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NONLINEAR QUASI-CONTRACTIONS OF ĆIRIĆ TYPE

CRISTINA DI BARI* AND PASQUALE VETRO*

*Università degli Studi di Palermo
Dipartimento di Matematica e Informatica
Via Archirafi, 34 - 90123 Palermo (Italy)
E-mail: dibari@math.unipa.it, vetro@math.unipa.it

Abstract. In this paper we obtain points of coincidence and common fixed points for two self mappings satisfying a nonlinear contractive condition of Ćirić type. As application, using the scalarization method of Du, we deduce a result of common fixed point in cone metric spaces.
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1. Introduction

Fixed point theory is an important and actual topic of nonlinear analysis. Then, for the most important contributions on the metric and non-metric setting, the reader can refer Goebel and Kirk [12], Kirk and Sims [17] (and the references therein), Kirk and Kang [16] and Rus et al. [20]. Let B be an ordered linear space with a cone K and a class of convergent sequences in B. For more details on convergence structures one can consider Zabrejko [23] and De Pascale, Marino and Pietramala [7].

Here, we start by the following definition.

Definition 1.1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \to B$ satisfies:

(i) $\theta \leqslant d(x, y)$, for all $x, y \in X$, and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a *cone metric* (*also a K-metric*) on X, and (X, d) is called a *cone metric space* (*also a K-metric space*).

A sequence $\{x_n\} \subset X$, is called convergent, if there exists an element $x \in X$ such that the sequence $\{d(x_n, x)\}$ is convergent to θ in the space B. Zabrejko [23] presented a very interesting revised version of the fixed point theory in K-metric and K-normed linear spaces and gave three fixed point theorems that cover numerous applications, e.g. in numerical methods and theory of integral equations (see also [18]). Clearly one can formulate specializations of these results for special types of K-metric. For more considerations on K-metric spaces, for example in terms of the weakly Picard operators, the reader can see [21]. In [21] the authors have also considered K-metrics induced by a functional.

Huang and Zhang [13] recently have considered cone metric space, where the set of real numbers is replaced by an ordered Banach space E with a class of convergent sequences. They have established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, other authors [1, 3, 4, 8, 9, 10, 15, 19, 22] have generalized the results of Huang and Zhang [13] and have studied the existence of common fixed points of a pair of self mappings satisfying a contractive type condition in the framework of cone metric spaces.

It is natural to ask the following question: given a cone metric space (X, d), is it possible to define a metric ρ on X such that (X, d) and (X, ρ) have the same class of convergent sequences?

A positive response was given recently by Du [10], that has investigated the equivalence of vectorial versions of fixed point theorems in cone metric spaces and scalar versions of fixed point theorems in metric spaces. Du showed that the Banach contraction principles in metric spaces and in cone metric spaces are equivalent, associating to every cone metric a metric with the same class of convergent sequences. Successively, also Feng and Mao [11] have investigated the equivalence of cone metric spaces and metric spaces.

It's well known that the contraction mapping principle, formulated and proved in the Ph. D. dissertation of Banach in 1920, which was published in 1922, is one of the most important theorems in classical functional analysis.

Ćirić [5] first introduced the notion of quasi-contractions and proved fixed point theorem for this class of mappings. Successively Ćirić's result was extended to nonlinear quasi-contractions by Ivanov [14] and Arandelović et al. [2].

In this paper we obtain points of coincidence and common fixed points for two self mappings satisfying a nonlinear contractive condition of Ćirić type. As application, using the scalarization method of Du, we deduce a result of common fixed point in cone metric spaces.

2. Preliminaries

By Ψ we denote the set of real functions $\psi : [0, +\infty[\to [0, +\infty[$ which have the following properties:

(i) ψ is nondecreasing;
(ii) $\psi(0) = 0$;
(iii) $\lim_{x \to +\infty}(x - \psi(x)) = +\infty$;
(iv) $\lim_{t \to r^+} \psi(t) < r$ for all $r > 0$.

