)$, then there exists $\lambda \in E_+$ such that $d(x, x_0) \leq \lambda r$. Since

$$\begin{aligned} d(Tx, x_0) &\leq d(Tx, Tx_0) + d(Tx_0, x_0) \\ &\leq \varphi[d(x, x_0)] + \lambda_0 r \leq \varphi(\lambda r) + \lambda_0 r \\ &\leq [\varphi(\lambda) + \lambda_0] r \end{aligned}$$

thus, there exists $\lambda' := [\varphi(\lambda) + \lambda_0] \in E_+$ such that $d(Tx, x_0) \leq \lambda' r$, i.e., $Tx \in X(x_0, r)$. Then, the conclusion follows by Theorem 2.3. □

Lemma 2.9. *If* $(y_n) \subset X(x_0, r)$ *and* $y_n \xrightarrow{d,E} y$, *then* $y \in X(x_0, r)$, *i.e.,* $X(x_0, r)$ *is* E*-closed in* X *with respect to the convergence* $\xrightarrow{d,E}$.

Proof. Let $y_n \in X(x_0, r), n \in \mathbb{N}^*$, then there exists $\lambda \in E_+$ such that $d(y_n, x_0) \leq \lambda r$. Since $y_n \xrightarrow{d,E} y$, there exists a sequence (ε_n) in E such that $\varepsilon_n \downarrow 0$ and $d(y_n, y) \leq \varepsilon_n r$, for any $n \in \mathbb{N}$. We have

$$d(y, x_0) \leq d(y, y_n) + d(y_n, x_0) \leq (\varepsilon_1 + \lambda) r$$

thus, there exists $\lambda' := (\varepsilon_1 + \lambda) \in E_+$ such that

$$d(y, x_0) \leq \lambda' r,$$

i.e., $y \in X(x_0, r)$. □

If we endow the space $X(x_0, r) \subset X$ with a metric $\rho : X(x_0, r) \times X(x_0, r) \to E_+$, given by

$$\rho(x, y) = \inf_{\lambda \in E_+} \{d(x, y) \leq \lambda r\},$$

we have the following:

Lemma 2.10. *Let* (X, d, E) *be an* E*-complete metric space with* E*-Archimedean. Then, the space* $X(x_0, r)$ *is* E*-complete with respect to* ρ.

Proof. It is easy to check that ρ is a metric of space $X(x_0, r)$ in the sense of Definition 1.8. We want to show that any E-Cauchy sequence in $X(x_0, r)$ is E-convergent and his limit is in $X(x_0, r)$.

Let (y_n) be an E-Cauchy sequence in $X(x_0, r)$. Thus, there exists a sequence (ε_n) in E such that $\varepsilon_n \downarrow 0$ and $\rho(y_n, y_{n+p}) \leq \varepsilon_n$, for any $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$. We have

$$d(y_n, y_{n+p}) \leq \left[\rho(y_n, y_{n+p}) + \frac{1}{k}\right] r \leq \left(\varepsilon_n + \frac{1}{k}\right) r,\ k \in \mathbb{N}^*.$$

Letting $k \to \infty$, we obtain that

$$d(y_n, y_{n+p}) \le \varepsilon_n r := c_n, \text{ for any } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*,$$

respectively by the fact that E is Archimedean, we have that $c_n \downarrow 0$. Thus, (y_n) is E-Cauchy in (X, d, E). By E-completeness of (X, d, E), it follows that there exists $y \in X$ such that $y_n \xrightarrow{d,E} y$.

By Lemma 2.9, $X(x_0, r)$ is E-closed in X with respect to the convergence $\xrightarrow{d,E}$. It follows that $d(y_n, y) \le c_n$, for any $n \in \mathbb{N}$ and

$$d(x_0, y) \le d(x_0, y_1) + d(y_1, y) \le \lambda_1 b + c_1 = (\lambda_1 + \varepsilon_1) r,$$

thus, $y \in X(x_0, r)$. □

If we use the same conditions as in Theorem 2.8, we can obtain another existence and uniqueness result in the E-metric space $X(x_0, b)$. Notice that, this time, the proof is based on the equivalence between the E-metrics ρ and d and thus, we will not apply Theorem 2.8 to show that the sequence of successive approximations of T converges with respect to $\xrightarrow{d,E}$ to the unique fixed point of T in $X(x_0, b)$.

