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NONLOCAL INITIAL VALUE PROBLEMS FOR FIRST ORDER DIFFERENTIAL SYSTEMS

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Abstract. The paper is devoted to existence of solutions to initial value problems for nonlinear first order differential systems with nonlocal conditions. The proof will rely on the Perov, Schauder and Leray-Schauder fixed point principles which are applied to a nonlinear integral operator. The novelty in this paper is that this approach is combined with the technique that uses convergent to zero matrices and vector norms.

Key Words and Phrases: Nonlinear differential system, nonlocal initial condition, fixed point, vector norm, matrix convergent to zero.
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1. Introduction

In this paper we deal with the nonlocal initial value problem for the first order differential system

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \qquad \text{(a.e. on } [0,1]) \\ x(0) = \alpha[x] \\ y(0) = \beta[y]. \end{cases} \tag{1.1}$$

Here $f_1, f_2 : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ are Carathéodory functions, $\alpha, \beta : C[0,1] \to \mathbf{R}$ are linear and continuous functionals such that $1 - \alpha[1] \neq 0$ and $1 - \beta[1] \neq 0$.

Problem (1.1) is equivalent to the following integral system in $C[0,1]^2$:

$$\begin{cases} x(t) = \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s, x(s), y(s))\, ds \\ y(t) = \frac{1}{1-\beta[1]}\beta[g_2] + \int_0^t f_2(s, x(s), y(s))\, ds, \end{cases}$$

where

$$g_1(t) := \int_0^t f_1(s, x(s), y(s))\, ds, \quad g_2(t) := \int_0^t f_2(s, x(s), y(s))\, ds.$$

This can be viewed as a fixed point problem in $C[0,1]^2$ for the completely continuous operator $T : C[0,1]^2 \to C[0,1]^2$, $T = (T_1, T_2)$, where T_1 and T_2 are given by

$$T_1(x,y)(t) = \tfrac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s, x(s), y(s))\, ds,$$
$$T_2(x,y)(t) = \tfrac{1}{1-\beta[1]}\beta[g_2] + \int_0^t f_2(s, x(s), y(s))\, ds.$$

Nonlocal problems were extensively discussed in the literature by different methods (see [2], [3], [5], [6], [9], [10], [12], [14], [15], [16], [17], [18] and references therein). For example, in [15], it is shown how the existence of multiple positive solutions of nonlinear second order differential equations of the form

$$u''(t) + p(t)u'(t) + q(t)u(t) + g(t)f(t, u(t)) = 0, \quad t \in (0, 1),$$

subject to various nonlocal boundary conditions can be established under a unified approach. The nonlocal boundary conditions are of the general form

$$au(0) - bu'(0) = \alpha[u], \quad cu(1) + du'(1) = \beta[u],$$

where $\alpha[u], \beta[u]$ are linear functionals given by Stieltjes integrals, not assumed to be positive for all $u \geq 0$. The well known multi-point boundary value problems are special cases, and it is allowed for some coefficients to have opposite signs. Then, in [16], it is established the existence of multiple positive solutions of nonlinear equations of the form

$$-u''(t) = g(t)f(t, u(t)), \quad t \in (0, 1),$$

where f, g are non-negative functions, subject to various nonlocal boundary conditions. The common feature is that each problem can be written as an integral equation in the space $C[0, 1]$, of the form

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s)g(s)f(s, u(s))\, ds,$$

where $\alpha[u]$ is a linear functional given by a Stieltjes integral.

A unified method of establishing the existence of multiple positive solutions for a large number of non-linear differential equations of arbitrary order with any allowed number of non-local boundary conditions is given in [17]. In particular, the authors determine the Green's function for these problems with very little explicit calculation, which shows that studying a more general version of a problem with appropriate notation can lead to a simplification in approach. They also obtain existence and non-existence results, some of which are sharp, and give new results for both non-local and local boundary conditions. The authors illustrate the theory with a detailed account of a fourth-order problem that models an elastic beam and also determine optimal values of the constants that appear in the theory. To be more specific, a typical example of the problems that are treated there is the weakly singular fourth-order equation

$$u^{(4)}(t) = g(t)f(t, u(t)), \quad t \in (0, 1),$$

where g and f are non-negative functions, typically f is continuous and $g \in L^1$ may have pointwise singularities, with the non-local boundary conditions (some $\beta_i[u]$ could be identically 0, and hence are omitted)

$$u(0) = \beta_1[u], \quad u'(0) = \beta_2[u], \quad u(1) = \beta_3[u], \quad u''(1) + \beta_4[u] = 0.$$

Here, the boundary conditions involve linear continuous functionals on $C[0,1]$, or equivalently Stieltjes integrals

$$\beta_j[u] = \int_0^1 u(s)dB_j(s)$$

with signed measures, that is B_j are functions of bounded variations.

