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FURI-PERA FIXED POINT THEOREMS FOR NONEXPANSIVE MAPS IN BANACH SPACES

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Abstract. In this work, we present some new fixed point theorems for nonexpansive maps, 1-set contractions, and demi-closed nonexpansive perturbations of nonexpansive maps defined on closed, convex, not necessarily bounded subsets of Banach spaces; the stress will be made on the so-called Furi-Pera boundary condition. The proofs use the Kuratowski measure of noncompactness and rely on a recent compactness result for the approximate fixed point set. To illustrate the results obtained, applications to a fixed point theorem in a Banach algebra and to an integral equation are provided.
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1. Introduction

Nonexpansive maps appear in many nonlinear problems modeled by Hammerstein and integral equations arising from mechanics, electricity and population dynamics. Given a Banach space $(E, \|.\|)$ and a mapping $f : E \longrightarrow E$, recall that f is called nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|,\ \forall\,(x, y) \in E^2$. Although the Banach fixed point theorem and the Schauder fixed point theorem as developed in [14, 32, 35] cannot be applied, the fixed point theory for such mappings has attracted much attention in the last couple of years (see [15, 19, 23, 26] and the references therein). In this respect, a fundamental existence theorem was discovered in 1965 (see [4, 21, 27]) for nonexpansive maps $f : Q \longrightarrow Q$ where Q is a nonempty, closed, bounded, convex subset of a uniformly convex Banach space E (see Theorem 2.3). This result was followed by an intensive research work developed in the rich recent literature; for a survey of some of these results, we refer the reader to [1, 2, 14, 18, 28] and to references therein. We should point out that the geometric structure of the Banach space E plays a decisive role in the development of the existence theory for nonexpansive mappings (see, e.g., [7, 8, 10, 20]).

In 1987, Furi and Pera [16] introduced the following boundary condition to replace $f(Q) \subset Q$ in some fixed point theorems where f is a continuous map:

$$(\mathcal{FP}) \begin{cases} \text{if } \{(x_j, \mu_j)\}_{j \geq 1} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{converging to } (x, \mu) \text{ with } x = \mu f(x) \text{ and } 0 \leq \mu < 1, \\ \text{then } \mu_j f(x_j) \in Q \text{ for j sufficiently large.} \end{cases}$$

Like the classical Leray-Schauder boundary condition, this condition has recently been extensively employed to obtain new fixed point theorems (see, e.g., [1, 12, 29]). It is the aim of this work to complement the existing rich literature by proving some new fixed point theorems for nonexpansive mappings satisfying $(\mathcal{FP})$ and defined on closed, convex, not necessarily bounded subsets of general Banach spaces. Indeed, if Q is a closed, convex, and unbounded subset of a Banach space enjoying the fixed point property, then it is not true, as the translation map with null vector shows, that a nonexpansive map $T : Q \longrightarrow Q$ has a fixed point. More precisely, Ray [31] proved that if Q is a closed, convex, unbounded subset of a Hilbert space, then there exist fixed-point free nonexpansive mappings (see also Sines's proof in [33]). Moreover, in [13] and [26], interesting fixed point theorems have been recently obtained in case Q is not necessarily bounded. The aim of this paper is complement some of these results and the plan is organized as follows.

Useful ingredients including important notions about retractions, contractions as well as recent results on nonexpansive maps are first gathered together in Section 2. Some new fixed point theorems for nonexpansive mappings and 1-set contractions defined in nonempty, closed, convex subsets of Banach spaces are then presented in Section 3. For this purpose, a recent compactness argument regarding the approximate fixed point set is used. Then we prove a fixed point theorem for a 1-set contraction mapping f satisfying $(\mathcal{FP})$ and such that $I - f$ is demi-closed in a reflexive Banach space where I is the identity operator. An existence result for nonexpansive maps is also derived. Comparison with already known results are provided. We end the paper in Section 4 with a fixed point theorem in a Banach algebra and an application to an integral equation. This paper is mainly inspired by the recent works by Agarwal *et al* [1], Isac-Németh [23], Kaewcharoen-Kirk [26], and the authors [13].

2. Preliminaries

2.1. **Basic notions.**

Definition 2.1. *Let E be a Banach space and $\mathcal{B} \subset \mathcal{P}_B(E)$ where $\mathcal{P}_B(E)$ denotes the set of all bounded subsets of E. For any subset $A \in \mathcal{B}$, define*

$$\alpha(A) = \inf\{\varepsilon > 0 : \ A = \bigcup_{i=1}^{i=n} \Omega_i, \ diam(\Omega_i) \leq \varepsilon, \ \forall\, i = 1, \ldots, n\}.$$

α is called the Kuratowski measure of noncompactness, $\alpha - MNC$ for brevity.

For more details on the main properties of $\alpha - MNC$, we refer the reader to [3, 10]. We point out that of particular importance is the regularity property stating that $\alpha(A) = 0$ if and only if A is relatively compact.