Theorem 2.11. *If all the assumptions of Theorem 2.8 holds and E is Archimedean, then:*

(i) T is a nonlinear φ-contraction in $X(x_0, r)$ with respect to ρ;

(ii) there exists a unique fixed point $z \in X(x_0, r)$ for T and for any $y_0 \in X(x_0, r)$ we have that $T^n(y_0) \xrightarrow{d,E} z$.

Proof. i) We prove that $\rho(Tx, Ty) \le \varphi[\rho(x, y)]$ for any $x, y \in X(x_0, r)$.

By the proof of Theorem 2.8, we get that $X(x_0, r)$ is invariant with respect to T. Let $x, y \in X(x_0, r)$, then there exists $\lambda \in E_+$ such that $x, y \in \bar{B}(x_0, \lambda r)$. We have

$$\begin{aligned} d(Tx, Ty) &\le \varphi[d(x, y)] \le \varphi\left[\left(\rho(x, y) + \frac{1}{k}\right) r\right] \\ &\le \varphi\left[\rho(x, y) + \frac{1}{k}\right] r, \ k \in \mathbb{N}^*. \end{aligned}$$

Letting $k \to \infty$, we obtain that

$$d(Tx, Ty) \le \varphi[\rho(x, y)] r,$$

which implies the inequality

$$\rho(Tx, Ty) \le \varphi[\rho(x, y)].$$

ii) Inductively, by i), we have

$$\rho\left(T^k(x), T^k(y)\right) \le \varphi^k[\rho(x, y)], \text{ for any } x, y \in X(x_0, r), \ k \in \mathbb{N}.$$

We fix $y_0 \in X(x_0, r)$ and we take $x = y_0$, $y = T(y_0)$, thus

$$\rho\left(T^k(y_0), T^{k+1}(y_0)\right) \le \varphi^k[\rho(y_0, T(y_0))].$$

On the other hand, since $\varphi^k(t) \xrightarrow{o} 0$ for any $t > 0$, we have that there exists $p \in \mathbb{N}$ such that

$$\varphi^p[\rho(y_0, T(y_0))] \le 1 - \varphi(1).$$

Then, we get that there exists $p \in \mathbb{N}$ such that

$$\rho\left(T^p(y_0), T^{p+1}(y_0)\right) \leq 1 - \varphi(1).$$

Let $z_0 = T^p(y_0)$, we define $V(z_0, 1) := \{x \mid x \in X(x_0, r), \rho(z_0, x) \leq 1\}$ and applying Theorem 2.4 to the operator $T : V(z_0, 1) \to V(z_0, 1)$, it follows that there exists a unique fixed point z for T in $V(z_0, 1)$ and $T^n(z_0) \xrightarrow{\rho,E} z$. We want to show that $T^n(z_0) \xrightarrow{d,E} z$. We have

$$\begin{aligned} d\left(T^n(z_0), T^{n+p}(z_0)\right) &\leq d\left(T^n(z_0), z\right) + d\left(z, T^{n+p}(z_0)\right) \\ &\leq \left(\rho\left(T^n(z_0), z\right) + \rho\left(z, T^{n+p}(z_0)\right) + \frac{1}{k}\right) r \\ &\leq \left(\varepsilon_n + \varepsilon_{n+p} + \frac{1}{k}\right) r \leq \left(2\varepsilon_n + \frac{1}{k}\right) r,\ k \in \mathbb{N}^*. \end{aligned}$$

Letting $k \to \infty$, we obtain that

$$d\left(T^n(z_0), T^{n+p}(z_0)\right) \leq 2\varepsilon_n r := d_n, \text{ for any } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*,$$

respectively by the fact that E is Archimedean, we have that $d_n \downarrow 0$. Thus, $(T^n(z_0))$ is an E-Cauchy sequence in $X(x_0, r)$. By E-completeness of $X(x_0, r)$, it follows that there exists $z' \in X(x_0, r)$ such that $d(T^n(z_0), z') \leq 2\varepsilon_n r$.
Moreover,

$$\begin{aligned} d(z, z') &\leq d\left(z, T^n(z_0)\right) + d\left(T^n(z_0), z'\right) \\ &\leq \left[\rho\left(z, T^n(z_0)\right) + \frac{1}{k}\right] r + 2\varepsilon_n r \\ &\leq 3\varepsilon_n r, \text{ when } k \to \infty. \end{aligned}$$

