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A NOTE ON STABILITY OF THE LINEAR FUNCTIONAL EQUATIONS OF HIGHER ORDER AND FIXED POINTS OF AN OPERATOR

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Abstract. We prove two general theorems, which appear to be very useful in investigation of the Hyers-Ulam stability of a higher order linear functional equation in single variable, with constant coefficients. We give several examples of their applications. In particular we show that we obtain in this way several fixed point results for a particular operator. The main tool in the proofs is a complexification of a real normed (or Banach) space X, which can be described as the tensor product $X \otimes \mathbb{R}^2$ endowed with the Taylor norm.
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1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}_+$, $\mathbb{R}$ and $\mathbb{C}$ stand, as usual, for the sets of positive integers, nonnegative integers, integers, nonnegative reals, reals and complex numbers, respectively. Moreover B^A denotes the family of all functions mapping a nonempty set A into a nonempty set B.

In what follows we assume that $m \in \mathbb{N}$, X is a normed space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$, $a_0, ..., a_{m-1} \in K$, S is a nonempty set, $D \subset S$ is nonempty, $f \in S^S$, and f^j denotes the j-th iterate of f for $j \in \mathbb{N}_0$.

The linear functional equation, of the form

$$\varphi(f^m(x)) = \sum_{j=0}^{m-1} a_j \varphi(f^j(x)) + G(x), \tag{1}$$

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where $G \in X^S$ is given and $\varphi \in X^S$ is the unknown function, is one of the most important functional equations in single variable and many results have been given (see [16] and [17] and the references therein) on continuity, convexity, differentiability and analyticity of solutions for it. One of the simplest examples of equation (1), with $S \in \{\mathbb{Z}, \mathbb{N}\}$, is the linear recurrence (or difference equation)

$$y_{n+m} = \sum_{j=0}^{m-1} a_j y_{n+j} + b_n, \quad \forall n \in S; \tag{2}$$

clearly (1) becomes (2) with $f(n) := n + 1$, $y_n := \varphi(n) = \varphi(f^n(0))$ and $b_n := G(n)$.

We prove two theorems, which appear to be very effective tools for improving and completing already known results concerning Hyers-Ulam stability of (1). We also show that we thus obtain some fixed point results for a particular operator (see Remarks 1, 3 and 4).

The problem of stability of functional equations was raised by S. Ulam and numerous authors have investigated it for different equations (for more details and further references see, e.g., [13, 14, 20]; for examples of some recent results see, e.g., [2, 6, 7, 8, 18, 19, 22, 26, 27, 28, 29, 31, 32]). Stability of (1) has been studied in particular in [1, 3, 4, 5, 11, 24, 33]; the problem of stability of (2) corresponds to the notions of shadowing (in dynamical systems and computer science) and controlled chaos (see, e.g., [12, 21, 23, 30]).

Let $L, \delta \in \mathbb{R}_+^S$. We say that equation (1) is (δ, L)-stable in X^S on D, provided for each $\gamma \in X^S$ with

$$\left\| \gamma(f^m(x)) - \sum_{j=0}^{m-1} a_j \gamma(f^j(x)) - G(x) \right\| \le \delta(x), \quad \forall x \in D, \tag{3}$$

there is a solution $\varphi \in X^S$ of (1) with

$$\|\gamma(x) - \varphi(x)\| \le L(x), \quad \forall x \in D. \tag{4}$$

If the function φ is unique, then we say that the equation is stable with uniqueness. If, for a given function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, equation (1) is $(\delta, \lambda \circ \delta)$-stable in X^S on D for each constant function $\delta \in \mathbb{R}_+^S$, then we say simply that (1) is λ-stable in X^S on D; for $D = S$ we omit the part 'on D'.

We say that equation (1) is nonstable in X^S on D, provided there is a function $\gamma \in X^S$ such that

$$\sup_{x \in D} \left\| \gamma(f^m(x)) - \sum_{j=0}^{m-1} a_j \gamma(f^j(x)) - G(x) \right\| < \infty \tag{5}$$

and

$$\sup_{x \in D} \|\gamma(x) - \varphi(x)\| = \infty$$

for each solution $\varphi \in X^S$ of (1).

