Abstract. We will apply the fixed point method for proving the stability and superstability of $J^*$–homomorphisms and $J^*$–derivations associated to a generalized Jensen type functional equation between $J^*$–algebras.

Key Words and Phrases: Approximate $J^*$–homomorphism; approximate $J^*$–derivation; $J^*$–algebra; alternative fixed point; generalized Jensen functional equation.

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1. Introduction

Our knowledge concerning the continuity properties of epimorphisms on Banach algebras, Jordan-Banach algebras, and, more generally, nonassociative complete normed algebras, is now fairly complete and satisfactory (see [24, 44, 45]). A basic continuity problem consists in determining algebraic conditions on a Banach algebra $A$ which ensure that derivations on $A$ are continuous. In 1996, Villena [45] proved that derivations on semisimple Jordan-Banach algebras are continuous. In [24], the authors dealt with derivations acting on Banach-Jordan pairs. By a $J^*$–algebra we mean a closed subspace $A$ of a C*-algebra such that $xx^*x \in B$ whenever $x \in B$. Several well known spaces have the structure of a $J^*$–algebra (cf.[17]). For example, (i) every Cartan factor of type $I$, i.e., the space of all bounded operators $B(H,K)$ between Hilbert spaces $H$ and $K$; (ii) every Cartan factor of type $IV$, i.e., a closed *–subspace $B$ of $B(H)$ in which the square of each operator in $B$ is scalar multiple of identity operator on $H$; (iii) every ternary algebra of operators [8, 18]. A $J^*$–homomorphism between $J^*$–algebras $A$ and $B$ is defined to be a C–linear mapping $H : A \to B$ such that

$$H(aa^*a) = H(a)H(a)^*H(a)$$
for all \( a \in A \), and a \( J^* \)-derivation on a \( J^* \)-algebras \( A \) is defined to be a \( C \)-linear mapping \( D : A \to A \) such that

\[
D(aa^*a) = D(a)a^*a + aD(a)^*a + aa^*D(a)
\]

for all \( a \in A \). In particular, every \( * \)-homomorphism between \( C^* \)-algebras is a \( J^* \)-homomorphism and every \( * \)-derivation on a \( C^* \)-algebra is a \( J^* \)-derivation.

The stability problem of functional equations originated from a question of Ulam [43] concerning the stability of group homomorphisms. Hyers [19] provided a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ theorem was generalized by T. Aoki [1] for additive mappings and by Th.M. Rassias [41] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [41] has provided a lot of influence in the development of what we now call generalized Hyers–Ulam stability or as Hyers–Ulam–Rassias stability of functional equations. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach. For more details about various results concerning such problems the reader is referred to [6, 9, 11, 14, 16, 20, 21, 22] and [37]–[42].

C. Park, J.C. Hou and Th.M. Rassias proved the stability of homomorphisms and derivations in Banach algebras, Banach ternary algebras, \( C^* \)-algebras, Lie \( C^* \)-algebras and \( C^* \)-ternary algebras [25]–[35]. Moreover, in [29], Park established the stability of \( * \)-homomorphisms of a \( C^* \)-algebra (see also [30]).

We note that a mapping \( f \) satisfying the following Jensen equation

\[
2f(\frac{x+y}{2}) = f(x) + f(y)
\]

is called Jensen. Stability of Jensen functional equation has been studied by using the direct method as well as the fixed point method at [3, 5, 20, 23, 42]. Recently, Eshaghi Gordji and Najati [12] proved the stability and superstability of \( J^* \)-homomorphisms between \( J^* \)-algebras for the Jensen type functional equation

\[
f(\frac{x+y}{2}) + f(\frac{x-y}{2}) - f(x) = 0.
\]

In addition, Eshaghi Gordji et al. [10] established the stability and superstability of \( J^* \)-derivations in \( J^* \)-algebras for the following Jensen type functional equation

\[
rf(\frac{x+y}{r}) + rf(\frac{x-y}{r}) - 2f(x) = 0.
\]

In this paper, we investigate the stability and superstability of \( J^* \)-homomorphisms and \( J^* \)-derivations in \( J^* \)-algebras for the generalized Jensen type functional equation

\[
\mu f(\frac{\sum_{i=1}^{n} x_i}{n}) + \mu \sum_{j=2}^{n} f(\frac{\sum_{i=1,i\neq j}^{n} x_i - (n-1)x_j}{n}) - f(\mu x_1) = 0 \tag{1}
\]

for all \( \mu \in \mathbb{T} = \{ \lambda \in \mathbb{C}; |\lambda| = 1 \} \), where \( n \geq 2 \).

Before proceeding to the main results, we recall a fundamental result in fixed point theory.