A similar approach for the study of existence of multiple positive solutions for semi-positone boundary value problems of arbitrary order is given by the same authors, Webb and Infante in [18]. The nonlocal boundary conditions are quite general, involving positive linear functionals on the space $C[0,1]$, given by Stieltjes integrals. With their general theory, one can, for the first time in semi-positone problems, allow any number of the boundary conditions to be nonlocal. More exactly, one approach to establishing the existence of positive solutions of a boundary value problem is to seek fixed points of a Hammerstein integral operator of the form

$$\widehat{S}u(t) = \int_0^1 k(t,s)f(s,u(s))ds, \tag{1.2}$$

in a cone of positive functions, where k is the corresponding Green's function. When seeking positive solutions of (1.2), it is usually required that the nonlinearity f and the kernel k are both positive. However, in some applications, f changes sign, which problems are called semi-positone or non-positone in the literature. For example, problems of the type

$$u^{(n)}(t) = \lambda f(t, u(t)),$$

occur as models for the concentration of a reactant inside a porous catalyst pellet. In applications one is often interested in showing the existence of positive solutions for λ small.

A completely different approach is given in [9] to the nonlocal initial value problem for the first order differential system

$$\begin{cases} x'(t) = f(t, x(t), y(t)) & \\ y'(t) = g(t, x(t), y(t)) & \text{(a.e. on } [0,1]) \\ x(0) + \sum\limits_{k=1}^{m} a_k x(t_k) = 0 & \\ y(0) + \sum\limits_{k=1}^{m} \widetilde{a}_k y(t_k) = 0. & \end{cases}$$

Here $f, g : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ are Carathéodory functions, t_k are given points with $0 \le t_1 \le t_2 \le \ ... \ \le t_m < 1$ and a_k, $\widetilde{a}_k$ are real numbers with $1 + \sum\limits_{k=1}^{m} a_k \ne 0$ and $1 + \sum\limits_{k=1}^{m} \widetilde{a}_k \ne 0$. The initial conditions that are imposed can be viewed of "functional type". Fixed point principles are applied to a nonlinear integral operator splitted into two parts like in [3], one of Fredholm type for the subinterval containing the points involved by the nonlocal condition, and an another one of Volterra type for the rest of the interval. The goal of this paper is to extend the results established in [9] to the case where the non-local initial conditions are more generally expressed in terms of two linear continuous functionals.

In the next sections three different fixed point principles will be used in order to prove the existence of solutions for the semilinear problem, namely the fixed point theorems of Perov, Schauder and Leray-Schauder (see [12]). In all three cases a key role will be played by the so called convergent to zero matrices. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as} \quad k \to \infty.$$

It is known that the property of being convergent to zero is equivalent to each of the following three conditions (for details see [12], [13]):

(a) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + ...$ (where I stands for the unit matrix of the same order as M);
(b) the eigenvalues of M are located inside the unit disc of the complex plane;
(c) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

The following lemma whose proof is immediate from characterization (b) of convergent to zero matrices will be used in the sequel:

Lemma 1.1. *If A is a square matrix that converges to zero and the elements of an other square matrix B are small enough, then $A + B$ also converges to zero.*

We finish this introductory section by recalling (see [1], [12]) the fundamental results which will be used in the next sections. Let X be a nonempty set. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbf{R}^n_+$ such that

(i) $d(u, v) \geq 0$ for all $u, v \in X$ and if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbf{R}^n$, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, ..., n$. We call the pair (X, d) a *generalized metric space*. For such a space convergence and completeness are similar to those in usual metric spaces.

An operator $T : X \to X$ is said to be *contractive* (with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that

$$d(T(u), T(v)) \leq M d(u, v) \quad \text{for all } u, v \in X.$$

Theorem 1.2. [Perov] *Let (X, d) be a complete generalized metric space and $T : X \to X$ a contractive operator with Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$d(T^k(u_0), u^*) \leq M^k (I - M)^{-1} d(u_0, T(u_0)) \text{ for all } k \in \mathbf{N}.$$

Theorem 1.3. [Schauder] *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

Theorem 1.4. [Leray–Schauder] *Let* $(X, ||\,.\,||_X)$ *be a Banach space,* $R > 0$ *and* $T : \overline{B}_R(0; X) \to X$ *a completely continuous operator. If* $||u||_X < R$ *for every solution* u *of the equation* $u = \lambda T(u)$ *and any* $\lambda \in (0, 1)$, *then* T *has at least one fixed point.*

2. Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here we show that the existence of solutions to problem (1.1) follows from Perov's fixed point theorem in case that f_1, f_2 satisfy Lipschitz conditions in x and y :

$$\begin{cases} |f_1(t,x,y) - f_1(t,\overline{x},\overline{y})| \le a_1 |x - \overline{x}| + b_1 |y - \overline{y}| \\ |f_2(t,x,y) - f_2(t,\overline{x},\overline{y})| \le a_2 |x - \overline{x}| + b_2 |y - \overline{y}|, \end{cases} \tag{2.1}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbf{R}$ and some $a_i, b_i > 0, i = 1, 2.$

Theorem 2.1. *If* f_1, f_2 *satisfy the Lipschitz conditions* (2.1) *and matrix*

$$M_{\alpha,\beta} = \begin{bmatrix} a_1 \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1 \right) & b_1 \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1 \right) \\ a_2 \left(\frac{\|\beta\|}{|1-\beta[1]|} + 1 \right) & b_2 \left(\frac{\|\beta\|}{|1-\beta[1]|} + 1 \right) \end{bmatrix} \tag{2.2}$$

converges to zero, then problem (1.1) *has a unique solution.*

Proof. We shall apply Perov's fixed point theorem in $C[0,1]^2$ endowed with the vector norm $\|.\|$ defined by

$$\|u\| = (\|x\|_\infty, \|y\|_\infty)$$

for $u = (x, y)$, where for $z \in C[0,1]$, we let

$$\|z\|_\infty = \max_{t\in[0,1]} |z(t)|.$$

We have to prove that T is a generalized contraction, more exactly that

$$\|T(u) - T(\overline{u})\| \le M_{\alpha,\beta} \|u - \overline{u}\| \tag{2.3}$$

for all $u = (x, y), \overline{u} = (\overline{x}, \overline{y}) \in C[0,1]^2$. Indeed, we have

$$\begin{aligned} & |T_1(x,y)(t) - T_1(\overline{x},\overline{y})(t)| \\ = \; & \left| \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s, x(s), y(s))\, ds \right. \\ & \left. - \frac{1}{1-\alpha[1]}\alpha[\overline{g}_1] - \int_0^t f_1(s, \overline{x}(s), \overline{y}(s))\, ds \right| \\ \le \; & \left| \frac{1}{1-\alpha[1]} \right| |\alpha[g_1 - \overline{g}_1]| + \int_0^t |f_1(s, x(s), y(s)) - f_1(s, \overline{x}(s), \overline{y}(s))|\, ds \\ \le \; & \frac{\|\alpha\|}{|1-\alpha[1]|} \|g_1 - \overline{g}_1\|_\infty + \int_0^t (a_1 |x(s) - \overline{x}(s)| + b_1 |y(s) - \overline{y}(s)|)\, ds. \end{aligned} \tag{2.4}$$

Taking the supremum, we have

$$\begin{aligned} & \|T_1(x,y) - T_1(\overline{x},\overline{y})\|_\infty \\ \le \; & \frac{\|\alpha\|}{|1-\alpha[1]|} \|g_1 - \overline{g}_1\|_\infty + a_1 \|x - \overline{x}\|_\infty + b_1 \|y - \overline{y}\|_\infty. \end{aligned}$$

Also

$$\begin{aligned} |g_1(t)-\overline{g}_1(t)| &\leq \int_0^t |f_1(s,x(s),y(s))-f_1(s,\overline{x}(s),\overline{y}(s))|\,ds \\ &\leq \int_0^t (a_1|x(s)-\overline{x}(s)|+b_1|y(s)-\overline{y}(s)|)\,ds \\ &\leq a_1\|x-\overline{x}\|_\infty + b_1\|y-\overline{y}\|_\infty , \end{aligned}$$

which gives

$$\|g_1-\overline{g}_1\|_\infty \leq a_1\|x-\overline{x}\|_\infty + b_1\|y-\overline{y}\|_\infty . \tag{2.5}$$

From (2.4) and (2.5), we obtain that

$$\begin{aligned} &\|T_1(x,y)-T_1(\overline{x},\overline{y})\|_\infty \\ \leq\ &\left(\frac{\|\alpha\|}{|1-\alpha[1]|}+1\right)(a_1\|x-\overline{x}\|_\infty + b_1\|y-\overline{y}\|_\infty). \end{aligned} \tag{2.6}$$