Remark 1. Assume that $a_0 \neq 0$ and define an operator $\mathcal{T} : X^S \to X^S$ by:

$$\mathcal{T}(\gamma)(x) := \frac{1}{a_0}\gamma(f^m(x)) - \sum_{j=1}^{m-1} \frac{a_j}{a_0}\gamma(f^j(x)) - \frac{1}{a_0}G(x), \quad \forall \gamma \in X^S, x \in S. \tag{6}$$

Let $L, \delta \in \mathbb{R}_+^S$. Note that equation (1) is (δ, L)-stable in X^S if and only if for each $\gamma \in X^S$ with

$$\Big\|\mathcal{T}(\gamma)(x) - \gamma(x)\Big\| \leq \frac{\delta(x)}{|a_0|}, \quad \forall x \in S, \tag{7}$$

there is a fixed point $\varphi \in X^S$ of $\mathcal{T}$ with

$$\|\gamma(x) - \varphi(x)\| \leq L(x), \quad \forall x \in S. \tag{8}$$

Therefore the stability results that we obtain in this paper yield some fixed point results for operator $\mathcal{T}$.

In the case $K = \mathbb{R}$, $\mathfrak{C}(X)$ denotes a complexification of X, i.e., the tensor product $X \otimes \mathbb{R}^2$ endowed with the Taylor norm (see [10]); it is known that $\mathfrak{C}(X)$ can be identified with X^2 endowed with a linear structure and norm $\|\cdot\|_T$, given by:

$$(x, y) + (z, w) := (x + z, y + w),$$

$$(\alpha + i\beta)(x, y) := (\alpha x - \beta y, \beta x + \alpha y),$$

$$\|(x, y)\|_T := \sup_{0 \leq \theta \leq 2\pi} \|(\cos\theta)x + (\sin\theta)y\|$$

for $x, y, z, w \in X$, $\alpha, \beta \in \mathbb{R}$. $\mathfrak{C}(X)$ is a complex normed space, which is a complex Banach space provided X is a Banach space (see e.g. [9, p. 39] or [15, 1.9.6, p. 66]). We define $p_1, p_2 : X^2 \to X$ by: $p_i(x_1, x_2) := x_i$ for $x_1, x_2 \in X$, $i = 1, 2$.

2. The main tools

The next two theorems are the main tools in this paper.

Theorem 1. (*i*) *Let $K = \mathbb{R}$, $F \in X^S$, $F_0(t) := (F(t), 0)$ for $t \in S$, and equation (1), with $G := F_0$, be (δ, L)-stable in $\mathfrak{C}(X)^S$ on D. Then equation (1) is (δ, L)-stable in X^S on D for $G := F$.*

(*ii*) *Let $K = \mathbb{R}$, $F \in X^S$, $\overline{F}(t) := (F(t), F(t))$ for $t \in S$, and equation (1), with $G := \overline{F}$, be nonstable in $\mathfrak{C}(X)^S$ on D. Then equation (1) is nonstable in X^S on D for $G := F$.*

Proof. (*i*) Let $\gamma \in X^S$ satisfy (3) with $G := F$. Write $\gamma_0(x) = (\gamma(x), 0)$ for $x \in S$. Then

$$\Big\|\gamma_0(f^m(x)) - \sum_{j=0}^{m-1} a_j\gamma_0(f^j(x)) - F_0(x)\Big\|_T \leq \delta(x), \quad \forall x \in D.$$

There is a solution $\varphi_0 \in \mathfrak{C}(X)^S$ of (1), with $G := F_0$, such that

$$\|\gamma_0(x) - \varphi_0(x)\|_T \leq L(x), \quad \forall x \in D.$$

Now, it is easily seen that $\varphi := p_1 \circ \varphi_0$ is a solution of (1), with $G := F$, such that

$$\|\gamma(x) - \varphi(x)\| \leq L(x), \quad \forall x \in D.$$

(ii) Take $\overline{\gamma} \in \mathfrak{C}(X)^S$ satisfying (5), with $G := \overline{F}$, and such that

$$\sup_{x\in D} \|\overline{\gamma}(x) - \overline{\varphi}(x)\|_T = \infty$$

for each solution $\overline{\varphi} \in \mathfrak{C}(X)^S$ of (1), with $G := \overline{F}$. Suppose there are solutions $\varphi_1, \varphi_2 \in X^S$ of (1), with $G := F$, such that

$$\sup_{x\in D} \|p_i(\overline{\gamma}(x)) - \varphi_i(x)\| < \infty, \quad i = 1, 2.$$

