Fixed Point Theory, 13(2012), No. 2, 475-480
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A NOTE ON OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS AND COMMON FIXED POINTS

DRAGAN ĐORIĆ*, ZORAN KADELBURG** AND STOJAN RADENOVIĆ***

*University of Belgrade, Faculty of Organizational Sciences
Jove Ilića 154, 11000 Beograd, Serbia
E-mail: djoricd@fon.bg.ac.rs

**University of Belgrade, Faculty of Mathematics
Studentski trg 16, 11000 Beograd, Serbia
E-mail: kadelbur@matf.bg.ac.rs

***University of Belgrade, Faculty of Mechanical Engineering
Kraljice Marije 16, 11120 Beograd, Serbia
E-mail: radens@beotel.net

Abstract. It is shown that, for single-valued maps, the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence (and a unique common fixed point) of the given maps. For hybrid pairs of maps this is not the case.
Key Words and Phrases: Weakly compatible mappings, occasionally weakly compatible mappings, common fixed point, hybrid pairs of mappings.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Investigation of common fixed points of pairs of mappings under certain contractive conditions began by using mappings that commute. Later on this condition was weakened in various ways. One of the conditions that was used most often was the weak compatibility, introduced by Jungck in [9]. In the paper [6], published in 2008, Al-Thagafi and Shahzad introduced an even weaker condition which they called occasionally weak compatibility. These and other authors (see, e.g., [10]–[8]) used this condition to obtain common fixed point results, sometimes trying to generalize results that were known to use (formally stronger) condition of weak compatibility. We shall show in this short note that in the presence of a unique point of coincidence (and a unique common fixed point) of the given mappings, occasionally weak compatibility actually reduces to weak compatibility. Thus, no generalization can be obtained in this way.

The situation is a bit different for so-called hybrid pairs of mappings, i.e., pairs whose first component is a single-valued selfmap, and the second is a set-valued map. Here, both of the mentioned kinds of compatibility can be defined (see [11] and [2]) and the connection between points of coincidence and common fixed points remains valid (see our Lemma 2.3). However, we shall show that in this case occasionally weak compatibility is a strictly weaker condition than weak compatibility, so one can hope to obtain more general results by using this condition (see, e.g., [3]–[5]).

2. Results

2.1. **Results for selfmaps.** We first recall some definitions.

Let X be a nonempty set and let f and g be two selfmaps on X. A point $x \in X$ is called a coincidence point (CP) of the pair (f, g) if $fx = gx\,(= w)$. The point w is then called a point of coincidence (POC) for (f, g). The set of coincidence points of (f, g) will be denoted as $C(f, g)$.

The pair (f, g) of selfmaps on X is called *weakly compatible* (WC) if they commute at each coincidence point, i.e., if $fgx = gfx$ for each $x \in C(f, g)$. It is *occasionally weakly compatible* (OWC) if they commute at some coincidence point, i.e., if there exists $x \in C(f, g)$ such that $fgx = gfx$.

Many authors simple state that if the pair (f, g) is WC then "obviously" it is OWC. Strictly speaking, this is true only if the set $C(f, g)$ is nonempty (if it is empty, then such pair is always WC but it is not OWC). In particular, if i_X is the identity mapping, then the pair (f, i_X) is always WC, but it is OWC if and only if f has a fixed point.

The use of WC and OWC pairs of mappings is based on the following simple lemma of Jungck and Rhoades. For the sake of completeness we give also a proof.

Lemma 2.1 [10] *If an OWC pair (f, g) of selfmaps on X has a unique POC, then it has a unique common fixed point.*

Proof. Since (f, g) is OWC, there exists $x \in C(f, g)$ such that $fx = gx =: w$ and $fw = fgx = gfx = gw$. Hence, $fw = gw$ is also a POC for (f, g), and since it must be unique, we have that $w = fw = gw$, i.e., w is a common fixed point for (f, g).

If z is any common fixed point for (f, g) (i.e., $fz = gz = z$) then, again by the uniqueness of POC, it must be $z = w$.