Remark 2.1 *From (iv) and $\psi(r) \leq \lim_{t \to r^+} \psi(t) < r$, we deduce that $\psi(r) < r$ for all $r > 0$. Furthermore (i) and (iv) imply that $\lim_{n \to +\infty} \psi^n(r) = 0$ for all $r > 0$.*

In this section we prove some results of common fixed point for ψ-quasi-contractions. Let (X, d) be a metric space and $f, g : X \to X$ be mappings, f and g are a ψ-quasi-contraction if there exists $\psi : [0, +\infty[\to [0, +\infty[$ such that

$$d(fx, fy) \leq \max\{\psi(d(gx, gy)), \psi(d(fx, gx)), \psi(d(fy, gy)), \psi(d(gx, fy)), \psi(d(fx, gy))\},$$

for all $x, y \in X$.

Suppose $f(X) \subset g(X)$. For every $x_0 \in X$ we consider the sequence $\{x_n\} \subset X$ defined by $gx_n = fx_{n-1}$ for all $n \in \mathbb{N}$, we say that $\{fx_n\}$ is a f-g-sequence of initial point x_0. Define $\mathcal{O}_n(x_0) = \{gx_0, fx_0, fx_1, \ldots, fx_n\}$, $\mathcal{O}(x_0) = \{gx_0, fx_0, fx_1, \ldots, fx_n, \ldots\}$.

The self-mappings f, g on X are said to be weakly compatible if they commute at their coincidence point (i.e. $fgx = gfx$ whenever $fx = gx$).

Theorem 2.2 *Let (X, d) be a metric space and let $f, g : X \to X$ be such that $f(X) \subset g(X)$. Suppose that f and g are a ψ-quasi-contraction with $\psi \in \Psi$. If $f(X)$ or $g(X)$ is a complete subspace of X and f and g are weakly compatible, then the mappings f and g have a unique common fixed point in X. Moreover for any $x_0 \in X$, the f-g-sequence $\{fx_n\}$ of initial point x_0 converges to the fixed point.*

Proof. Let $x_0 \in X$ be fixed. As $f(X) \subset g(X)$, one can choose a f-g-sequence $\{fx_n\}$ of initial point x_0. Now we prove that $\mathcal{O}(x_0)$ is bounded. To this end, we show that for each $n \geq 1$ there exists $0 \leq k \leq n$, such that $\delta(\mathcal{O}_n(x_0)) = d(gx_0, fx_k)$. Since f and g are a ψ-quasi-contraction with $\psi \in \Psi$, for every $0 \leq i, j \leq n$, we have

$$d(fx_i, fx_j) \leq \psi(\delta(\mathcal{O}_n(x_0))) < \delta(\mathcal{O}_n(x_0)).$$

This implies that $\delta(\mathcal{O}_n(x_0)) = d(gx_0, fx_k)$, for some $0 \leq k \leq n$.

By property (iii) of the function ψ there is a $c > h = d(gx_0, fx_0)$, such that $x - \psi(x) > h$, for all $x > c$. So we obtain that for each $n \geq 1$:

$$\delta(\mathcal{O}_n(x_0)) = d(gx_0, fx_k) \leq h + d(fx_0, fx_k) \leq h + \psi(\delta(\mathcal{O}_n(x_0))).$$

Therefore

$$\delta(\mathcal{O}_n(x_0)) - \psi(\delta(\mathcal{O}_n(x_0))) \leq h$$

for $n = 1, 2, \ldots$, and, consequently, the set $\mathcal{O}(x_0)$ is bounded.

Define $\mathcal{O}(fx_k) = \{fx_k, \ldots, fx_n, \ldots\}$ for every $k \geq 1$. Obviously $\delta(\mathcal{O}(fx_k)) \leq \psi(\delta(\mathcal{O}(fx_{k-1})))$. Consequently

$$\delta(\mathcal{O}(fx_k)) \leq \psi^k(\delta(\mathcal{O}(fx_0))),$$

and, hence, the f-g-sequence $\{fx_n\}$ of initial point x_0 is a Cauchy sequence.