**Theorem 1.1.** [7]. Suppose that we are given a complete generalized metric space \((\Omega, d)\) and a strictly contractive function \( T : \Omega \to \Omega \) with Lipschitz constant \( L \). Then for each given \( x \in \Omega \), either


\[ d(T^m x, T^{m+1} x) = \infty \quad \text{for all } m \geq 0, \]

or there exists a natural number \( m_0 \) such that

- \( d(T^m x, T^{m+1} x) < \infty \) for all \( m \geq m_0 \);
- the sequence \( \{ T^m x \} \) is convergent to a fixed point \( y^* \) of \( T \);
- \( y^* \) is the unique fixed point of \( T \) in the set \( \Lambda = \{ y \in \Omega : d(T^{m_0} x, y) < \infty \} \);
- \( d(y, y^*) \leq 1 - L d(T y, y) \) for all \( y \in \Lambda \).

Radu and Cădariu \([2, 3, 36]\) applied the fixed point method to the investigation of functional equations (see also \([4, 13, 22]\)).

This paper is organized as follows: By using the fixed point method, in Section 2, we prove the superstability and stability of \( J^* \)-homomorphisms in \( J^* \)-algebras for the functional equation (1), and also using Gajda’s example \([14]\) to give a counterexample for a singular case. In Section 3, we prove the superstability and stability of \( J^* \)-derivations on \( J^* \)-algebras for the functional equation (1), and also we present a counterexample for a singular case.

Throughout this paper assume that \( A, B \) are two \( J^* \)-algebras. For convenience, we use the following abbreviation for given a mapping \( f : A \to B \),

\[
\triangle f(x_1, x_2, \ldots, x_n, a) = \mu f\left(\frac{\sum_{i=1}^{n} x_i + aa^* a}{n}\right) + \mu \sum_{j=2}^{n} f\left(\frac{\sum_{i=1 \neq j} x_i - (n-1)x_j + aa^* a}{n}\right) - f(\mu x_1)
\]

for all \( \mu \in \mathbb{T} \) and all \( x_1, x_2, \ldots, x_n, a \in A \), where \( n \geq 2 \).

2. Approximation of \( J^* \)-homomorphisms in \( J^* \)-algebras

We will use the following lemma:

**Lemma 2.1.** Let both \( X \) and \( Y \) be real vector spaces. If a mapping \( f : X \to Y \) satisfies (1) with \( \mu = 1 \), then \( f : X \to Y \) is additive.

**Proof.** Letting \( x_i = 0 \) (\( 1 \leq i \leq n \)) in (1), we obtain \( f(0) = 0 \). Setting \( x_1 = x \) and \( x_i = 0 \) (\( 2 \leq i \leq n \)) in (1), we get

\[
n f\left(\frac{x}{n}\right) = f(x)
\]

for all \( x \in X \). Setting \( x_i = 0 \) (\( 3 \leq i \leq n \)) in (1) and using (2), we get

\[
n - 1 \frac{1}{n} f(x_1 + x_2) + \frac{1}{n} f(x_1 - (n-1)x_2) = f(x_1)
\]

for all \( x_1, x_2 \in X \). Putting \( x_1 = x_1 + (n-1)x_2 \) in (3), we get

\[
n - 1 \frac{1}{n} f(x_1 + nx_2) + \frac{1}{n} f(x_1) = f(x_1 + (n-1)x_2)
\]

for all \( x_1, x_2 \in X \). Replacing \( x_1 \) by 0 and \( x_2 \) by \( x \) in (4) and using (2), we get

\[
f ((n-1)x) = (n-1)f(x)
\]
for all \( x \in X \). Replacing \( x_1 \) by 0 and \( x_2 \) by \( x \) in (3) and using (5), we get \( f(-x) = -f(x) \) for all \( x \in X \), i.e., \( f \) is an odd function. Setting \( x_2 = x_1 - x \) in (3), we get

\[
\frac{n-1}{n} f(x_2) + \frac{1}{n} f(n(x_1 - (n-1)x_2)) = f(x_1)
\]

(6)

for all \( x_1, x_2 \in X \). Replacing \( x_1 \) by \( \frac{x_1}{n} \) and \( x_2 \) by \( \frac{-x_2}{n-1} \) in (6), by (2), (5) and the oddness of \( f \), we obtain

\[
f(x_1 + x_2) = f(x_1) + f(x_2)
\]

for all \( x_1, x_2 \in X \). So \( f \) is additive. \( \square \)

In the following we formulate and prove a theorem in superstability of \( J^* \)-homomorphisms for the functional equation (1).