Write $\overline{\varphi}_0(x) := (\varphi_1(x), \varphi_2(x))$ for $x \in S$. Then $\overline{\varphi}_0$ is a solution of (1), with $G := \overline{F}$, and

$$\begin{aligned}
\sup_{x\in D} &\|\overline{\gamma}(x) - \overline{\varphi}_0(x)\|_T \\
&\le \sup_{x\in D} (\|p_1(\overline{\gamma}(x)) - p_1(\overline{\varphi}_0(x))\| + \|p_2(\overline{\gamma}(x)) - p_2(\overline{\varphi}_0(x))\|) \\
&\le \sup_{x\in D} \|p_1(\overline{\gamma}(x)) - \varphi_1(x)\| + \sup_{x\in D} \|p_2(\overline{\gamma}(x)) - \varphi_2(x)\| < \infty.
\end{aligned}$$

This is a contradiction, which shows that there is $j \in \{1, 2\}$ such that

$$\sup_{x\in D} \|p_j(\overline{\gamma}(x)) - \varphi(x)\| = \infty$$

for each solution $\varphi \in X^S$ of (1), with $G := F$. Now, it is enough to take $\gamma := p_j \circ \overline{\gamma}$. $\square$

In what follows $r_1, \ldots, r_m$ denote the complex roots of the characteristic equation

$$r^m - \sum_{j=0}^{m-1} a_j r^j = 0;$$

if $m > 1$, then

$$r^m - \sum_{j=0}^{m-1} a_j r^j = (r - r_m) \left(r^{m-1} - \sum_{j=0}^{m-2} b_j r^j \right)$$

for some unique $b_0, \ldots, b_{m-2} \in \mathbb{C}$.

Remark 2. Clearly,

$$a_{m-1} = r_m + b_{m-2}, \quad a_0 = -r_m b_0$$

and, in the case $m > 3$,

$$a_j = -r_m b_j + b_{j-1}$$

for $j = 1, \ldots, m-2$. Observe yet that, if $r_m, a_0, \ldots, a_{m-1} \in \mathbb{R}$, then we have $b_0, \ldots, b_{m-2} \in \mathbb{R}$.

Theorem 2. *Assume that one of the following two conditions is valid:*

(a) $|r_i| > 1$ *for each* $i \in \{1, \ldots, m\}$*;*

(b) f *is surjective and* $|r_i| \neq 1$ *for* $i = 1, \ldots, m$.

Then, for every $F \in X^S$ *and* $\gamma \in X^S$*, there exists at most one solution* $\varphi \in X^S$ *of (1), with* $G := F$*, such that*

$$\sup_{x\in S} \|\gamma(x) - \varphi(x)\| < \infty.$$

Proof. First consider the case $K = \mathbb{C}$. We show, by induction on m, that if $\psi, \psi' \in X^S$ are solutions of (1) with $G := F$, such that

$$\sup_{x \in S} \|\psi(x) - \psi'(x)\| =: M < \infty,$$

then $\psi = \psi'$. For $m = 1$ we have $r_1 = a_1$ and

$$|a_1|^n \|\psi(x) - \psi'(x)\| = \|\psi(f^n(x)) - \psi'(f^n(x))\| \leq M$$

for $n \in \mathbb{N}$, $x \in S$, whence $\psi = \psi'$ in the case $|r_1| > 1$. If f is surjective, then for each $n \in \mathbb{N}$ and $x \in S$ there is $x_n \in S$ with $f^n(x_n) = x$ and, consequently,

$$\|\psi(x) - \psi'(x)\| = |a_1|^n \|\psi(x_n) - \psi'(x_n)\| \leq |a_1|^n M,$$

which yields $\psi = \psi'$ for $|r_1| < 1$, too.

So, fix now $k \in \mathbb{N}$ and assume that the inductive statement is true for $m = k$. We are to show that this is also the case for $m = k + 1$. To this end take solutions $\psi, \psi' \in X^S$ of (1), with $G := F$, such that

$$M := \sup_{x \in S} \|\psi(x) - \psi'(x)\| < \infty.$$

Write

$$\eta(x) := \psi(f(x)) - r_{k+1}\psi(x), \quad \eta'(x) := \psi'(f(x)) - r_{k+1}\psi'(x)$$

for $x \in S$. Then $\eta, \eta' \in X^S$ are solutions of (1) with $G := F$, $m = k$ and $a_0, \ldots, a_{k-1}$ replaced by $b_0, \ldots, b_{k-1}$. Moreover,

$$\sup_{x \in S} \|\eta(x) - \eta'(x)\| \leq (1 + |r_{k+1}|)M.$$

Consequently, in view of the inductive hypothesis, $\eta = \eta'$ and consequently, arguing analogously as in the case $m = 1$, we get $\psi = \psi'$.