Suppose that $f(X)$ is a complete subspace of X, then there exists $y \in f(X) \subset g(X)$ such that $fx_n \to y$ and also $gx_n \to y$. (This holds also if $g(X)$ is complete with $y \in g(X)$). Let $z \in X$ be such that $gz = y$. We show that $fz = gz$. If $d = d(fz, gz) > 0$, then for n sufficiently large, we have

$$d(gx_n, gz), d(gx_n, fx_n), d(gz, fx_n) \leq d.$$

From

$$\begin{aligned} d(fx_n, fz) \leq \max\{&\psi(d(gx_n, gz)), \psi(d(gx_n, fx_n)), \\ &\psi(d(fz, gz)), \psi(d(gx_n, fz)), \psi(d(gz, fx_n))\}, \end{aligned}$$

we deduce that

$$\begin{aligned} d(fx_n, fz) &\leq \max\{\psi(d(gz, fz)), \psi(d(fx_{n-1}, fz))\} \\ &= \psi(\max\{d(gz, fz), d(fx_{n-1}, fz)\}). \end{aligned}$$

As $n \to +\infty$, $\max\{d(gz, fz), d(fx_{n-1}, fz)\} \to d^+$, and so

$$\begin{aligned} d(gz, fz) &= \lim_{n\to+\infty} d(fx_n, fz) \\ &\leq \lim_{n\to+\infty} \psi(\max\{d(gz, fz), d(fx_{n-1}, fz)\}) \\ &< d(gz, fz), \end{aligned}$$

which is a contradiction. Therefore $fz = gz$.

Now, we show that y is a common fixed point for f and g. Since f and g are weakly compatible we deduce that

$$fy = fgz = gfz = gy.$$

If $fy \neq y$ from

$$\begin{aligned} d(fy, fz) &\leq \max\{\psi(d(gy, gz)), \psi(d(gy, fy)), \psi(d(gz, fz)), \psi(d(gy, fz)), \psi(d(fy, gz))\} \\ &= \psi(d(fy, fz)) < d(fy, fz), \end{aligned}$$

which is a contradiction, we deduce that $fy = gy = y$. The uniqueness of the common fixed point is immediate from the definition of ψ-quasi-contraction. □

Now, we denote with Φ the set of real functions $\varphi : [0, +\infty[\to [0, +\infty[$ which have the following properties:

(i) $\varphi(0) = 0$;
(ii) $\varphi(r) < r$ for all $r > 0$;
(iii) $\limsup_{t\to r} \varphi(t) < r$ for all $r > 0$;
(iv) $\lim_{x\to+\infty}(x - \varphi(x)) = +\infty$.

Remark 2.3 (Lemma 1, [2]) *If $\varphi \in \Phi$, then the function $\psi : [0, +\infty[\to [0, +\infty[$ defined by $\psi(x) = \sup_{t\in[0,x]} \varphi(t) \in \Psi$.*

From Theorem 2.2 we obtain the following corollary.

Corollary 2.4 *Let (X, d) be a metric space and let $f, g : X \to X$ be such that $f(X) \subset g(X)$. Suppose that f and g are a ψ-quasi-contraction with $\psi \in \Phi$. If $f(X)$ or $g(X)$ is a complete subspace of X and f and g are weakly compatible, then the mappings f and g have a unique common fixed point in X. Moreover for any $x_0 \in X$, the f-g-sequence $\{fx_n\}$ of initial point x_0 converges to the fixed point.*

Remark 2.5 *If in Theorem 2.2 and Corollary 2.4 we choose $g = Id_X$, the identity mapping on X, we obtain Theorems 1-2 of Aranđelović et al.* [2].

3. Nonlinear quasi-contractions in cone metric space

We recall the definition of cone metric space and the notion of convergence introduced by Huang and Zhang [13]. Let E be a real Banach space and P be a subset of E. The subset P is called an *order cone* if it has the following properties:

(i) P is non-empty, closed and $P \neq \{\theta\}$;
(ii) $0 \leqslant a, b \in \mathbb{R}$ and $x, y \in P \Rightarrow ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leqslant$ on E with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leqslant y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in \mathrm{Int}P$, where $\mathrm{Int}P$ denotes the interior of P.

In the following we always suppose that E is a real Banach space and P is an order cone in E with $\mathrm{Int}P \neq \emptyset$ and $\leqslant$ is the partial ordering with respect to P.

Definition 3.1 Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(i) $\theta \leqslant d(x,y)$, for all $x, y \in X$, and $d(x,y) = \theta$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \leqslant d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X,d) is called a *cone metric space*.