Definition 2.2. *Let E_1, E_2 be two Banach spaces and $f : E_1 \longrightarrow E_2$ a mapping which maps bounded subsets of E_1 into bounded subsets of E_2.*
(a) *f is called $\alpha-$Lipschitz with constant k (or a $k-$set contraction) if there exists some constant $k \geq 0$ such that $\alpha(f(A)) \leq k\alpha(A)$, for any bounded subset $A \subset E_1$.*
(b) *f is an $\alpha-$contraction whenever f is an $\alpha-$Lipschitz with constant $0 < k < 1$.*

Note that a nonexpansive map is a 1-set contraction (see Lemma 1.1 and Remark 1.5 in [13]). In 1955, G. Darbo [9] proved that if Q is a closed, convex and bounded subset of a Banach space, then every $\alpha-$contraction $f : Q \longrightarrow Q$ has at least one fixed point. But this theorem does not apply for 1-set contractions.

Definition 2.3.
(a) *A subset A of a Banach space E is a nonexpansive retract of E if there exists a nonexpansive mapping $r : E \longrightarrow A$ such that $rx = x$ for all $x \in A$. The map r is called a nonexpansive retraction. Using the Minkowski functional (see [32], Lemma 4.2.5, p.27), r may be chosen such that $r(x) \in \partial A$ whenever $x \notin A$.*
(b) *We say that a Banach space E has the nonexpansive retract property (shortly NRP) if each of its nonempty closed convex subsets is a nonexpansive retract of E.*

In a Hilbert space, the ball retraction is nonexpansive (see [10] or [24], Thm. 6.1.4). The fixed point set of any nonexpansive map $f : A \longrightarrow A$ is a nonempty nonexpansive retract of A (see Bruk [5, 6]). Any uniformly convex Banach space is reflexive [34] and has the nonexpansive retract property [7, 8, 10]. Recall that a space E is said to be uniformly convex if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $(x, y) \in E^2$, we have

$$\|x - y\| \geq \varepsilon, \ \|x\| \leq 1, \|y\| \leq 1 \ \Rightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

For instance, Hilbert spaces and Lebesgue spaces $L^p(\Omega)$ $(1 < p < \infty)$ are uniformly convex spaces (see, e.g., [34]).

2.2. **Approximate and $\delta-$fixed points.** Basic approach to nonexpansive mappings is through approximation by contractive mappings. The following notation will be used throughout this paper. Let $(E, \| \cdot \|)$ be a Banach space, Q a nonempty subset of E, and $f : Q \longrightarrow E$ a mapping. Define the set of the approximate fixed points

$$S = S(f, Q) = \{(x_n)_{n \in \mathbb{N}} \subset Q : \ x_n = \left(1 - \frac{1}{n}\right) f(x_n), \, \forall \, n \in \mathbb{N}^*\}. \tag{2.1}$$

By the Banach fixed point theorem, it is clear that for each $n \in \mathbb{N}^*$, S is nonempty whenever f is nonexpansive and Q is a nonempty, closed, convex subset. Let $A \subset E$ be a nonempty bounded subset and α the Kuratowski measure of noncompactness. For some real parameters $\varepsilon > 0$ and $c > 0$ with $0 < c < \alpha(A) + \varepsilon$, define the sets (see [13])

$$N_\varepsilon(A) = \{(x, y) \in A^2 : \alpha(A) - \varepsilon \leq \|x - y\| \leq \alpha(A) + \varepsilon\}. \tag{2.2}$$

$$N_\varepsilon^c(A) = \{(x, y) \in A^2 : c \leq \|x - y\| \leq \alpha(A) + \varepsilon\}. \tag{2.3}$$

We will say that a map f is an $\alpha-$contraction on $N_\varepsilon(A)$ if there exists $0 < k < 1$ such that $\alpha(f(B)) \leq k\alpha(B)$ for every subset B with $B \times B \setminus \Delta_B \subset N_\varepsilon(A)$; here

$\Delta_B = \{(x,x),\, x \in B\}$ is the diagonal of B. Finally, for some positive δ and $A \subseteq Q$, following Kaewcharoen and Kirk [26] (see also [19], [32]), denote by

$$F_\delta(f, A) = \{x \in A : \|x - f(x)\| \leq \delta\}, \tag{2.4}$$

the set of the $\delta-$fixed points of f in A. Note that if f is bounded, then $S(f, Q)$ is contained in some δ-fixed point set $F_\delta(f, Q)$. For the sake of completeness, recall that Bruck [6] has proved that $F_\delta(f, Q)$ is rectifiably path-wise connected whenever $f : A \longrightarrow A$ is nonexpansive and A is a nonempty bounded convex subset of a Banach space. The following theorem tells us when this set is nonempty.