Now, to complete the proof fix $F, \gamma \in X^S$. Let $\varphi_1, \varphi_2 \in X^S$ be solutions of (1), with $G := F$, such that

$$\sup_{x \in S} \|\gamma(x) - \varphi_i(x)\| < \infty$$

for $i = 1, 2$. Then

$$\sup_{x \in S} \|\varphi_1(x) - \varphi_2(x)\| < \infty.$$

Suppose that $K = \mathbb{R}$ and write

$$F_0(x) := (F(x), 0), \quad \hat{\varphi}(x) := (\varphi(x), 0), \quad \hat{\varphi}'(x) := (\varphi'(x), 0)$$

for $x \in S$, $i = 1, 2$. Note that $\hat{\varphi}, \hat{\varphi}'$ are solutions of (1), with $G = F_0$, and

$$\sup_{x \in S} \|\hat{\varphi}(x) - \hat{\varphi}'(x)\|_T < \infty.$$

Hence, by the first part of the proof, $\hat{\varphi} = \hat{\varphi}'$ and consequently $\varphi = \varphi'$. $\square$

3. Applications

Now we present some examples of applications of Theorems 1 and 2. We show how to derive from them complements and improvements of some main outcomes in [3, 4, 14, 25, 33]. We also present some fixed point results.

Corollary 1. *Let $\lambda_1, \ldots, \lambda_m : \mathbb{R}_+ \to \mathbb{R}_+$. Assume that, for every complex normed space Y, $i \in \{1, \ldots, m\}$ and $F \in Y^S$, the equation*

$$\psi(f(x)) = r_i \psi(x) + F(x) \tag{9}$$

is λ_i-stable in Y^S. Then equation (1) is $\lambda_m \circ \ldots \circ \lambda_1$-stable in X^S for every $G \in X^S$; moreover, if one of conditions (a), (b) of Theorem 2 holds, then this stability is with uniqueness.

Proof. According to [4, Theorem 1], equation (1) is $\lambda_m \circ \ldots \circ \lambda_1$-stable in Y^S for every complex normed space Y and $G \in Y^S$ (in the proof of that theorem it is easily seen that it is not necessary to assume that Y is a Banach space). Hence Theorems 1(i) and 2 end the proof. □

Corollary 1 complements [4, Theorem 1], which has been proved under the additional assumption that $r_1, \ldots, r_m$ are real when $K = \mathbb{R}$. The next remark expresses that corollary in the terms of fixed points.

Remark 3. Let $\lambda_1, \ldots, \lambda_m : \mathbb{R}_+ \to \mathbb{R}_+$. Let $a_0 \neq 0$ (which means that $r_i \neq 0$ for $i = 1, ..., m$). Assume that, for every complex normed space Y, $i \in \{1, \ldots, m\}$ and $F \in Y^S$, the operator $\mathcal{T}_i : Y^S \to Y^S$, given by

$$\mathcal{T}_i(\gamma)(x) := \frac{1}{r_i}\gamma(f(x)) - \frac{1}{r_i}F(x), \quad \forall \gamma \in Y^S, x \in S,$$

has the following property.

($\mathcal{H}_i$) For every $\gamma \in Y^S$ with

$$\sup_{x \in S} \|\mathcal{T}_i(\gamma)(x) - \gamma(x)\| := d_i < \infty,$$

there is a fixed point $\gamma_i \in Y^S$ of $\mathcal{T}_i$ with

$$\sup_{x \in S} \|\gamma_i(x) - \gamma(x)\| \leq \lambda_i(|r_i| d_i).$$

Then the operator $\mathcal{T} : X^S \to X^S$, defined by (6), has the property:

($\mathcal{H}$) for every $\gamma \in X^S$ with

$$\sup_{x \in S} \|\mathcal{T}(\gamma)(x) - \gamma(x)\| := d < \infty,$$

there is a fixed point $\gamma_0 \in Y^S$ of $\mathcal{T}$ with

$$\sup_{x \in S} \|\gamma_0(x) - \gamma(x)\| \leq \lambda_m \circ \ldots \circ \lambda_1(|a_0| d);$$

moreover, if one of conditions (a), (b) of Theorem 2 holds, then such fixed point is unique.