Thus, we conclude that $z = z'$ and $T^n(z_0) \xrightarrow{d,E} z$, respectively by the properties of φ, it follows that there exists in $X(x_0, r)$ at most one fixed point of T. Hence, there exists a unique fixed point z for T in $X(x_0, r)$ and for any $y_0 \in X(x_0, r)$, $T^n(y_0) \xrightarrow{d,E} z$. □

Remark 2.12. The equality between the E-metric spaces X and $X(x_0, r)$ can hold and depends of the space X and r. If $X = C(I, B)$ and $E = C(I, \mathbb{R}_+)$ then the mentioned equality holds if $\inf[r(t) : t \in I] > 0$.

Next, we recall some preliminary topological results to Krasnoselskii's theorem in E-Banach spaces.

Definition 2.13. ([2], [8]) Let (X, d, E) be an E-metric space. A subset $A \subset X$ is called E-open if for any $x \in A$, there exists some $r > 0$ in E such that $B(x, r) \subset A$, where $B(x, r) = \{y \in X : d(x, y) < r\}$. Any E-open ball is an E-open set and the collection of all E-open subsets of X represents the E-metric topology on X denoted by $\tau_{d,E}$.

Definition 2.14. ([2], [30]) Let (X, d, E) be an E-metric space. A subset C of X is called E-compact if every E-open cover of C has a finite subcover. Equivalently, a subset C of X is E-sequentially compact if every sequence in C contains an E-convergent subsequence with limit in C.

A subset C of X is said to be E-totally bounded if for each $\varepsilon \in E_+, \varepsilon > 0$, there exists a finite number of elements $x_1, x_2, \ldots, x_n$ in X such that $C \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. The set $\{x_1, x_2, \ldots, x_n\}$ is called a finite ε-net.

A set C of a topological space is said to be E-relatively compact if its closure is E-compact, i.e., $\bar{C}$ is E-compact. Equivalently, C is E-sequentially relatively compact if every sequence in C contains an E-convergent subsequence (the limit need not be an element of C), i.e., $\bar{C}$ is E-sequentially compact.

Proposition 2.15. *([30]) If C is a subset of E, then we have:*

(i) E-compact $\Leftrightarrow$ E-sequentially compact $\Leftrightarrow$ E-closed and E-totally bounded;
(ii) E-relatively compact $\Leftrightarrow$ E-sequentially relatively compact $\Leftrightarrow$ E-totally bounded.

If a, b are elements of E, then the set $\{x \mid x = (1-\lambda)a + \lambda b, 0 \leq \lambda \leq 1\}$ is called the line segment joining a to b. Then a subset $K \subset E$ is called E-convex if for each pair $a, b \in K$, the line segment joining them lies in K (see [2] and [12]). For any subset $A \subset E$, the intersection of all E-convex sets containing A is called the convex hull $co(A)$ of A, i.e., the smallest E-convex set containing A.

Definition 2.16. ([10]) Let X be a linear space and E be a Riesz space. Then, an E-norm on X is a function $||\cdot|| : X \to E$ satisfying the following properties:
(a) $||x|| \geq 0$, for all $x \in X$;
(b) $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in X$.
Moreover, the triple $(X, ||\cdot||, E)$ is called an E-normed space.

Remark 2.17. ([10]) If $||\cdot||$ is an E-norm on X, then the function $d : X \times X \to E$, $d(x, y) = ||x - y||$ is an E-metric on X and d is called the E-metric generated by the E-norm $||\cdot||$.

Definition 2.18. ([10]) An E-normed space $(X, ||\cdot||, E)$ is called a vector Banach space or E-Banach space if any E-Cauchy sequence in X is E-convergent with respect to $||\cdot||$.