Theorem 2.1. *Let E be a Banach space, $C \subset E$ a nonempty closed convex and $f\colon C \longrightarrow C$ a nonexpansive mapping. Then for any $\delta > 0$, f has a $\delta-$fixed point in C.*

Indeed, assume that $C = \overline{B(0, R)}$. Then for any $\lambda \in (0, 1)$, the mapping λf is a contraction and then admits a fixed point $x_\lambda \in C$. We have

$$0 \leq \|f(x_\lambda) - x_\lambda\| = \|f(x_\lambda) - \lambda f(x_\lambda)\| = (1 - \lambda)\|f(x_\lambda)\| \leq (1 - \lambda)R.$$

The claim then follows by passing to the limit as $\lambda \to 1^-$. As an easy consequence, we derive

Theorem 2.2. *Let $C \subset E$ be a closed subset of a Banach space and $f\colon C \longrightarrow E$ a compact, nonexpansive mapping. Then f has a fixed point in C.*

In fact, when we only know that f is compact, continuous but not necessarily nonexpansive, then the existence of fixed points is equivalent to the existence of $\delta-$fixed points for each positive δ (see [[14], Proposition 3.1]). However, when f is nonexpansive, the assumption that f is compact may be removed if the Banach space has a special geometric structure. The following result has been proved independently by Browder [4], Göhde [21] and Kirk [27] in 1965 first in case of a Hilbert space and then in uniformly convex Banach spaces (for the proof, see e.g., either [Theorem 1.3, [14]] or [Thm. 10.A, [35]] or [17, 18]).

Theorem 2.3. *Let Q be a nonempty, closed, bounded, convex subset in a uniformly convex space E. Then each nonexpansive map $f : Q \longrightarrow Q$ has at least one fixed point.*

For the sake of completeness, recall that the assumption that the Banach space E is uniformly convex space can be relaxed if the nonempty, closed, bounded, convex subset Q has a normal structure (for the proof, we refer to the above references or to [25]). Q is said to have a normal structure if for every closed, bounded, convex subset $Q' \subset Q$ which contains at least two points, there is a point $x_0 \in Q'$ such that $\sup_{x \in Q'} \|x - x_0\| < \mathrm{diam}\,(\mathrm{Q}')$. For instance, every compact, convex subset of a Banach space has a normal structure.

2.3. **On the boundedness of $F_\delta(f, Q)$.** Let Q be a nonempty closed convex subset of a Banach space E such that $0 \in Q$ and let $f : Q \longrightarrow E$ be a mapping.

Definition 2.4. *We say that f has the property $(\mathcal{K})$ if there exists a nonempty, bounded, closed, convex subset $K \subset E$ such that $f(K \cap Q) \subset K$.*

In [26], Kaewcharoen and Kirk obtained some necessary and sufficient condition for the set $F_\delta(f,Q)$ defined in (2.4) to be bounded.

Lemma 2.4. *([26], Lemma 2.1) Suppose $f : Q \longrightarrow E$ is asymptotically contractive ([30]) i.e.*

$$\limsup_{\|x-y\|\to+\infty} \frac{\|f(x)-f(y)\|}{\|x-y\||} < 1. \tag{2.5}$$

Then for each $\delta > 0$, $F_\delta(f,Q)$ is bounded.

Lemma 2.5. *([26], Lemma 2.2) Suppose $f : Q \longrightarrow Q$ is nonexpansive and there exists $\delta > 0$ such that $F_\delta(f,Q)$ is nonempty and bounded. Then there exists $p \in Q$ such that $(f^n(p))_n$ is a bounded subset of Q.*

Lemma 2.6. *([26], Lemma 2.3) Suppose $f : Q \longrightarrow E$ is nonexpansive and $(f^n(p))_n$ is a bounded subset of Q for some $p \in Q$. Then f satisfies the property $(\mathcal{K})$. In particular if $f(Q) \subset Q$, then there is a bounded, closed, convex subset of Q which is mapped into itself by f.*

We end this section with a useful result for the sequel.

2.4. **A compactness result.** Let S be given by (2.1) and define the set

$$S_K = S \cap K, \tag{2.6}$$

where K is a closed, bounded, convex subset.

Lemma 2.7. *([13], Lemma 3.1) Let E be a Banach space, $Q \ni 0$ a convex, closed subset of E, and $f : Q \longrightarrow Q$ a nonexpansive mapping satisfying the property $(\mathcal{K})$. Assume that there exist $\delta_0, \varepsilon_0 > 0$ such that for all $c \in (0, \alpha(S_K) + \varepsilon_0)$, we have*

$$[F_{\delta_0}(f,S_K) \times F_{\delta_0}(f,S_K)] \cap N^c_{\varepsilon_0}(f,S_K) = \emptyset. \tag{2.7}$$

Then $\alpha(S_K) = 0$.