It is easy to construct simple examples of OWC pairs of mappings that are not WC. For instance, one can take $X = [1, \infty)$, $fx = 3x - 2$, $gx = x^2$ and it is obvious that $C(f, g) = \{1, 2\}$ and that $fg1 = gf1$, $fg2 \neq gf2$. Thus, several authors claim that assertions on common fixed points that use OWC of a pair of mappings as an assumption are stronger than respective assertions that use WC as an assumption. However, if the conclusion of such assertion is that there is a unique fixed point of the given pair, then it is not actually a generalization, because in that case OWC-condition reduces to WC-condition. Namely, the following simple proposition holds true.

Proposition 2.2 *Let a pair of mappings (f, g) has a unique POC. Then it is WC if and only if it is OWC.*

Proof. In this case we have only to prove that OWC implies WC. Let $w_1 = fx = gx$ is the given POC and let (f, g) be OWC. Let $y \in C(f, g)$, $y \neq x$. We have to prove that $fgy = gfy$.

Now $w_2 = fy = gy$ is a POC for the pair (f, g), By the assumption, $w_2 = w_1$, i.e. $fy = gy = fx = gx$. Since, by Lemma 2.1, w_1 is a unique common fixed point of the pair (f, g), it follows that $w_1 = fw_1 = fgy$ and $w_1 = gw_1 = gfy$, hence $fgy = gfy$. The pair (f, g) is WC.

The example given above shows that the assumption about the uniqueness of POC cannot be removed.

2.2. **Results for hybrid pairs of maps.** Let X be a nonempty set and $N(X)$ the collection of all nonempty subsets of X. We shall say that (f, F) is a *hybrid pair of maps* if $f : X \to X$ is a selfmap and $F : X \to N(X)$ is a set-valued map on X. A point $x \in X$ is called a coincidence point (CP) of (f, F) (written as $x \in C(f, F)$) if $fx \in Fx$. In that case, $y = fx$ is called a point of coincidence (POC) of (f, F). The point x is called a fixed point of F if $x \in Fx$ and a strict fixed point (also stationary point or endpoint) if $Fx = \{x\}$.

The pair (f, F) is called *weakly compatible* (WC) if $fFx = Ffx$ for each $x \in C(f, F)$ [11] and *occasionally weakly compatible* (OWC) if there exists $x \in C(f, F)$ such that $fFx \subset Ffx$ [2].

Again there are easy examples showing that a pair can be OWC and not WC. For instance (see [2]), one can take $X = [0, \infty)$,

$$fx = \begin{cases} 0, & 0 \le x < 1, \\ 2x, & 1 \le x < \infty \end{cases} \quad \text{and} \quad Fx = \begin{cases} \{x\}, & 0 \le x < 1, \\ [1, 1 + 4x], & 1 \le x < \infty \end{cases}$$

and it is easy to show that $0, 1 \in C(f, F)$, $fF0 = Ff0 = \{0\}$, but $fF1 \ne Ff1$.

We could not find the following simple extension of Lemma 2.1 in the literature.

Lemma 2.3 *If (f, F) is an OWC hybrid pair of maps on a nonempty set X which has a unique POC, then (f, F) has a unique common fixed point (i.e., there exists a unique $y \in X$ such that $y = fy \in Fy$).*

Proof. Since (f, F) is OWC, there exists $x \in C(f, F)$, so that $y := fx \in Fx$ and $fFx \subset Ffx = Fy$. Then $fy \in fFx \subset Fy$ and fy is also a POC of (f, F). By the assumption on uniqueness of POC it must be $y = fy \in Fy$. We have proved that y is a common fixed point for the pair (f, F).

Let z be any common fixed point of (f, F), i.e., $z = fz \in Fz$. Then fz is a POC of (f, F), and again by the uniqueness, $z = y$. Thus, the common fixed point of (f, F) is unique.

The previous Lemma does not hold without the assumption of OWC as the next simple example shows.

Example 2.4 Let $X = \{1, 2, 3\}$,

$$f : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \quad \text{and} \quad F : \begin{pmatrix} 1 & 2 & 3 \\ \{2\} & \{2, 3\} & \{1\} \end{pmatrix}.$$

Then $1 = f3 \in F3$ is a unique POC for (f, F), i.e., $C(f, F) = \{3\}$. However, $fF3 = f(\{1\}) = \{3\} \ne Ff3 = F1 = \{2\}$ and (f, F) is not WC (neither OWC). Obviously, f has no fixed point, so the pair (f, F) has no common fixed points. Hence, OWC-assumption cannot be removed from the previous lemma.