**Theorem 2.2.** Let \( \ell \in \{-1,1\} \) be given and let \( 0 \neq \ell |s| < \ell \). Assume \( f : A \rightarrow B \) is a mapping for which \( f(sx) = sf(x) \) for all \( x \in A \). Suppose there exists a function \( \phi : A^{n+1} \rightarrow [0,\infty) \) such that

\[
\|\Delta f(x_1, x_2, \ldots, x_n, a) - \mu f(a) f(a)^* f(a)\| \leq \phi(x_1, x_2, \ldots, x_n, a)
\]

(7)

for all \( x_1, \ldots, x_n, a \in A \). If there exists an \( L < 1 \) such that

\[
\phi(x_1, x_2, \ldots, x_n, a) \leq \frac{L}{|s|^2} \phi(s^f x_1, s^f x_2, \ldots, s^f x_n, s^f a)
\]

(8)

for all \( x_1, \ldots, x_n, a \in A \), then \( f \) is a \( J^* \)-homomorphism.

**Proof.** It follows from (8) that

\[
\lim_{m \to \infty} |s|^{m\ell} \phi\left(\frac{x_1}{s^{m\ell}}, \frac{x_2}{s^{m\ell}}, \ldots, \frac{x_n}{s^{m\ell}}, \frac{a}{s^{m\ell}}\right) = 0
\]

(9)

for all \( x_1, \ldots, x_n, a \in A \). Setting \( \mu = 1 \) and \( x_i = 0 \) (1 \( \leq i \leq n \)) in (7), we obtain

\[
\|f(aa^*) - f(a) f(a)^* f(a)\| = \lim_{m \to \infty} |s|^{3m\ell} \|f((\frac{a}{s^{m\ell}})(\frac{a^*}{s^{m\ell}}))
\]

\[
- f((\frac{a}{s^{m\ell}}) f((\frac{a}{s^{m\ell}})^* f((\frac{a}{s^{m\ell}})))
\]

\[
\leq \lim_{m \to \infty} |s|^{3m\ell} \phi(0, 0, \ldots, \frac{a}{s^{m\ell}}) \leq \lim_{m \to \infty} |s|^{m\ell} \phi(0, 0, \ldots, \frac{a}{s^{m\ell}}) = 0
\]

for all \( a \in A \). So

\[
f(aa^*) = f(a) f(a)^* f(a)
\]

for all \( a \in A \). Similarly put \( a = 0 \) in (7), then

\[
\|\mu f(\frac{\sum_{i=1}^n x_i}{n}) + \mu \sum_{j=2}^n f(\frac{\sum_{i=1,i\neq j}^n x_i - (n-1)x_j}{n}) - f(\mu x_1)\|
\]

\[
= \lim_{m \to \infty} |s|^{m\ell} \|\mu f(\frac{\sum_{i=1}^n x_i}{s^{m\ell} n}) + \mu \sum_{j=2}^n f(\frac{\sum_{i=1,i\neq j}^n x_i - (n-1)x_j}{s^{m\ell} n}) - f(\mu x_1)\|
\]

\[
\leq \lim_{m \to \infty} |s|^{m\ell} \phi\left(\frac{x_1}{s^{m\ell}}, \frac{x_2}{s^{m\ell}}, \ldots, \frac{x_n}{s^{m\ell}}, 0\right) = 0
\]
for all \( x_1, \ldots, x_n \in A \). So
\[
\mu f\left( \sum_{i=1}^{n} \frac{x_i}{n} \right) + \mu \sum_{j=2}^{n} f\left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j}{n} \right) = f(\mu x_1)
\]
for all \( \mu \in T \) and all \( x_1, \ldots, x_n \in A \). Thus by Lemma 2.1, the mapping \( f \) is additive.

Letting \( x_i = x \) (1 \( \leq i \leq n \)) and \( a = 0 \) in (7), we have
\[
\|f(\mu x) - f(x)\| = \lim_{m \to \infty} |s|^{mk} \|f(\mu \frac{x}{s^{mk}}) - f(\frac{x}{s^{mk}})\|
\]
\[
\leq \lim_{m \to \infty} |s|^{mk} \|f(\frac{x}{s^{mk}}, \ldots, \frac{x}{s^{mk}}, 0) - f(\frac{x}{s^{mk}}, \ldots, \frac{x}{s^{mk}}, 0)\| = 0
\]
for all \( \mu \in T \) and all \( x \in A \). One can show that the mapping \( f : A \to B \) is \( C \)-linear, and we conclude that \( f \) is a \( J^* \)-homomorphism.

**Corollary 2.3.** Let \( \ell \in \{-1, 1\} \) be given and let \( 0 \neq \ell |s| < \ell, \ell p < \ell \) and \( \delta, \theta, p \) be non-negative real numbers. Suppose that \( f : A \to B \) is a mapping satisfying (7) for all \( x \in A \), and the following inequality
\[
\|\Delta f(x_1, x_2, \ldots, x_n, a) - \mu f(a) \phi(a)^n f(a)\| \leq \frac{1 + \ell}{2} \delta + \theta \left( \sum_{i=1}^{n} \|x_i\|^p + \|a\|^p \right)
\]
for all \( \mu \in T \) and all \( x_1, x_2, \ldots, x_n, a \in A \), then \( f \) is a \( J^* \)-homomorphism.