Huang and Zhang have defined convergence in terms of interior points of P. Let $\{x_n\}$ be a sequence in X, and $x \in X$. If for every $c \in E$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be *convergent*, $\{x_n\}$ converges to x and x is the *limit* of $\{x_n\}$. We denote this by $\lim_{n\to+\infty} x_n = x$, or $x_n \to^c x$, as $n \to +\infty$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is called a *complete cone metric space.*

Let (X,d) be a cone metric space. In [10] Du have defined on X a metric ρ with the property that the class of convergent sequence in (X,d) is the same in (X,ρ). He considered the nonlinear scalarization $\xi_e : E \to \mathbb{R}$, where $e \in \mathrm{Int}P$ is fixed, defined as follows:

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}, \qquad \text{for all } y \in E.$$

In the following lemma are resumed the properties of the function ξ_e.

Lemma 3.2 (Lemma 1.1, [10]) *For each $r \in \mathbb{R}$ and $y \in E$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \Leftrightarrow y \in re - P$;
(ii) $\xi_e(y) > r \Leftrightarrow y \notin re - P$;
(iii) $\xi_e(y) \geq r \Leftrightarrow y \notin re - IntP$;
(iv) $\xi_e(y) < r \Leftrightarrow y \in re - IntP$;
(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on E;*
(vi) *if $y_2 \leq y_1$, then $\xi_e(y_2) \leq \xi_e(y_1)$;*
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$.

Let (X,d) be a cone metric space and let $\rho : X \times X \to \mathbb{R}$ defined by $\rho = \xi_e \circ d$. The following results are consequences of Lemma 3.

Theorem 3.3 (Theorem 2.1, [10]) *Let (X,d) be a cone metric space. Then $\rho = \xi_e \circ d$ is a metric on X.*

From the proof of Theorem 2.2 of [10], we deduce the following theorem.

Theorem 3.4 *Let (X,d) be a cone metric space, $e \in IntP$, $\rho = \xi_e \circ d$, $\{x_n\} \subset X$ and $x \in X$. Then*

(i) $x_n \to^c x \Leftrightarrow \rho(x_n, x) \to 0$;
(ii) x_n *is a Cauchy sequence in* (X, d) *if and only if* x_n *is a Cauchy sequence in* (X, ρ)*;*
(iii) (X, d) *is a complete cone metric space if and only if* (X, ρ) *is a complete metric space.*

Now we denote with Φ_P the set of all functions $\varphi : P \to P$ which have the following properties:

(i) $\varphi(\theta) = \theta$;
(ii) $\varphi(t) \ll t$ for all $t \in \mathrm{Int}P$;
(iii) $\lim_{t\to+\infty}[\xi_e(te) - \xi_e(\varphi(te))] = +\infty$ for some $e \in \mathrm{Int}P$;
(iv) if $x_n \to x$, $x_n, x \in \mathrm{Int}P$, then there exists $\lambda(x) \in]0, 1[$ and $n_0 \in \mathbb{N}$ such that $\varphi(x_n) \leq \lambda(x)x$ for all $n \geq n_0$.

Let (X, d) be a cone metric space and $f, g : X \to X$ be mappings, f and g are a φ-quasi-contraction if there exists $\varphi : P \to P$ such that for all $x, y \in X$

$$d(fx, fy) \leq u,$$

where

$$u \in \{\varphi(d(gx, gy)), \varphi(d(fx, gx)), \varphi(d(fy, gy)), \varphi(d(gx, fy)), \varphi(d(fx, gy))\}. \tag{3.1}$$

Theorem 3.5 *Let (X, d) be a cone metric space and* $f, g : X \to X$ *be such that* $f(X) \subset g(X)$. *Suppose that* f *and* g *are a* φ*-quasi-contraction with* $\varphi \in \Phi_P$. *If* $f(X)$ *or* $g(X)$ *is a complete subspace of* X *and* f *and* g *are weakly compatible, then the mappings* f *and* g *have a unique common fixed point in* X. *Moreover for any* $x_0 \in X$, *the* f*-*g*-sequence* $\{fx_n\}$ *of initial point* x_0 *converges to the fixed point.*

Proof. We choose $e \in \mathrm{Int}P$ such that $\lim_{t\to+\infty}[\xi_e(te) - \xi_e(\varphi(te))] = +\infty$. Define $\psi : [0, +\infty[\to [0, +\infty[$ as $\psi(t) = \xi_e(\varphi(te))$. The function ψ has the following properties:

(i) $\psi(0) = 0$;
(ii) $\psi(t) = \xi_e(\varphi(te)) < t$ by (iv) of Lemma 3.2;
(iii) $\lim_{t\to+\infty}[t - \psi(t)] = \lim_{t\to+\infty}[\xi_e(te) - \xi_e(\varphi(te))] = +\infty$;
(iv) If $t_n \to r$, $r, t_n > 0$, then there exists $\lambda(r) \in]0, 1[$ and $n_0 \in \mathbb{N}$ such that $\varphi(t_n e) \leq \lambda(r)re$ for all $n \geq n_0$ and hence $\psi(t_n) = \xi_e(\varphi(t_n e)) \leq \xi_e(\lambda(r)re) = \lambda(r)r$. Consequently, $\limsup_{t\to r} \psi(t) < r$.

Then $\psi \in \Phi$. Now, in the metric space (X, ρ) the mappings f and g are a ψ-quasi-contraction. In fact, from

$$d(fx, fy) \leq u,$$

where u satisfies (3.1), we deduce that

$$\rho(fx, fy) = \xi_e(d(fx, fy)) \leq \xi_e(u),$$

where u satisfies (3.1). This assures that f and g are a ψ-quasi-contraction. We note that, by Theorem 3.4, $f(X)$ or $g(X)$ is a complete subspace of (X, ρ). Finally, from Corollary 2.4 it follows that f and g have a unique common fixed point.

References

[1] M. Abbas, G. Jungck, *Common fixed point results for non commuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341**(2008), 416-420.
[2] I.D. Arandelović, M. Rajović, V. Kilibarda, *On nonlinear quasi-contractions*, Fixed Point Theory, **9**(2008), 387-394.
[3] M. Arshad, A. Azam, P. Vetro, *Some common fixed point results in cone metric spaces*, Fixed Point Theory Appl., **2009**(2009), 11 pp., Article ID 493965.
[4] A. Azam, M. Arshad, I. Beg, *Common fixed points of two maps in cone metric spaces*, Rend. Circ. Mat. Palermo, **57**(2008), 433-441.
[5] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45**(1974), 267-273.
[6] Lj. B. Ćirić, *On Ivanov open problem related to the nonlinear quasi-contraction mappings*, (in Russian), Publ. Inst. Math. (Beograd), **41**(1987), 149-150.
[7] E. De Pascale, G. Marino, P. Pietramala, *The use of the E-metric spaces in the search for fixed points*, Le Matematiche, **48**(1993), 367-376.
[8] C. Di Bari, P. Vetro, *φ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, **57**(2008), 279-285.
[9] C. Di Bari, P. Vetro, *Weakly φ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, **58**(2009), 125-132.
[10] W.-S. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal., **72**(2010), 2259-2261.
[11] Y. Feng, W. Mao, *The equivalence of cone metric spaces and metric spaces*, Fixed Point Theory, **11**(2010), 259-264.
[12] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.
[13] L.-G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332**(2007), 1468-1476.
[14] A.A. Ivanov, *Fixed points of mappings on metric spaces*, (in Russian), Studies in topology II - Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), **66** (1976), 5-102.
[15] G. Jungck, S.Radenović, S. Radojevic, V. Rakocevic, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory Appl., **2009**(2009), 13 pp., Article ID 643840.
[16] W.A. Kirk, B.G. Kang, *A fixed point theorem revisited*, J. Korean Math. Soc., **34**(1997), 285-291.
[17] W.A. Kirk, B. Sims (Eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.
[18] M.A. Krasnoselskii, P.P. Zabrejko, *Geometrical Methods in Nonlinear Analysis*, Springer, 1984.
[19] S. Rezapour, R. Hamlbarani, *Some notes on paper "Cone metric spaces and fixed point theorems of contractive mappings."*, J. Math. Anal. Appl., **345**(2008), 719-724.
[20] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008.
[21] I.A. Rus, A. Petruşel, M.A. Serban, *Weakly Picard operators: equivalent definitions, applications and open problems*, Fixed Point Theory, **7**(2006), 3-22.
[22] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, **56**(2007), 464-468.
[23] P.P. Zabrejko, *K-metric and K-normed spaces: survey*, Collect. Math., **48**(1997), 825-859.

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