Remark 2.19. ([10]) If $|\cdot|$ represents the absolute value of the Riesz space E, then $(E, |\cdot|, E)$ is an E-Banach space.

Remark 2.20. ([30]) Any E-normed Riesz space is Archimedean and thus, an E-Banach space is obviously Archimedean.

Notice that Ky Fan's Lemma and Schauder's Theorem can be proved, in an E-Banach space X, by a similar method to the classical case, where we assume that E is order complete and $Y \subset X$ is an E-bounded set (thus, the order completeness guarantees that $\inf_{x \in Y} ||x - f(y_0)||$ exists in E). More precisely, we have the following results.

Lemma 2.21. *Let X be an order complete E-normed space, let $Y \subset X$ be an E-compact and E-convex set and let $f : Y \to X$ be an E-continuous operator. Then* $||y_0 - f(y_0)|| = \inf_{x \in Y} ||x - f(y_0)||$.

Definition 2.22. Let X, Y be two E-normed spaces, $K \subset X$ and $f : K \to Y$ an operator. Then, we say that f is:

(i) E-compact, if for any E-bounded subset $A \subset K$ we have $f(A)$ is E-relatively compact or $\overline{f(A)}$ is E-compact;

(ii) E-complete continuous, if f is E-continuous and E-compact;

(iii) with E-relatively compact range, if f is E-continuous and $f(K)$ is E-relatively compact.

Theorem 2.23. *Let $(X, ||\cdot||, E)$ be an E-Banach space with E order complete, let $Y \subset X$ be an E-bounded, E-closed and E-convex set and let $f : Y \to Y$ be an operator with E-relatively compact value. Then f has at least one fixed point in Y.*

Remark 2.24. For another Schauder type theorem in Hausdorff Archimedean vector lattice, see T. Kawasaki, M. Toyoda, T. Watanabe [14].

In the context of E-Banach spaces, we will prove a nonlinear version of Krasnoselskii's fixed point theorem and we will present an existence result for the solution of a Fredholm-Volterra type integral equation.

Theorem 2.25. *Let $(X, ||\cdot||, E)$ be an E-Banach space with E order complete and let Y be a nonempty, E-bounded, E-convex and E-closed subset of X. Assume that the operators $f, g : Y \to X$ satisfy the properties:*

(i) f is a nonlinear φ-contraction and the operator $\psi : E_+ \to E_+$ defined by $\psi(t) = t - \varphi(t)$ satisfies the following relation:

$$\text{if } (\psi(t_n)) \downarrow 0 \text{ as } n \to +\infty, \text{ then } (t_n) \downarrow 0 \text{ as } n \to +\infty.$$

(ii) g is E-continuous;

(iii) $g(Y)$ is E-relatively compact and $f(x) + g(y) \in Y$, for any $x, y \in Y$.

Then $f + g$ has a fixed point in Y.

Proof. We show that for any $x \in Y$, the operator $u_x : Y \to Y, u_x(y) = f(y) + g(x)$ is a nonlinear φ-contraction. We have that

$$||u_x(y_1) - u_x(y_2)|| = ||f(y_1) - f(y_2)|| \leq \varphi(||y_1 - y_2||), \text{ for any } y_1, y_2 \in Y.$$

Thus, u_x is a nonlinear φ-contraction, for each $x \in Y$. By Theorem 2.3, it follows that for each $x \in Y$, there exists a unique $y_x^* \in Y$ such that $f(y_x^*) + g(x) = y_x^*$. We define $c : Y \to Y, c(x) = y_x^*$. Thus,

$$c(x) = f[c(x)] + g(x), \text{ for any } x \in Y.$$

We prove that c is E-continuous.