**Proof.** Let \( \phi(x_1, x_2, \ldots, x_n, a) := \frac{1}{n} \phi\left( \frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n}, \frac{a}{n} \right) \) for all \( x_1, x_2, \ldots, x_n, a \in A \) in Theorem 2.2. Then we choose \( L = |s|^{(1-p)} \) and we get the desired result. \( \square \)

We prove the following generalized Hyers–Ulam stability problem for \( J^* \)-homomorphisms on \( J^* \)-algebras for the functional equation (1).

**Theorem 2.4.** Let \( f : A \to B \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \phi : A^{n+1} \to [0, \infty) \) satisfying (7). If there exists an \( L < 1 \) such that
\[
\phi(x_1, x_2, \ldots, x_n, a) \leq n \phi\left( \frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n}, \frac{a}{n} \right)
\]
for all \( x_1, \ldots, x_n, a \in A \), then there exists a unique \( J^* \)-homomorphism \( H : A \to B \) such that
\[
\|f(x) - H(x)\| \leq \frac{1}{n(1-L)} \phi(nx, 0, 0, \ldots, 0)
\]
for all \( x \in A \).

**Proof.** Letting \( \mu = 1, x_1 = x, x_i = 0 \) (2 \( \leq i \leq n \)) and \( a = 0 \) in (7), we obtain
\[
\|nf(x) - f(x)\| \leq \phi(x, 0, \ldots, 0)
\]
for all \( x \in A \). Replacing \( x \) by \( nx \) in (12), we get
\[
\|\frac{1}{n} f(nx) - f(x)\| \leq \frac{1}{n} \phi(nx, 0, \ldots, 0)
\]
for all \( x \in A \). Consider the set \( X := \{g \mid g : A \to B\} \) and introduce the generalized metric on \( X \) as follows:
\[
d(g, h) := \inf \left\{ C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C \phi(nx, 0, \ldots, 0), \forall x \in A \right\}.
\]
It is easy to show that \((X,d)\) is a generalized complete metric space \([3, 4]\).

Now we define the linear mapping \(T : X \to X\) by 
\[
T(h)(x) = \frac{1}{n} h(nx) \quad \text{for all } x \in A.
\]

It is easy to see that 
\[
d(T(g), T(h)) \leq Ld(g, h)
\]
for all \(g, h \in X\). It follows from (13) that
\[
d(f, T(f)) \leq \frac{1}{n} < \infty.
\] (14)
By Theorem 1.1, \(T\) has a unique fixed point in the set 
\[
X_1 := \{g \in X : d(f, g) < \infty\}.
\]

Let \(H\) be the fixed point of \(T\). \(H\) is the unique mapping with 
\[
H(nx) = nH(x) \quad \text{for all } x \in A,
\]
such that there exists \(C \in (0, \infty)\) satisfying
\[
\|f(x) - H(x)\| \leq C\phi(nx, 0, \ldots, 0)
\]
for all \(x \in A\). On the other hand we have \(\lim_{m \to \infty} d(T^m(f), H) = 0\). It follows that 
\[
\lim_{m \to \infty} \frac{1}{n^m} f(n^m x) = H(x)
\] (15)
for all \(x \in A\). Also by Theorem 1.1, we have
\[
d(f, H) \leq \frac{1}{1 - L}d(f, T(f))
\] (16)
It follows from (14) and (16), that
\[
d(f, H) \leq \frac{1}{n(1 - L)}
\]
This implies inequality (11). It follows from (10) that
\[
\lim_{m \to \infty} \frac{1}{n^m} \phi(n^m x_1, n^m x_2, \ldots, n^m x_n, n^m a) = 0
\] (17)
for all \(x_1, \ldots, x_n, a \in A\). By the same reasoning as the proof of Theorem 2.2, One can show that the mapping \(H : A \to B\) is \(C\)-linear. It follows from (7), (15) and (17) that
\[
\|H(aa^* a) - H(a)H(a)^* H(a)\| = \lim_{m \to \infty} \frac{1}{n^m} \|H((n^m a)(n^m a^*)(n^m a))
\]
\[
- H(n^m a)H(n^m a)^* H(n^m a)\|
\]

\[
\leq \lim_{m \to \infty} \frac{1}{n^m} \phi(0, 0, \ldots, n^m a)
\]
\[
\leq \lim_{m \to \infty} \frac{1}{n^m} \phi(0, 0, \ldots, n^m a) = 0
\]
for all \(a \in A\). Thus
\[
H(aa^* a) = H(a)H(a)^* H(a)
\]
for all \(a \in A\). Hence \(H : A \to B\) is a \(J^*\)-homomorphism. \(\square\)