Remark 2.1. *If we look closely in the proof of this lemma in [13], we notice that the condition that f maps Q onto itself has only been used to prove that the set S_K is nonempty. Thus if $f : Q \longrightarrow E$ and $S_K \neq \emptyset$, then the conclusion of the lemma remains true.*

Remark 2.2. *Even when $F_\delta(f,Q)$ is bounded, we do not know whether or not the set S is bounded; thus we cannot define $\alpha(S)$. To this end we have used the boundedness of K, which exists by Lemma 2.6, to define $\alpha(K \cap S)$ and then the set $N^c_\varepsilon(f,S_K)$ for some $c, \varepsilon > 0$. However the set S is bounded and hence we may take $S_K = S$ whenever either one of the following condition is satisfied:*

(a) *$f(Q)$ is bounded.*
(b) *f is a contraction.*
(c) *f is asymptotically contractive.*
(d) *f verifies the asymptotic condition ([22]):*

$$\limsup_{x\in Q,\ \|x\|\longrightarrow+\infty} \frac{G(fx,x)}{\|x\|^2} < 1, \tag{2.8}$$

where $G : E \times E \longrightarrow \mathbb{R}$ is a map such that

(G1) *$G(\lambda x, y) = \lambda G(x, y)$ for all $x, y \in E$ and $\lambda > 0$.*
(G2) *$\|x\|^2 \leq G(x, x)$ for all $x \in E$.*

3. Main results

3.1. **Nonexpansive maps with the Furi-Pera condition.** In the general setting of Banach spaces satisfying the NRP, we can state our first existence result. In the sequel, the closed convex set Q is not required to be bounded.

Theorem 3.1. *Let E be a Banach space satisfying the NRP and $Q \ni 0$ a convex closed subset of E. Let $f : Q \longrightarrow E$ be a nonexpansive mapping satisfying the property $(\mathcal{K})$. Assume that (2.7) both with the Furi-Pera condition hold. Then f has a fixed point in Q.*

Proof.

Step 1. Approximate fixed points for fr_K. Let $r : E \longrightarrow Q \cap K$ be a nonexpansive retraction where K is a closed convex subset and, for each $n \in \mathbb{N}^*$, consider the nonlinear equation

$$x = (1 - 1/n)fr_K(x), \tag{3.1}$$

where $r_K = r_{|K}$ is the restriction of r on the set K and $fr_k = f \circ r_K : K \longrightarrow K$. Without loss of generality, assume that $0 \in K \cap Q$. Indeed, in case $0 \notin K$, one may take any $p \in K \cap Q$ and instead of equation (3.1) rather consider the equation $x = (1-1/n)fr_K(x)+p/n,\ n \in \mathbb{N}^*$. Now, since $f(K\cap Q) \subset K$ and $r_K : K \longrightarrow K\cap Q$ we have that $fr_K : K \longrightarrow K$. By convexity of K and the fact that $p \in K \cap Q \subset K$, we deduce that for all $x \in K$, $(1 - 1/n)fr_K(x) + p/n \in K$. Also, since f and r are nonexpansive, then for each $n \in \mathbb{N}^*$, the mapping $f_n : K \longrightarrow K$ defined by $f_n(x) = (1-1/n)fr_K(x)+p/n$ is a contraction. By the Banach fixed point theorem, for each $n \in \mathbb{N}^*$, f_n admits a unique fixed point $x_n \in K$. This implies that the equation (3.1) has a unique solution x_n for each $n \in \mathbb{N}^*$.

Step 2. Approximate fixed points for f. Our aim now is to prove that, for each $n \in \mathbb{N}^*$, the equation

$$x = (1 - 1/n)f(x) \tag{3.2}$$

has a solution. For this, it is enough to prove that the sequence $(x_n)_n$ lies in Q where, for each $n \in \mathbb{N}^*$, x_n is a solution of the equation (3.1). Arguing by contradiction, assume that $(x_n)_n \not\subset Q$. Let $x_{n_0} \notin Q$ for some $n_0 \in \mathbb{N}^*$. Since Q is closed, there exists $0 < \delta < dist(x_{n_0}, Q)$. Following the proof of [[1], Thm. 5.10], choose an integer $N \in \mathbb{N}^*$ such that $N > 1/\delta$; then, for all integer $i \geq N$, consider the open set $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\}$. It is clear that $dist(x_{n_0}, Q) > \delta$ and $1/i < \delta$ imply that $x_{n_0} \notin \overline{U}_i$. In addition, for each $i \geq N$, $U_i \cap K \neq \emptyset$ because $Q \cap K \subset U_i \cap K$ and, by definition, $Q \cap K \neq \emptyset$. Thus, the mapping $(1 - 1/n_0)fr_K : \overline{U}_i \longrightarrow K$ is well defined and it is further a contraction; in addition $(1-1/n_0)fr_K(\overline{U}_i)$ is bounded since $r_K(\overline{U}_i\cap K) \subset Q\cap K$ and, by the property $(\mathcal{K})$, $f(Q\cap K) \subset K$ with K bounded. Since $x_{n_0} \notin \overline{U}_i$, a nonlinear alternative (Theorem 3.2 of [1]) guarantees that there exists $(y_i, \mu_i) \in \partial U_i \times (0, 1)$ such that $y_i = \mu_i(1 - 1/n_0)fr_K(y_i)$. Note that since $x_{n_0} \notin \overline{U}_i$, then the map $(1 - 1/n_0)fr$ has no fixed point in $\overline{U}_i$. As a consequence

$$\mu_i(1 - 1/n_0)fr_K(y_i) \notin Q,\ \forall\, i \geq N. \tag{3.3}$$

Now define the set

$$D_{n_0} = \{x \in E : \exists\, \mu \in [0,1],\ x = \mu(1 - 1/n_0) f r_K(x)\}.$$