Similarly

$$\begin{aligned} &\|T_2(x,y)-T_2(\overline{x},\overline{y})\|_\infty \\ \leq\ &\left(\frac{\|\beta\|}{|1-\beta[1]|}+1\right)(a_2\|x-\overline{x}\|_\infty + b_2\|y-\overline{y}\|_\infty). \end{aligned} \tag{2.7}$$

Now, (2.6), (2.7) can be put together and be rewritten as

$$\begin{bmatrix} \|T_1(x,y)-T_1(\overline{x},\overline{y})\|_\infty \\ \|T_2(x,y)-T_2(\overline{x},\overline{y})\|_\infty \end{bmatrix} \leq M_{\alpha,\beta} \begin{bmatrix} \|x-\overline{x}\|_\infty \\ \|y-\overline{y}\|_\infty \end{bmatrix},$$

that is (2.3) holds. Since $M_{\alpha,\beta}$ is assumed to be convergent to zero, the result follows from Perov's fixed point theorem. □

3. Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here we show that the existence of solutions to problem (1.1) follows from Schauder's fixed point theorem in case that f_1, f_2 satisfy instead of the Lipschitz condition, the more relaxed condition of growth at most linear:

$$\begin{cases} |f_1(t,x,y)| \leq a_1|x|+b_1|y|+c_1 \\ |f_2(t,x,y)| \leq a_2|x|+b_2|y|+c_2, \end{cases} \tag{3.1}$$

for all $x,y,\overline{x},\overline{y}\in \mathbf{R}$ and some $a_i,b_i,c_i>0, i=1,2$.

Theorem 3.1. *If f_1, f_2 satisfy* (3.1) *and matrix* (2.2) *converges to zero, then problem* (1.1) *has at least one solution.*

Proof. In order to apply Schauder's fixed point theorem, we look for a nonempty, bounded, closed and convex subset B of $C[0,1]^2$ so that $T(B)\subset B$. Let x,y be any elements of $C[0,1]$.

We have

$$\begin{aligned}
|T_1(x,y)(t)| &= \left| \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1\left(s, x\left(s\right), y(s)\right) ds \right| \\
&\leq \left| \frac{1}{1-\alpha[1]} \right| |\alpha[g_1]| + \int_0^t \left(a_1 \left|x(s)\right| + b_1 \left|y(s)\right| + c_1\right) ds \\
&\leq \frac{\|\alpha\|}{|1-\alpha[1]|} \left\|g_1\right\|_\infty + \int_0^t \left(a_1 \left|x(s)\right| + b_1 \left|y(s)\right| + c_1\right) ds \\
&\leq \frac{\|\alpha\|}{|1-\alpha[1]|} \left\|g_1\right\|_\infty + a_1 \left\|x\right\|_\infty + b_1 \left\|y\right\|_\infty + c_1. \qquad (3.2)
\end{aligned}$$

Also

$$\begin{aligned}
|g_1(t)| &\leq \int_0^t \left|f_1\left(s, x\left(s\right), y(s)\right)\right| ds \\
&\leq \int_0^t \left(a_1 \left|x(s)\right| + b_1 \left|y(s)\right| + c_1\right) ds \\
&\leq a_1 \left\|x\right\|_\infty + b_1 \left\|y\right\|_\infty + c_1,
\end{aligned}$$

which gives

$$\left\|g_1\right\|_\infty \leq a_1 \left\|x\right\|_\infty + b_1 \left\|y\right\|_\infty + c_1. \qquad (3.3)$$

From (3.2) and (3.3), we obtain that

$$\left\|T_1(x,y)\right\|_\infty \leq \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1\right)\left(a_1 \left\|x\right\|_\infty + b_1 \left\|y\right\|_\infty\right) + \widetilde{c}_1, \qquad (3.4)$$

where $\widetilde{c}_1 := c_1 \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1\right)$. Similarly

$$\left\|T_2(x,y)\right\|_\infty \leq \left(\frac{\|\beta\|}{|1-\beta[1]|} + 1\right)\left(a_2 \left\|x\right\|_\infty + b_2 \left\|y\right\|_\infty\right) + \widetilde{c}_2, \qquad (3.5)$$

with $\widetilde{c}_2 := c_2 \left(\frac{\|\beta\|}{|1-\beta[1]|} + 1\right)$. Now (3.4), (3.5) can be put together as

$$\begin{bmatrix} \left\|T_1(x,y)\right\|_\infty \\ \left\|T_2(x,y)\right\|_\infty \end{bmatrix} \leq M_{\alpha,\beta} \begin{bmatrix} \left\|x\right\|_\infty \\ \left\|y\right\|_\infty \end{bmatrix} + \begin{bmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{bmatrix},$$

where matrix $M_{\alpha,\beta}$ is given by (2.2) and converges to zero. Next, we look for two positive numbers R_1, R_2 such that if $\left\|x\right\|_\infty \leq R_1$, $\left\|y\right\|_\infty \leq R_2$, then $\left\|T_1(x,y)\right\|_\infty \leq R_1$, $\left\|T_2(x,y)\right\|_\infty \leq R_2$. To this end it is sufficient that

$$M_{\alpha,\beta} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \geq \left(I - M_{\alpha,\beta}\right)^{-1} \begin{bmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \end{bmatrix}.$$