**Corollary 2.5.** Let \(\theta, p\) be non–negative real numbers such that \(p < 1\). Suppose that a function \(f : A \to B\) satisfies
\[
\|\triangle f(x_1, x_2, \ldots, x_n, a) - \mu f(a)f(a)^* f(a)\| \leq \theta \sum_{i=1}^{n} (\|x_i\|^p + \|a\|^p)
\]
for all $\mu \in T$ and all $x_1, \ldots, x_n, a \in A$. Then there exists a unique $J^*$-homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{\theta}{n!-1} \|x\|^p$$

for all $x \in A$.

The case in which $p = 1$ was excluded in Corollary 2.5. Indeed this result is not valid when $p = 1$. Here we use Gajda’s example [14] to construct a Counterexample.

Example 2.6. Let $\phi : C \to C$ be defined by

$$\phi(x) := \begin{cases} x & \text{for } |x| < 1; \\ 1 & \text{for } |x| \geq 1. \end{cases}$$

Consider the function $f : C \to C$ to be defined by the formula

$$f(x) := \sum_{n=0}^{\infty} n^{-m}\phi(n^m x)$$

Let

$$D_\mu f(x_1, \ldots, x_n, a) := \mu f(\sum_{i=1}^{n} x_i + a\alpha_n) + \mu \sum_{j=2}^{n} f(\sum_{i=1}^{n} x_i - (n-1)x_j + a\alpha_n) - f(\mu x_1) - \mu f(a)f(a)$$

for all $\mu \in T$ and all $x_1, x_2, \ldots, x_n, a \in C$. Then $f$ satisfies

$$|D_\mu f(x_1, \ldots, x_n, a)| \leq \frac{n^4 + n^3 + 6n^2 - 7n + 2}{(n - 1)^2} \left(\sum_{i=1}^{n} |x_i| + |a|\right)$$

for all $\mu \in T$ and all $x_1, x_2, \ldots, x_n, a \in C$, and the range of $|f(x) - A(x)|/|x|$ for $x \neq 0$ is unbounded for each additive function $A : C \to C$.

Proof. It is clear that $f$ is bounded by $\frac{n}{n+1}$ on $C$. If $\sum_{i=1}^{n} |x_i| + |a| = 0$ or $\sum_{i=1}^{n} |x_i| + |a| \geq 1$, then

$$|D_\mu f(x_1, \ldots, x_n, a)| \leq \frac{n^4 - n^2 + n}{(n - 1)^3} \leq \frac{n^4 - n^2 + n}{(n - 1)^3} \left(\sum_{i=1}^{n} |x_i| + |a|\right)$$

Now suppose that $0 < \sum_{i=1}^{n} |x_i| + |a| < 1$. Then there exists an integer $k \geq 0$ such that

$$\frac{1}{n^{k+1}} \leq \sum_{i=1}^{n} |x_i| + |a| < \frac{1}{n^k}$$

Therefore

$$n^k \left|\sum_{i=1}^{n} x_i + a\alpha_n\right|, n^k \left|\sum_{i=1}^{n} x_i + a\alpha_n - (n - 1)x_j\right|, n^k |\mu x_1|, n^k |a| < 1$$

for all $j = 2, 3, \ldots, n$ and all $t = 0, 1, \ldots, k - 1$. From the definition of $f$ and (19), we have

$$|f(a)| \leq k|a| + \sum_{m=k}^{\infty} n^{-m} |\phi(n^m a)| \leq k|a| + \frac{n}{n^k(n - 1)}.$$
\begin{align*}
|D_\mu f(x_1, \ldots, x_n, a)| & \leq k|a|^3 + \frac{n(n + 1)}{n^k(n - 1)} + |f(a)|^3 \\
& \leq (k + k^3)|a|^3 + \frac{n^2 + 2n}{n^k(n - 1)} + 3n(n - 1)k^2 + 3n^2k|a| \\
& \leq \frac{(n - 1)^2k^3 + 3n(n - 1)k^2 + ((n - 1)^2 + 3n^2)k|a| + n^2 + 2n}{n^k(n - 1)^2} \\
& \leq \frac{2(n - 1)^2 + 3n(n - 1) + 3n^2}{(n - 1)^2} |a| + \frac{n^3 + 2n^2}{(n - 1)} \left( \sum_{i=1}^{n} |x_i| + |a| \right)
\end{align*}