D_{n_0} is nonempty because it contains $0, x_{n_0}$ and y_i, for all $i \geq N$. Moreover, the set D_{n_0} is compact. Indeed

$$D_{n_0} \subseteq \overline{co}\,((1 - 1/n_0) f r_K(D_{n_0}) \cup \{0\})$$

yields that

$$\alpha(D_{n_0}) \leq \alpha\left(\overline{co}((1 - 1/n_0) f r_K(D_{n_0}) \cup \{0\})\right),$$

where α is the Kuratowski MNC. However, since f and r are nonexpansive, then we have the estimates

$$\begin{array}{rcl} \alpha(D_{n_0}) & \leq & \alpha\big(\overline{co}\big((1 - 1/n_0) f r_K(D_{n_0})\big)\\ & \leq & (1 - 1/n_0)\alpha(r_K(D_{n_0}))\\ & \leq & (1 - 1/n_0)\alpha(D_{n_0}). \end{array}$$

Then $\alpha(D_{n_0}) = 0$ yielding that D_{n_0} is compact since it is closed. Now, for each $i \geq N$ and $0 \leq \mu_i \leq 1$, we have that $d(y_i, Q) = \frac{1}{i}$ since $y_i \in \partial U_i \cap D_{n_0}$. Then, up to a subsequence, $\mu_i \longrightarrow \mu^* \in [0,1]$ and, by the compactness of D_{n_0}, $y_i \longrightarrow y^* \in Q$, as $i \to +\infty$. Moreover $y_i = \mu_i(1 - 1/n_0) f r_K(y_i)$ tends to $\mu^*(1 - 1/n_0) f r_K(y^*)$ by continuity. Hence $y^* = \mu^*(1 - 1/n_0) f r_K(y^*)$. In addition $x_{n_0} \notin Q$ yields that $\mu^* \neq 1$, otherwise we have by uniqueness $y_* = x_{n_0}$, which is a contradiction. Therefore $0 \leq \mu^* < 1$. Finally $r_K(y_i) \in \partial Q$ follows from $y_i \notin Q$ and the definition of the retraction r. In addition $y^* = r_K(y^*)$ and $\mu'_i = (1 - 1/n_0)\mu_j$, $\mu' = (1 - 1/n_0)\mu^*$. Since f satisfies the Furi-Pera condition, we infer that $\mu_i(1 - 1/n_0) f r_K(y_i) \in Q$ for i sufficiently large. This contradicts (3.3) and the fact that $y_i \notin Q$, for $i \geq N$.

Thus, for each $n \in \mathbb{N}^*$, $x_n \in Q \cap K$. Hence $r_K(x_n) = x_n$ and $x_n = (1 - 1/n) f r_K(x_n) = (1 - 1/n) f(x_n)$. To sum up, we have proved that the equation $x_n = (1 - 1/n) f(x_n)$ has a solution for each $n \in \mathbb{N}^*$.

Step 3. Passing to the limit. It remains to prove that the sequence $(x_n)_n$, where x_n is a solution of equation (3.2), is convergent. Let $S_K = \{x_n \in Q \cap K : x_n = (1 - 1/n) f(x_n),\ \forall\, n \in \mathbb{N}\} = S \cap K$. Owing to Steps 1, 2, the set S_K is a nonempty bounded set. Lemma 2.7 (see also Remark 2.1) both with (2.7) imply that the set $\overline{S}_K$ is compact, hence sequentially compact. Therefore we can extract a sequence converging to x. Finally, by continuity of f, we obtain that x is a fixed point of f. □

Remark 3.1. *With the condition (2.7), Theorem 3.1 extends a result obtained in [[1], Thm. 5.11] for E a Hilbert space and $Q \subset E$ a bounded subset. Moreover let us mention that (2.7) allows us to obtain [[1], Thm. 5.10] for nonexpansive maps instead of $k-$set contractions with $0 \leq k < 1$ (see Theorem 3.3 below).*

3.2. **The case of 1-set contractions.** We have

Theorem 3.2. *Let E be a Banach space satisfying the NRP and $Q \ni 0$ a convex closed subset. Let $f : Q \longrightarrow E$ be a continuous 1-set contraction map satisfying the property $(\mathcal{K})$ and the $(\mathcal{FP})$ condition. Assume that there exists $\varepsilon_0 > 0$ such that f is an $\alpha-$contraction on $N_{\varepsilon_0}(S_K)$. Then f has a fixed point in Q.*