Let (x_n) be a sequence in Y such that $x_n \overset{||\cdot||,E}{\longrightarrow} x$. Then there exists (ε_n) in E such that $\varepsilon_n \downarrow 0$ and $||x_n - x|| \leq \varepsilon_n$, for any $n \in \mathbb{N}$. Since f is a nonlinear φ-contraction, we have that φ is increasing. Now, by the properties of φ and the following estimation

$$||f(x_n) - f(x)|| \leq \varphi(||x_n - x||) \leq \varphi(\varepsilon_n) \leq \varepsilon_n \downarrow 0,$$

we have that $f(x_n) \overset{||\cdot||,E}{\longrightarrow} f(x)$, when $n \to \infty$. Thus, f is E-continuous. Since g is E-continuous, for any sequence (x_n) in Y such that $x_n \overset{||\cdot||,E}{\longrightarrow} x$, there exists (a_n) in

E such that $a_n \downarrow 0$ and $||g(x_n) - g(x)|| \leq a_n$, for any $n \in \mathbb{N}$. Now, we prove that the mapping c defined below is E-continuous. Indeed, we have:

$$\begin{aligned} ||c(x_n) - c(x)|| &= ||f[c(x_n)] + g(x_n) - f[c(x)] - g(x)|| \\ &\leq ||f[c(x_n)] - f[c(x)]|| + ||g(x_n) - g(x)|| \\ &\leq \varphi(||c(x_n) - c(x)||) + ||g(x_n) - g(x)||, \end{aligned}$$

we get that

$$\psi(||c(x_n) - c(x)||) \leq ||g(x_n) - g(x)|| \leq a_n \downarrow 0, \text{ when } n \to +\infty.$$

By the properties of ψ, we have

$$||c(x_n) - c(x)|| \downarrow 0, \text{ when } n \to +\infty. \tag{2}$$

Since $g(Y)$ is E-relatively compact, we have that $g(Y)$ is E-totally bounded and thus, for any $r \in E_+$ with $r > 0$, there exists $Z = \{x_1, \ldots, x_n\} \subset Y$ such that $g(Y) \subset \{z_1, \ldots, z_n\} + \bar{B}(0, \psi(r)) = \{g(x_1), \ldots, g(x_n)\} + \bar{B}(0, \psi(r))$, where $z_i = g(x_i)$, for any $i = 1, 2, \ldots, n$. Thus, in view of relation (2), we have that $c(Y) \subset \{c(x_1), \ldots, c(x_n)\} + \bar{B}(0, r)$. Then, the set $c(Y)$ is E-totally bounded.

Since X is a Banach vector space, we have that $c(Y)$ is E-relatively compact and by Theorem 2.23, it follows that there exists $x^* \in Y$ with $c(x^*) = x^*$, i.e., $f(x^*) + g(x^*) = x^*$ and hence, the theorem is proved. □

It is known that the classical form of Theorem 2.25 has many interesting applications, see, for example, [7], [32], etc.

Our next purpose is to give an application for our result in an E-Banach space. Using Theorem 2.25 we can obtain existence results for some integral and differential equations. For this purpose we also need an extended version of Cantor's intersection theorem and of Cesaro's lemma.

Lemma 2.26. *Let (X, d, E) be an E-complete metric space with the property that for every descending sequence $\{F_n\}_{n\geq 1}$ of nonempty E-closed subsets of X we have that $\delta(F_n) \longrightarrow 0$ as $n \to \infty$. Then the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point.*

Proof. For each positive integer n, let x_n be any point in F_n. Then, by the hypothesis, $x_n, x_{n+1}, x_{n+2}, \ldots$ all lie in F_n. Given $\varepsilon_n > 0$ in E with $\varepsilon_n \downarrow 0$, there exists some integer n_0 such that $\delta(F_{n_0}) < \varepsilon_n$. Now, $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots$ all lie in F_{n_0}. For $m, n \geq n_0$, we have that $d(x_m, x_n) \leq \delta(F_{n_0}) < \varepsilon_n$. This shows that the sequence $\{x_n\}_{n\geq 1}$ is an E-Cauchy sequence in the E-complete metric space X. So, it is E-convergent. Let $x \in X$ be such that $\lim_{n\to\infty} x_n = x$. Now for any given n, we have that $x_n, x_{n+1}, \ldots \subset F_n$. In view of this, $x = \lim_{n\to\infty} x_n \in \bar{F}_n = F_n$, since F_n is E-closed. Thus, $x \in \bigcap_{n=1}^{\infty} F_n$. If $y \in \bigcap_{n=1}^{\infty} F_n$ and $y \neq x$, then $d(y, x) = \alpha > 0$. There exists $n \in \mathbb{N}$ large enough such that $\delta(F_n) < \alpha = d(y, x)$, which ensures that $y \notin F_n$. Thus, y cannot be in $\bigcap_{n=1}^{\infty} F_n$ and hence, the intersection contains only one point. □