Therefore \( f \) satisfies (18). Let \( A : \mathbb{C} \to \mathbb{C} \) be an additive function such that

\[
|f(x) - A(x)| \leq \alpha |x|
\]

for all \( x \in \mathbb{C} \), where \( \alpha > 0 \) is a constant. Then there exists a constant \( c \in \mathbb{C} \) such that \( A(x) = cx \) for all rational numbers \( x \). Thus we have

\[
|f(x)| \leq (\alpha + |c|)|x|
\]

for all rational numbers \( x \). Let \( t \in \mathbb{N} \) with \( t > \alpha + |c| \). If \( x \) is a rational number in \( (0, n^{1-t}) \), then \( n^m x \in (0, 1) \) for all \( m = 0, 1, \ldots, t - 1 \). Therefore

\[
f(x) \geq \sum_{m=0}^{t-1} n^{-m} \phi(n^m x) = tx > (\alpha + |c|)x
\]

which contradicts (20).

3. Approximation of \( J^* \)-derivations in \( J^* \)-algebras

In this section, we prove the superstability and stability of \( J^* \)-derivations on \( J^* \)-algebras for the functional equation (1).

**Theorem 3.1.** Let \( \ell \in \{-1, 1\} \) be given and let \( 0 \neq |s| \ell > \ell \). Suppose \( f : A \to A \) is a mapping for which \( f(sx) = sf(x) \) for all \( x \in A \). Suppose there exists a function \( \psi : A^{n+1} \to [0, \infty) \) such that

\[
\|\triangle f(x_1, x_2, \ldots, x_n, a) - \mu f(a)a^*a - \mu a f(a)^*a - \mu aa^*f(a)\| \leq \psi(x_1, x_2, \ldots, x_n, a) \tag{21}
\]

for all \( x_1, \ldots, x_n, a \in A \). If there exists an \( L < 1 \) such that

\[
\psi(x_1, x_2, \ldots, x_n, a) \leq \ell|s|\psi\left(\frac{x_1}{s^\ell}, \frac{x_2}{s^\ell}, \ldots, \frac{x_n}{s^\ell}, \frac{a}{s^\ell}\right) \tag{22}
\]

for all \( x_1, \ldots, x_n, a \in A \), then \( f \) is a \( J^* \)-derivation.

**Proof.** By using equation \( f(sx) = sf(x) \) and (21), we have \( f(0) = 0 \) and

\[
\|\mu f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) + \mu \sum_{j=2}^{n} f\left(\sum_{i=1, i \neq j}^{n} x_i - \frac{(n - 1)x_j}{n}\right) - f(\mu x_1)\|
\]
It is easy to see that for all \( x \) for all \( \psi \) function \( \text{Theorem 3.3.} \)

\[ \|f(aa^*a) - f(a)^*a - af(a)^*a - aa^*f(a)\| \leq |s|^{-3m} \psi(0, 0, \ldots, 0, s^{m^*}a) \]

for all \( x_1, \ldots, x_n, a \in A \) and all integers \( m \). It follows from (22), that

\[ \lim_{m \to \infty} |s|^{-m} \psi(s^{m^*}x_1, s^{m^*}x_2, \ldots, s^{m^*}x_n, s^{m^*}a) = 0 \]

for all \( x_1, \ldots, x_n, a \in A \). Hence, we get from (23), (24) and (25) that

\[ \mu f(\sum_{i=1}^{n} x_i) + \mu \sum_{j=2}^{n} f(\sum_{i=1, i \neq j}^{n} x_i - (n - 1)x_j) = f(\mu x_1), \]

\[ f(aa^*a) = f(a)^*a + af(a)^*a + aa^*f(a) \]

for all \( x_1, \ldots, x_n, a \in A \). Therefore \( f \) is additive and \( f(\mu x) = \mu f(x) \) for all \( \mu \in T \) and \( x \in A \). By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping \( f : A \to A \) is \( C \)-linear, and we conclude that \( f \) is a \( J^* \)-derivation.

**Corollary 3.2.** Let \( \ell \in \{-1, 1\} \) be given and let \( 0 \neq \ell|s| > \ell, \ell p > \ell \) and \( \beta, \epsilon, p \) be non-negative real numbers. Suppose that \( f : A \to A \) is a mapping satisfying \( f(sx) = sf(x) \) for all \( x \in A \), and the following inequality

\[ \|\Delta f(x_1, x_2, \ldots, x_n, a) - \mu f(\mu a^*a - \mu af(a)^*a - \mu aa^*f(a)) \|
\]

\[ \leq \frac{1 + \ell}{2} \beta + \epsilon \left( \sum_{i=1}^{n} \|x_i\|^p + \|a\|^p \right) \]

for all \( \mu \in T \) and all \( x_1, x_2, \ldots, x_n, a \in A \), then \( f \) is a \( J^* \)-derivation.