The next corollary generalizes [14, Theorem 3.1], which concerns stability of the functional equation

$$\psi(x) = p\psi(x-1) - q\psi(x-2), \tag{10}$$

and, in some cases (e.g., $a < 0$ or $b < 0$), improves the estimations obtained in [4, Theorem 2] (see [14]). It seems that the fixed point result contained in that corollary is quite clear.

Corollary 2. *Let X be a Banach space, $p, q \in \mathbb{R}$, $p^2 \neq 4q$, $a, b \in \mathbb{C}$ be the roots of the equation $z^2 - pz + q = 0$ with $|a| > 1$ and $0 < |b| < 1$, and*

$$\lambda(t) := \frac{t(|a| - |b|)}{|a-b|(|a|-1)(1-|b|)}, \quad \forall t \in \mathbb{R}_+.$$

Then functional equation (10) is λ-stable in $X^{\mathbb{R}}$, with uniqueness.

Proof. The case where X is a complex normed space has been proved in [14, Theorem 3.1]. So, applying Theorems 1(i) and 2 we also obtain the statement in the case where X is a real normed space; it is just enough to notice that the equation can be rewritten in the form:

$$\psi(x+2) = p\psi(x+1) - q\psi(x).$$

$\square$

Corollary 3. *Let*

$$\lambda(t) := \frac{t}{|1 - |r_1|| \cdot \ldots \cdot |1 - |r_m||}, \quad \forall t \in \mathbb{R}_+.$$

Then the following two statements are valid.

(*i*) *If $|r_i| < 1$ for $i = 1, \ldots, m$, then (2) is λ-stable in $X^{\mathbb{N}}$ for any $(b_n) \in X^{\mathbb{N}}$.*

(*ii*) *If $|r_i| \neq 1$ for $i = 1, \ldots, m$ and X is a Banach space, then (2) is λ-stable in $X^{\mathbb{N}}$ for any $(b_n) \in X^{\mathbb{N}}$; moreover that stability is with uniqueness when $|r_i| > 1$ for $i = 1, \ldots, m$.*

Proof. The case $K = \mathbb{C}$ results from [25, Theorem 2.3 and Remark 2.5] (from the proof of this theorem it follows that the assumption that X is a Banach space is superfluous if $|r_i| < 1$ for $i = 1, \ldots, m$). Hence Theorems 1(i) and 2 complete the proof. $\square$

Corollary 4. *Let $(b_n) \in X^{\mathbb{N}}$ and $|r_i| = 1$ for some $i \in \{1, \ldots, m\}$. Then (2) is nonstable in $X^{\mathbb{N}}$.*

Proof. The case $K = \mathbb{C}$ results from [3, Theorem 4]. So, it is enough to use Theorem 1(ii). $\square$

Remark 4. Let $a_0 \neq 0$, $G : \mathbb{N} \to X$ and $\mathcal{T}$ be defined by (6) with $S = \mathbb{N}$ and $f(n) = n+1$ for $n \in \mathbb{N}$. Then, in the terms of fixed points, Corollary 4 states that, in the case where $|r_i| = 1$ for some $i \in \{1, \ldots, m\}$, there is a sequence $\gamma : \mathbb{N} \to X$ with

$$\sup_{n \in \mathbb{N}} \left\| \mathcal{T}(\gamma)(n) - \gamma(n) \right\| < \infty, \tag{11}$$

such that for every fixed point $y : \mathbb{N} \to X$ of $\mathcal{T}$ we have

$$\sup_{n\in\mathbb{N}} \left\| \gamma(n) - y(n) \right\| = \infty. \tag{12}$$

We end this paper with a corollary that improves [33, Theorem 3.1], which has been proved only in the case for $K = \mathbb{C}$. In view of Remark 1 it is easy to derive from it a fixed point result analogously as in Remark 3.

Corollary 5. *Let X be a Banach space, $a_0 \neq 0$, $\xi_0 \in \mathbb{R}_+^S$ and*

$$\xi_i(x) := \sum_{k=0}^{\infty} \xi_{i-1}(f^k(x))|r_i|^{-k-1} < \infty$$

for $i \in \{1, \dots, m\}$, $x \in S$. Then equation (1) is (ξ_0, ξ_m)-stable in X^S for any $G \in X^S$.

Proof. In view of [33, Theorem 3.1], the statement is true when X is a complex Banach space. Hence Theorem 1(i) completes the proof. □

In a similar way, as in Corollary 5, also [33, Theorem 3.3] can be extended for real Banach spaces.

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