The analogue of Proposition 2.2 for hybrid pairs does not hold, as the following simple example shows.

Example 2.5 Let $X = \{1, 2, 3\}$,

$$f : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad F : \begin{pmatrix} 1 & 2 & 3 \\ \{1\} & \{1,3\} & \{1\} \end{pmatrix}.$$

Now $1 = f1 \in F1$ is obviously a unique common fixed point of (f, F) and also $1 = f1 = f2$ is a unique POC since $1 = f1 \in F1$ and $1 = f2 \in F2$, but $C(f, F) = \{1, 2\}$. Since $fF1 = \{1\} = Ff1$ and $fF2 = \{1, 2\} \neq \{1\} = Ff2$, the pair (f, F) is OWC but it is not WC.

Hence, in the case of hybrid pairs, OWC-condition does not reduce to the WC-condition, even in the presence of a unique POC and a unique common fixed point.

Finally, we note that, unlike to the situation with single-valued maps, a hybrid pair can have both unique POC and unique common fixed point, without being even OWC.

Example 2.6 Let $X = \{1, 2, 3\}$,

$$f : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad F : \begin{pmatrix} 1 & 2 & 3 \\ \{1,3\} & \{1,3\} & \{1\} \end{pmatrix}.$$

Here again 1 is both a unique common fixed point of (f, F) and a unique POC and also $C(f, F) = \{1, 2\}$. But now $fF1 = \{1, 2\} \neq \{1, 3\} = Ff1$ and $fF2 = \{1, 2\} \neq \{1\} = Ff2$, and the pair (f, F) is not OWC (neither WC).

Acknowledgement. The authors are thankful to the Ministry of Science and Technological Development of Serbia.

References

[1] M. Abbas, B.E. Rhoades, *Common fixed point theorems for occasionally weakly compatible mappings satisfying generalized contractive conditions*, Math. Commun. **13**(2008), 295–301.

[2] M. Abbas, B.E. Rhoades, *Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type*, Fixed Point Theory Appl., Vol. 2007, Article ID 54101, 9 pages, doi: 10.1155/2007/54101.

[3] M. Abbas, B.E. Rhoades, *Erratum: Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type revisited*, Fixed Point Theory Appl., Vol. 2008, Article ID 274793, 2 pages, doi: 10.1155/2007/274793.

[4] M. Abbas, A. Rahim Khan, *Common fixed points of generalized contractive hybrid pairs in symmetric spaces*, Fixed Point Theory Appl., Vol. 2009, Article ID 867497, 11 pages, doi: 10.1155/2009/869407.

[5] A. Aliouche, V. Popa, *General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications*, Novi Sad J. Math., **39**(2009), no. 1, 89–109.

[6] M.A. Al-Thagafi, N. Shahzad, *Generalized I-nonexpansive selfmaps and invariant approximations*, Acta Math. Sinica, English Ser., **24**(2008), no. 5, 867–876.

[7] A. Bhatt, H. Chandra, D.R. Sahu, *Common fixed point theorems for occasionally weakly compatible mappings under relaxed conditions*, Nonlinear Anal., **73**(2010), 176–182.

[8] H. Bouhadjera, Ch. Godet-Thobie, *Common fixed point theorems for occasionally weakly compatible maps*, Reprint.arXiv 0812.3734.

[9] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci., **4**(1996), 199–215.

[10] G. Jungck, B.E. Rhoades, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory **7**(2006), no. 2, 287–296.

[11] G. Jungck, B.E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., **29**(1998), no. 3, 227–248.

[12] Z. Kadelburg, S. Radenović, V. Rakočević, *Topological vector space valued cone metric spaces and fixed point theorems*, Fixed Point Theory Appl., Vol. 2010, Article ID 170253, 17 pages, doi:10.1155/2010/170253.

Received: August 12, 2010; Accepted: October 28, 2010.