**Theorem 3.3.** Let \( f : A \to A \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \psi : A^{n+1} \to [0, \infty) \) satisfying (21). If there exists an \( L < 1 \) such that

\[ \psi(x_1, x_2, \ldots, x_n, a) \leq nL \psi \left( \frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n}, a \right) \]

for all \( x_1, \ldots, x_n, a \in A \), then there exists a unique \( J^* \)-derivation \( D : A \to A \) such that

\[ \|f(x) - D(x)\| \leq \frac{L}{1 - L} \psi(x, 0, 0, \ldots, 0) \]

for all \( x \in A \).

**Proof.** It follows from (26) that

\[ \lim_{m \to \infty} \frac{1}{m^m} \psi(n^m x_1, n^m x_2, \ldots, n^m x_n, n^m a) = 0 \]

for all \( x_1, \ldots, x_n, a_1, \ldots, a_n \in A \). Consider the set \( X' := \{g : A \to X\} \) and introduce the generalized metric on \( X' \) as follows:

\[ d(g, h) := \inf \{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C \psi(x, 0, \ldots, 0), \forall x \in A\} \]

It is easy to show that \( (X', d) \) is a generalized complete metric space.

Now we define the linear mapping \( J : X' \to X' \) by \( J(h)(x) = \frac{1}{n} h(nx) \) for all \( x \in A \). It is easy to see that

\[ d(J(g), J(h)) \leq Ld(g, h) \]

for all \( g, h \in X' \).
Letting \( \mu = 1, x_1 = x, x_i = 0 \) \((2 \leq i \leq n)\) and \( a = 0 \) in (21), we obtain
\[
\|nf(x_n^N) - f(x)\|_X \leq \psi(x,0,\ldots,0)
\]
for all \( x \in A \). Thus by using (26), we obtain
\[
\|1/nf(nx) - f(x)\|_X \leq 1/n\psi(nx,0,\ldots,0) \leq L\psi(x,0,\ldots,0)
\]
for all \( x \in A \), that is,
\[
d(f,J(f)) \leq L < \infty.
\]
By Theorem 1.1, \( J \) has a unique fixed point in the set \( X_2 := \{ h \in X' : d(f,h) < \infty \} \).
Let \( D \) be the fixed point of \( J \). We note that \( D \) is the unique mapping with \( D(nx) = nD(x) \) for all \( x \in A \), such that there exists \( C \in (0,\infty) \) satisfying
\[
\|f(x) - D(x)\| \leq C\psi(x,0,\ldots,0)
\]
for all \( x \in A \). On the other hand we have
\[
\lim_{m \to \infty} d(J^{mN}(f),D) = 0,
\]
so
\[
\lim_{m \to \infty} \frac{1}{nm} f(n^{mN}x) = D(x)
\]
for all \( x \in A \). Also by Theorem 1.1, we have
\[
d(f,D) \leq \frac{1}{1-L} d(f,J(f))
\]
It follows from (31) and (32), that
\[
d(f,D) \leq \frac{L}{1-L}
\]
This implies inequality (27). By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping \( f : A \to A \) is \( C \)-linear. It follows from (21) and (28) that
\[
\|D(aa^*a) - D(a)a^*a - aD(a)^*a - aa^*D(a)\|
\]
\[
= \lim_{m \to \infty} \frac{1}{nm} \left\| D((n^{mN}a)(n^{mN}a^*)(n^{mN}a)) - D(n^{mN}a)(n^{mN}a)^*(n^{mN}a) - (n^{mN}a)D(n^{mN}a)^*(n^{mN}a) - (n^{mN}a)(n^{mN}a^*)D(n^{mN}a) \right\|
\]
\[
\leq \frac{1}{n^{3m}} \psi(0,0,\ldots,0,n^{mN}a) \leq \frac{1}{n^m} \psi(0,0,\ldots,0,n^{mN}a) = 0
\]
for all \( a \in A \). Therefore
\[
D(aa^*a) = D(a)a^*a + aD(a)^*a + aa^*D(a)
\]
for all \( a \in A \). Hence \( D : A \to A \) is a \( J^* \)-derivation. \( \square \)

**Corollary 3.4.** Let \( \varepsilon, p \) be non–negative real numbers such that \( p < 1 \). Suppose that a function \( f : A \to A \) satisfies
\[
\|\triangle f(x_1,x_2,\ldots,x_n,a) - \mu f(a)a^*a - \mu af(a)^*a - \mu a^*f(a)\|
\]
for all $\mu \in \mathbb{T}$ and all $x_1, \ldots, x_n, a \in A$. Then there exists a unique $J^*$-derivation $D : A \to A$ such that

$$
\| f(x) - D(x) \| \leq \frac{n^{p-1} \varepsilon}{1 - n^{p-1}} \| x \|^p
$$

for all $x \in A$.