Lemma 2.27. *Let (X, d, E) be an E-complete metric space such that E is Archimedean and let (x_n) be an E-bounded sequence in X. Then, there exists an E-convergent subsequence (x_{n_k}) in X.*

Proof. Since (x_n) is E-bounded in X there exists $x \in X$ and $r \in E_+$ with $r > 0$ such that

$$d(x_n, x) \leq r, \text{ for any } n \in \mathbb{N}.$$

Let $a_0 := x$ and consider now the sets $\bar{B}\left(a_0, \frac{1}{2}r\right)$ and $\bar{B}(a_0, r) \setminus B\left(a_0, \frac{1}{2}r\right)$.

We choose x_{n_1} in the set which contains an infinite number of elements of (x_n), let $x_{n_1} \in \bar{B}\left(a_0, \frac{1}{2}r\right)$. Next, let $a_1 \in \partial\bar{B}\left(a_0, \frac{1}{2^2}r\right)$ such that the set $\bar{B}\left(a_1, \frac{1}{2^2}r\right)$ contains an infinite number of elements of (x_n).

We choose $x_{n_2} \in \bar{B}\left(a_1, \frac{1}{2^2}r\right)$, $n_2 > n_1$ and inductively, let $a_k \in \partial\bar{B}\left(a_{k-1}, \frac{1}{2^k}r\right)$ such that the set $\bar{B}\left(a_k, \frac{1}{2^k}r\right)$ contains an infinite number of elements of (x_n).

We choose $x_{n_k} \in \bar{B}\left(a_{k-1}, \frac{1}{2^k}r\right)$, $n_k > n_{k-1}$, for any $k \in \mathbb{N}^*$. By Lemma 2.26, we get that there exists a unique $l \in \bar{B}\left(a_{k-1}, \frac{1}{2^k}r\right)$, for any $k \in \mathbb{N}^*$.

But $x_{n_k} \in \bar{B}\left(a_{k-1}, \frac{1}{2^k}r\right)$, for any $k \in \mathbb{N}^*$. Then, since E is Archimedean, we have that $d(x_{n_k}, l) \leq \frac{1}{2^k}r \downarrow 0$ as $k \to \infty$. Hence, $x_{n_k} \xrightarrow{d,E} l$ as $k \to \infty$. □

Theorem 2.28. *Let E be an order complete Riesz space, $r \in E_+$ with $r > 0$ and let $I := [0, a]$ (where $a > 0$) be an order interval of E. We consider the following Fredholm-Volterra type integral equation in $C(I, E)$:*

$$x(t) = \int_I k(t, s, x(s))\, ds + \int_0^t l(t, s, x(s))\, ds,\ t \in I. \tag{3}$$

We assume that:

(i) $k \in C\left(I^2 \times E, E\right)$ and $l \in C\left(I^2 \times E, E\right)$ are two o-continuous operators;

(ii) there exists $\omega \in C\left(I^2, E_+\right)$ with $\sup\limits_{t \in I} \int_I \omega(t, s)\, ds \leq 1$, such that

$$|k(t, s, x) - k(t, s, y)| \leq \omega(t, s)\, \varphi(|x - y|), \text{ for any } t, s \in I,\ x, y \in E,$$

where $\varphi : E_+ \to E_+$ is an o-comparison operator and the operator $\psi : E_+ \to E_+$, defined by $\psi(t) = t - \varphi(t)$ satisfies the following relation:

$$\text{if } (\psi(t_n)) \downarrow 0 \text{ as } n \to +\infty, \text{ then } (t_n) \downarrow 0 \text{ as } n \to +\infty.$$

(iii) we have that $M_l := \sup\limits_{t \in I} \int_0^t l(t, s, x(s))\, ds \leq \frac{1}{2}r$ and $\psi(r) \geq \delta$, where $\delta := \sup\limits_{x \in \bar{B}(0,r)} \left| \sup\limits_{t \in I} \int_I k(t, s, x(s))\, ds \right| \in E_+$.