Proof. Let $r : E \longrightarrow Q \cap K$ be a nonexpansive retraction and, for each $n \in \mathbb{N}^*$, consider the equation

$$x = (1 - 1/n) f r_K(x), \tag{3.4}$$

where $r_K = r_{|K}$ is the restriction of r on the set K. Since r is nonexpansive and f is a 1-set contraction satisfying the property $(\mathcal{K})$, then the map $(1 - 1/n) f r_K : K \longrightarrow K$ is a $(1 - 1/n)-$set contraction. By Darbo's fixed point theorem [9], for every $n \in \mathbb{N}^*$, (3.4) has at least one solution x_n. Then, for every fixed $n \in \mathbb{N}^*$, consider the nonempty set $S_n = \{x \in E : x = (1 - 1/n) f r_K(x)\}$. Clearly S_n is closed and even compact; indeed

$$\alpha(S_n) \leq (1 - 1/n_0)\alpha(r_K(S_n)) \leq (1 - 1/n_0)\alpha(S_n).$$

Hence we may choose $\delta < dist(Q, S_n)$, $N > 1/\delta$ and follow the proof of Theorem 3.1 to prove, by contradiction, that $Q \cap S_n \neq \emptyset$. The only difference is that here, we rather apply [[1], Thm. 5.7] instead of [[1], Thm. 3.2]. By picking one $x_n \in Q \cap S_n$, for each $n \in \mathbb{N}^*$, we then construct in this way a sequence $(x_n)_{n \in \mathbb{N}} \subset Q$ such that

$$x_n = (1 - 1/n) f(x_n), \ \forall\, n \in \mathbb{N}^*,$$

yielding that $S \neq \emptyset$, where $S = (x_n)_n$. According to the proof of [[13], Thm. 3.5], we obtain that $\alpha(S_K) = 0$, where $S_K = S \cap K$. Consequently $\overline{S}_K$ is compact; then we may extract from $(x_n)_n$ a sequence converging to some limit x. By continuity of f, we deduce that f has at least one fixed point, as claimed. □

Remark 3.2. *In Theorem 3.2, we have considered* 1*-set contractions which are* α*-contractions on some subset of Banach spaces with the NRP; this theorem could be compared with the following result.*

Theorem 3.3. *(*[1]*, Thm. 5.10, p.59) Let X be a Hilbert space, $Q \ni 0$ a closed, convex subset of X, and let $f : Q \longrightarrow E$ be a continuous $k-$contraction with $0 \leq k < 1$. Assume that $f(Q)$ is a bounded set in E and that the Furi-Pera condition holds. Then f has at least one fixed point in Q.*

3.3. **The case $I - f$ is demi-closed.** First we start with

Definition 3.1. *Let A be a subset of a Banach space E. A mapping $g : A \longrightarrow E$ is said to be demi-closed if for any $y \in E$ and any sequence $(x_n)_{n \in \mathbb{N}} \subset A$, the condition (x_n) converges weakly to x and $\|g(x_n) - y\| \to 0$ imply that $x \in A$ and $g(x) = y$.*

When Q is a closed, convex, and bounded subset of a Banach space, the following special existence result has been proved by O'Regan in [[29], Thm. 2.9]:

Theorem 3.4. *Let $(E, \|\cdot\|)$ be a Banach space and $Q \ni 0$ a convex closed bounded subset. Let $f : Q \longrightarrow E$ be a* 1*-set contraction with $(I - f)(Q)$ closed. If the Furi-Pera condition holds, then f has a fixed point.*

Now, we state and prove our final existence result in which again Q is an arbitrary closed convex subset and $(I - f)(Q)$ closed is replaced by $I - f$ demi-closed.

Theorem 3.5. *Let $(E, \|\cdot\|)$ be a reflexive Banach space satisfying the NRP and $Q \ni 0$ a convex closed subset of E. Let $f : Q \longrightarrow E$ be a* 1*-set contraction with $I - f$ demi-closed. If the Furi-Pera condition and the property $(\mathcal{K})$ are satisfied, then f has a fixed point.*

Proof. Let $r : E \longrightarrow Q \cap K$ be a nonexpansive retraction and, for each $n \in \mathbb{N}^*$, consider the equation

$$x = (1 - 1/n) f r_K(x), \tag{3.5}$$

where $r_K = r_{|K}$ is the restriction of r on the set K. Without loss of generality, assume that $0 \in K \cap Q$. Indeed, in case $0 \notin K$, one may take any $p \in K \cap Q$ and, instead of equation (3.5), rather consider the equation $x = (1 - 1/n) f r_K(x) + p/n,\ n \in \mathbb{N}^*$ instead. Now, since $f(K \cap Q) \subset K$ and $r_K : K \longrightarrow K \cap Q$, we have that $f r_K : K \longrightarrow K$. By convexity of K and the fact that $p \in K \cap Q \subset K$, we deduce that for every $x \in K$, we have that $(1 - 1/n) f r_K(x) + p/n \in K$. Also, since f is a 1-set contraction and r is nonexpansive, then for each $n \in \mathbb{N}^*$, the mapping $f_n : K \longrightarrow K$ defined by $f_n(x) = (1-1/n) f r_K(x) + p/n$ is an $\alpha-$contraction with constant $1-1/n$. By Darbo's fixed point theorem [9], f_n admits at least one fixed point $x_n \in K$. This implies that the equation (3.5) has a solution for each $n \in \mathbb{N}^*$. Arguing as in the proof of Theorem 3.2, we can prove that for each $n \in \mathbb{N}^*$, the equation