For the case $p = 1$, similar to the Example 2.6, we have the following counterexample.

**Example 3.5.** Let $\psi : \mathbb{C} \to \mathbb{C}$ be defined by

$$
\psi(x) := \begin{cases} 
  x & \text{for } |x| < 1; \\
  1 & \text{for } |x| \geq 1.
\end{cases}
$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ to be defined by the formula

$$
f(x) := \sum_{m=0}^{\infty} n^{-m} \psi(n^m x)
$$

Let

$$
D_\mu f(x_1, \ldots, x_n, a) := 
\mu f\left(\frac{\sum_{i=1}^{n} x_i + a\overline{a}}{n}\right) 
+ \mu \sum_{j=2}^{n} f\left(\frac{\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j + a\overline{a}}{n}\right) - f(\mu x_1)
$$

for all $\mu \in \mathbb{T}$ and all $x_1, x_2, \ldots, x_n, a \in \mathbb{C}$. Then $f$ satisfies

$$
|D_\mu f(x_1, \ldots, x_n, a)| \leq \frac{n^3 + n^2 + 7n - 4}{n-1} \left(\sum_{i=1}^{n} |x_i| + |a|\right)
$$

for all $\mu \in \mathbb{T}$ and all $x_1, x_2, \ldots, x_n, a \in \mathbb{C}$, and the range of $|f(x) - A(x)|/|x|$ for $x \neq 0$ is unbounded for each additive function $A : \mathbb{C} \to \mathbb{C}$.

**Proof.** It is clear that $f$ is bounded by $\frac{n}{n-1}$ on $\mathbb{C}$. If $\sum_{i=1}^{n} |x_i| + |a| = 0$ or $\sum_{i=1}^{n} |x_i| + |a| \geq 1$, then

$$
|D_\mu f(x_1, \ldots, x_n, a)| \leq \frac{n^2 + (1 + 3|a|^2)n}{(n-1)} \leq \frac{n^2 + (1 + 3|a|^2)n}{(n-1)} \left(\sum_{i=1}^{n} |x_i| + |a|\right)
$$

Now suppose that $0 < \sum_{i=1}^{n} |x_i| + |a| < 1$. Then there exists an integer $k \geq 0$ such that

$$
\frac{1}{n^{k+1}} \leq \sum_{i=1}^{n} |x_i| + |a| < \frac{1}{n^k}
$$

(34)
Therefore
\[ n' \sum_{i=1}^{n} x_i + a \sum_{i=1}^{n} x_i + a(a - (n-1)x_j), n'|x_1|, n'|a| < 1 \]
for all \( j = 2, 3, \ldots, n \) and all \( t = 0, 1, \ldots, k - 1 \). From the definition of \( f \) and (34), we have
\[
|f(a)| \leq k|a| + \sum_{m=k}^{\infty} n^{-m} |\psi(n^m a)| \leq k|a| + \frac{n}{n^k(n-1)},
\]
\[
|D_n f(x_1, \ldots, x_n, a)| \leq k|a|^3 + \frac{n(n+1)}{n^k(n-1)} + 3|a|^2 |f(a)|
\]
\[
\leq 4k|a|^3 + \frac{n^2 + n}{n^k(n-1)} + \frac{3n}{n^k(n-1)} |a|^2
\]
\[
\leq \frac{4(n-1)k + 3n}{n^k(n-1)} |a|^2 + \frac{n^2 + n}{n^k(n-1)}
\]
\[
\leq \frac{4(n-1)k + 3n}{n^k(n-1)} |a| + \frac{n^3 + n^2}{n-1} \left( \sum_{i=1}^{n} |x_i| + |a| \right)
\]
\[
\leq \frac{n^3 + n^2 + 7n - 4}{(n-1)} \left( \sum_{i=1}^{n} |x_i| + |a| \right)
\]
Therefore \( f \) satisfies (33). Let \( A : \mathbb{C} \to \mathbb{C} \) be an additive function such that
\[
|f(x) - A(x)| \leq \alpha|x|
\]
for all \( x \in \mathbb{C} \), where \( \alpha > 0 \) is a constant. Then there exists a constant \( c \in \mathbb{C} \) such that \( A(x) = cx \) for all rational numbers \( x \). Thus we have
\[
|f(x)| \leq (\alpha + |c|)|x| \quad (35)
\]
for all rational numbers \( x \). Let \( t \in \mathbb{N} \) with \( t > \alpha + |c| \). If \( x \) is a rational number in \( (0, n^{1-t}) \), then \( n^m x \in (0, 1) \) for all \( m = 0, 1, \ldots, t - 1 \). Hence
\[
f(x) \geq \sum_{m=0}^{t-1} n^{-m} \phi(n^m x) = tx > (\alpha + |c|)x
\]
which contradicts (35).  \( \square \)

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