Notice that $I - M_{\alpha,\beta}$ is invertible and its inverse $\left(I - M_{\alpha,\beta}\right)^{-1}$ has nonnegative elements since $M_{\alpha,\beta}$ converges to zero. Thus, if

$$B = \left\{(x,y) \in C[0,1]^2 : \left\|x\right\|_\infty \leq R_1, \left\|y\right\|_\infty \leq R_2\right\},$$

then $T(B) \subset B$ and Schauder's fixed point theorem can be applied. □

4. More general nonlinearities. Application of the Leray-Schauder principle

We now consider that nonlinearities f_1, f_2 satisfy more general growth conditions, namely:

$$\begin{cases} |f_1(t,u)| \leq \omega_1(t, |u|_e) \\ |f_2(t,u)| \leq \omega_2(t, |u|_e) \end{cases}, \quad \text{for } t \in [0,1], \tag{4.1}$$

for all $u = (x,y) \in \mathbf{R}^2$, where by $|u|_e$ we mean the euclidean norm in $\mathbf{R}^2$. Here, ω_1, ω_2 are Carathéodory functions on $[0,1] \times \mathbf{R}_+$, nondecreasing in their second argument.

Theorem 4.1. *Assume that condition* (4.1) *holds. Denote* $A_1 := \frac{\|\alpha\|}{|1-\alpha[1]|} + 1$, $A_2 := \frac{\|\beta\|}{|1-\beta[1]|} + 1$. *In addition assume that there exists a positive number* R_0 *such that for* $\rho = (\rho_1, \rho_2) \in (0,\infty)^2$

$$\begin{cases} \frac{1}{\rho_1} \int_0^1 \omega_1(t, |\rho|_e) dt \geq \frac{1}{A_1} \\ \frac{1}{\rho_2} \int_0^1 \omega_2(t, |\rho|_e) dt \geq \frac{1}{A_2} \end{cases} \quad \textit{implies} \quad |\rho|_e \leq R_0. \tag{4.2}$$

Then problem (1.1) *has at least one solution.*

Proof. The result will follow from the Leray-Schauder fixed point theorem once we have proved the boundedness of the set of all solutions to equations $u = \lambda T(u)$, for $\lambda \in [0,1]$. Let $u = (x,y)$ be such a solution. Hence, we apply Theorem 1.4 considering $X = C[0,1]^2$, $T = (T_1, T_2)$, R any real number with $R > R_0$ and the norm in $C[0,1]^2$ defined by

$$\|u\|_X = \left(\|x\|_\infty^2 + \|y\|_\infty^2 \right)^{1/2}.$$

Then, for $t \in [0,1]$, we have

$$\begin{aligned} |x(t)| &= |\lambda T_1(x,y)(t)| \\ &= \lambda \left| \frac{1}{1-\alpha[1]} \alpha[g_1] + \int_0^t f_1(s, x(s), y(s))\, ds \right| \\ &\leq \frac{\|\alpha\|}{|1-\alpha[1]|} \|g_1\|_\infty + \int_0^t |f_1(s, x(s), y(s))|\, ds \\ &\leq \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1 \right) \int_0^1 \omega_1(s, |u(s)|_e) ds \\ &= A_1 \int_0^1 \omega_1(s, |u(s)|_e) ds. \end{aligned} \tag{4.3}$$

Similarly,

$$|y(t)| \leq A_2 \int_0^1 \omega_2(s, |u(s)|_e) ds. \tag{4.4}$$

Let $\rho_1 = \|x\|_\infty$, $\rho_2 = \|y\|_\infty$. Then from (4.3), (4.4), we deduce

$$\begin{cases} \rho_1 \leq A_1 \int_0^1 \omega_1(s, |\rho|_e) ds \\ \rho_2 \leq A_2 \int_0^1 \omega_2(s, |\rho|_e) ds. \end{cases}$$

From our assumption (4.2),

$$|\rho|_e \leq R_0. \tag{4.5}$$

Since $|\rho|_e = \|u\|_X$, one has $\|u\|_X < R$ as we wished and the result follows from Leray-Schauder's fixed point theorem. □

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