$$x = (1 - 1/n) f(x) \tag{3.6}$$

has at least one solution. Finally, since $(x_n)_n \subset K$, K is bounded, and E is reflexive, then up to a subsequence, $x_n \rightharpoonup x$, as $n \to +\infty$. Since $I - f$ is demi-closed, we conclude that the equation $x = f(x)$ has a solution, proving the theorem. $\square$

Since nonexpansive maps are 1-set contractions, from Theorem 3.5, we infer the following one:

Corollary 3.6. *Let $(E, \|\cdot\|)$ be a reflexive Banach space satisfying the NRP and $Q \ni 0$ a convex closed subset. Let $f : Q \longrightarrow E$ be a nonexpansive map with $I - f$ demi-closed and such that $(\mathcal{FP})$ and the property $(\mathcal{K})$ hold. Then f has a fixed point.*

Remark 3.3. *Corollary 3.6 may be compared with the following recent result by Isac and Németh. Indeed we have replaced the stronger condition $F(Q) \subset Q$ by $(\mathcal{FP})$ and the condition (2.8) is dropped and substituted by the weaker property $(\mathcal{K})$ (see Lemmas 2.4, 2.5, 2.6).*

Theorem 3.7. *([23], Thm. 3.1) Let $(E, \|.\|)$ be a reflexive Banach space and $Q \subseteq E$ a nonempty unbounded closed convex set with $0 \in Q$. Let $f : Q \longrightarrow E$ be a nonexpansive map such that $f(Q) \subset Q$ and $I - f$ is demi-closed. If f satisfies the condition (2.8), then f has a fixed point.*

Remark 3.4. *Note that if the Banach space E were uniformly convex, then $I - f$ demi-closed follows from $f : Q \longrightarrow E$ nonexpansive, whenever $Q \subset E$ is any closed, bounded, convex subset (F. Browder's demi-closedness principle [4]) (see also [Prop. 10.9, [35]]). In this case, we need not to assume that $I - f$ is demi-closed.*

4. Applications

We begin with

Definition 4.1. *Let E be a Banach space and $f : E \longrightarrow E$ a mapping. f is called $\mathcal{D}-$Lipschitzian with $\mathcal{D}-$function ϕ_f if there exists a continuous nondecreasing function $\phi_f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi_f(0) = 0$ and*

$$\|f(x) - f(y)\| \leq \phi_f(\|x - y\|),\ \forall\, (x, y) \in E^2.$$

In particular, if $\phi_f(r) = kr$ for some constant $0 < k < 1$, then f is a contraction.

4.1. **A fixed point theorem in Banach algebras.**

Theorem 4.1. *Let $Q \ni 0$ be a convex, closed, bounded subset of a Banach algebra E and let $A, B : Q \longrightarrow E$ be two mappings satisfying*

(a) *A is $\mathcal{D}-$Lipschitzian with $\mathcal{D}-$function ϕ_A.*
(b) *B is completely continuous.*
(c) *$AB(Q)$ is bounded and AB satisfies the Furi-Pera condition.*

Then $f = AB$ satisfies the property $(\mathcal{K})$ and the equation $x = AxBx$ has a solution whenever $M\phi_A(r) \leq r,\ \forall\, r > 0$ and $M\phi_A(r) < r,\ \forall\, r \in (0, \alpha(S_K) + \omega]$, for some $\omega > 0$. Here $M := \|B(Q)\|$ and S_K is as defined in (2.6).

Proof. According to the proof of [[13], Lemma 4.1], the mapping $f = AB$ is a continuous 1-set contraction. Similarly, we can see that f is an $\alpha-$contraction on $N_\omega(S_K)$, since $M\phi_A(r) < r,\ \forall\, r : 0 < r \leq \alpha(S_K) + \omega$. Applying Theorem 3.2, we conclude that the map AB has a fixed point. $\square$

Theorem 4.1 improves [[12], Theorem 3.1] and a similar result obtained in [11]. As a result, the following example of application is a slight improvement of [[12], Example 2].