Then, the equation (3) has a solution x^* in $\bar{B}(0, r) \subset C(I, E)$.

Proof. We define

$$f, g : \bar{B}(0,r) \to C(I,E), \quad x \longmapsto f(x), \; x \longmapsto g(x),$$

$$f(x)(t) := \int_I k(t,s,x(s))\,ds, \text{ for any } t \in I,$$

$$g(x)(t) := \int_0^t l(t,s,x(s))\,ds, \text{ for any } t \in I.$$

From $i)$, the operators f and g are well defined. Obviously, x^* is a solution for (3) if and only if x^* is a fixed point for $f+g$. We need to show that the operators f and g satisfies the assumptions of Theorem 2.25. Let $x, y \in C(I,E)$. We have

$$\begin{aligned} |f(x)(t) - f(y)(t)| &\le \int_0^a |k_i(t,s,x(s)) - k_i(t,s,y(s))|\,ds \\ &\le \int_0^a \omega(t,s)\,\varphi(|x(s)-y(s)|)\,ds \\ &\le \varphi(||x-y||_\infty) \int_I \omega(t,s)\,ds. \end{aligned}$$

If we consider the abstract norm $||\cdot||_\infty$ on $C(I,E)$ defined in a similar way with Example 1.2, i.e.,

$$||u||_\infty := \{\sup |u(t)| : t \in I\}$$

we get that

$$||f(x) - f(y)||_\infty \le \varphi(||x-y||_\infty).$$

Thus, f is a nonlinear φ-contraction with respect to $||\cdot||_\infty$.

We have to show that g is E-continuous. Since l is o-continuous, it follows immediately that g is E-continuous.

Since $X := C(I,E)$, let us consider $Y = \bar{B}(0,r) \subset X$. We prove that we can choose $r \in E_+$ with $r > 0$ such that $f(x) \subset \bar{B}\left(0, \frac{1}{2}r\right)$, for any $x \in \bar{B}(0,r)$. Since

$$\begin{aligned} |f(x)(t)| &\le \int_I |k(t,s,x(s))|\,ds \le \int_I \omega(t,s)\,\varphi(|x(s)|)\,ds \\ &\le \varphi(||x||_\infty) \int_I \omega(t,s)\,ds \le \varphi(r) \end{aligned}$$

and passing to the norm $||\cdot||_\infty$, we get that

$$||f(x)||_\infty \le \varphi(r), \text{ for any } x \in \bar{B}(0,r). \tag{4}$$

On the other hand, we consider $\delta := \sup\limits_{x \in \bar{B}(0,r)} ||f(x)||_\infty \in E_+$, thus

$$\delta \le \varphi(r). \tag{5}$$

By $iii)$, we get that

$$\varphi(r) \le r - \delta. \tag{6}$$

Thus, from (5) and (6), we have that $\delta \le \frac{1}{2}r$.

In view of relation (4), it follows that

$$||f(x)||_\infty \le \delta \le \frac{1}{2}r, \text{ for any } x \in \bar{B}(0,r).$$

Clearly,

$$|g(x)(t)| \leq \int_0^t |l(t,s,x(s))|\,ds \leq M_l \leq \frac{1}{2}r$$

and so,

$$g(x) \in \bar{B}\left(0, \frac{1}{2}r\right), \text{ for any } x \in Y.$$

Thus, $f(Y) + g(Y) \subset Y$, i.e., $f(x) + g(y) \in Y$, for any $x, y \in Y$.

Let (x_n) be a sequence in $\bar{B}(0,r)$. Then, (x_n) is E-bounded and so, it has a subsequence (x_{n_k}) which is E-convergent to a certain $y \in \bar{B}(0,r)$. Then, by the E-continuity of g, we have that $g(x_{n_k}) \xrightarrow{d,E} g(y) \in X$. This means that the sequence $(g(x_n)) \subset g\left(\bar{B}(0,r)\right)$ has a subsequence which is E-convergent. Thus, $g\left(\bar{B}(0,r)\right)$ is E-sequentially relatively compact and by Proposition 2.15 is E-relatively compact. Hence, the conclusion follows by Theorem 2.25. □

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