4.2. **An integral equation.** **(a)** Consider the Banach space

$$X = C_0(\mathbb{R}, \mathbb{R}) = \{x \in C(\mathbb{R}, \mathbb{R}),\ \lim_{|t| \to +\infty} x(t) = 0\}$$

endowed with the sup-norm

$$\|x\|_X = \sup_{t \in \mathbb{R}} \{|x(t)|\}.$$

Let a continuous function $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy

$$|f(s, x) - f(s, y)| \leq \theta(|x - y|),\ s \in [0, 1],$$

where

$$\theta(r) \leq r, \forall\, r > 0 \ \text{ and } \ \theta(r) < r, \forall\, r \in (0, \alpha(S_K) + \omega), \ \text{ for some } \omega > 0.$$

In [12], this latter condition on θ was assumed for every positive real number r.
(b) Let $Ax(t) = f(t, x(t))$ be the Nemytskii operator and B the mapping defined by

$$Bx(t) = \int_{-\infty}^{+\infty} G(t, s) h\big(s, x(s)\big)\, ds,$$

where the nonlinear function $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and verifies the growth condition:

$$|h(t, x)| \leq q(t)\Psi(|x|), \quad \forall\, t, x \in \mathbb{R} \tag{4.1}$$

where $q \in C_0(\mathbb{R}, \mathbb{R}^+) \cap L^1(\mathbb{R}, \mathbb{R}^+)$ and $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous nondecreasing function. We assume that the kernel $G : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and satisfies

$$\exists\, \varrho > 0,\, \sigma > 0,\ |G(t,s)| \leq \varrho \exp^{-\sigma|t-s|}, \quad \forall s, t \in \mathbb{R}, \tag{4.2}$$

with $\frac{2\varrho}{\sigma}\|q\|_1\Psi(R) \leq 1$. We can show that B is completely continuous (see the proof of [[12], Thm. 2.1] for the details). Finally, consider the bounded, closed and convex subset of X :

$$\Omega = \{x \in X :\ \|x\| \leq R\},$$

where the positive constant R is to be selected later on.

(c) Assume that for any compact subset $K \subset \mathbb{R}$, there exists a positive constant $M_K > 0$ such that for any $x \in X$ and $\lambda \in [0, 1)$

$$x = \lambda AxBx \implies (|x(t)| \leq M_K,\ \forall\, t \in K). \tag{4.3}$$

(d) The mapping F satisfies the Furi-Pera condition. Indeed, let $(x_j, \lambda_j) \in \partial\Omega \times [0, 1]$ be a sequence such that, as $j \to +\infty$, $\lambda_j \longrightarrow \lambda$ and $x_j \longrightarrow x$ with $\lambda F(x) = x$ and $0 \leq \lambda < 1$. We show that $\lambda_j F(x_j) \in \Omega$ where $F(x_j) = Ax_jBx_j$. Since Ψ is nondecreasing, we get

$$|Bx(t)| \leq \Psi(R) \int_{-\infty}^{+\infty} G(t,s)q(s)ds := \gamma(t).$$

Moreover, for each j, we have

$$\begin{aligned} \|\lambda_i F(x_j)\| &\leq \|Ax_j\| \cdot \|Bx_j\| \\ &\leq (\theta(\|x_j\|) + \|f(\cdot, 0)\|)\,\gamma(t), \\ &\leq (R + \|f(\cdot, 0)\|)\,\gamma(t). \end{aligned}$$

Since $\lim\limits_{t\to\pm\infty} \gamma(t) = 0$, there exist some t_1, t_2 $(t_1 < t_2)$ and a sufficiently small positive constant $M_1 > 0$ such that

$$\|\lambda_j F(x_j)(t)\| \leq M_1,\ \forall\, t \in (-\infty, t_1) \cup (t_2, +\infty). \tag{4.4}$$

In addition, for $t \in K = [t_1, t_2]$ and $(x_j)_{j\in\mathbb{N}} \subset \partial S$, the sequence $(x_j)_{j\in\mathbb{N}}$ converges to $x = \lambda F(x)$ uniformly, as $j \to +\infty$. Then for j large enough and $t \in K$, we have, from conditions (4.3) and (4.4), the estimate

$$\|\lambda_j F(x_j)(t) \leq \lambda|F(x)(t)| + \frac{1}{2} \leq M_K + \frac{1}{2}\cdot \tag{4.5}$$

Combining (4.4) and (4.5) and taking $R = \max(M_1, M_K + \frac{1}{2})$, we arrive at the estimate

$$\|\lambda_j F(x_j)(t)\| \leq R,\ \forall\, t \in \mathbb{R},\ \forall\, j \in \mathbb{N},$$

showing that the Furi-Pera condition is fulfilled.

(e) We have that $M\phi(r) \leq \|q\|_1\Psi(R)\theta(r) \leq \theta(r),\ \forall\, r > 0$ because $\frac{2\varrho}{\sigma}\|q\|_1\Psi(R) \leq 1$. Therefore all conditions of Theorem 4.1 are met, which implies that the nonlinear equation

$$(f(t, x(t))) \left(\int_{-\infty}^{+\infty} G(t,s)h\big(s, x(s)\big)\, ds \right) = x(t), \quad t \in \mathbb{R}$$

admits at least one solution $x \in C_0(\mathbb{R}, \mathbb{R})$.

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