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A COMMON FIXED POINT FOR WEAK ϕ-CONTRACTIONS ON b-METRIC SPACES

HASSEN AYDI*, MONICA-FELICIA BOTA**, ERDAL KARAPINAR***
AND SIROUS MORADI****

*Université de Sousse,
Institut Supérieur d'Informatique et des Technologies
de Communication de Hammam Sousse,
Route GP1-4011 H. Sousse, Tunisie
E-mail: hassen.aydi@isima.rnu.tn

**Department of Mathematics, Babeş-Bolyai University,
Kogălniceanu Street No.1, 400084, Cluj-Napoca, Romania
E-mail: bmonica@math.ubbcluj.ro

***Department of Mathematics,
Atilim University 06836, İncek, Ankara, Turkey
E-mail: erdalkarapinar@yahoo.com, ekarapinar@atilim.edu.tr

***Department of Mathematics, Faculty of Science,
Arak University, Arak 38156-8-8349, Iran
E-mail: s-moradi@araku.ac.ir, sirousmoradi@gmail.com

Abstract. In this paper, we give a common fixed point result for single-valued and multi-valued mappings satisfying a weak ϕ-contraction in b-metric spaces. Presented theorems extend, generalize and improve some existing results in the literature. Some examples are also given.
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1. Introduction and Preliminaries

The concept of a b-metric space appeared in some works, such as Bakhtin [1] and Czerwik [9]. For instance, Czerwik [9] presented a generalization of the well known Banach's [2] fixed point theorem in b-metric spaces. We recall the following notations and definitions from [9, 10].

Definition 1.1. ([9, 10]) *Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to [0, +\infty)$ is called a b-metric provided that, for all x, y, $z \in X$,*
(bm-1) $d(x, y) = 0$ if and only if $x = y$,
(bm-2) $d(x, y) = d(y, x)$,
(bm-3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

Throughout this paper, the letters $\mathbb{R}$ and $\mathbb{N}^*$ will denote the set of all real numbers and the set of all positive integer numbers, respectively.

For more considerations and examples of $b-$metric spaces see [7, 9, 4, 10, 11, 12, 20, 22].

The study of fixed points for multi-valued contractive mappings using the Hausdorff metric was initiated by Markin [16] and Nadler [17]. Later, an interesting and rich fixed point theory for such mappings was developed which has found applications in control theory, convex optimization, differential inclusion and economics (see, [14] and references cited therein).

Definition 1.2. *Let X be a nonempty set. An element x in X is said to be a common fixed point of a single-valued $T : X \to X$ and a multi-valued mapping $S : X \to P(X)$ if $x = Tx \in Sx$, where $P(X)$ denotes the collection of all nonempty subsets of X.*

Let (X, d) be a b-metric space. Let $P_{cl,b}(X)$ be the collection of all nonempty closed bounded subsets of X. Again as in [4, 10], for $A, B \in P_{cl,b}(X)$, we define

$$H(A, B) = \max\{\rho(A, B),\ \rho(B, A)\}, \tag{1.1}$$

where

$$\rho(A, B) = \sup\{D(a, B),\ a \in A\}, \quad \delta(B, A) = \sup\{D(b, A),\ b \in B\}$$

with

$$D(a, C) = \inf\{d(a, x),\ x \in C\}, \quad C \in P_{cl,b}(X).$$

By definition H is called the Pompeiu-Hausdorff functional.

We recall the following lemmas.

Lemma 1.3. ([9, 20]) *Let (X, d) be a b-metric space. For any $A, B, C \in P_{cl,b}(X)$ and any $x, y \in X$, we have the following:*
(i) $D(x, B) \le d(x, b)$ for any $b \in B$,
(ii) $\rho(A, B) \le H(A, B)$,
(iii) $d(x, B) \le H(A, B)$ for all $x \in A$,
(iv) $H(A, A) = 0$,
(v) $H(A, B) = H(B, A)$,
(vi) $H(A, C) \le s(H(A, B) + H(B, C))$,
(vii) $D(x, A) \le s(d(x, y) + D(y, A))$.

Lemma 1.4. ([9, 20]) *Let (X, d) be a b-metric space. Let A and B be in $P_{cl,b}(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that*

$$d(a, b) \le H(A, B) + \alpha.$$

Lemma 1.5. ([9, 20]) *Let (X, d) be a b-metric space. For $A \in P_{cl,b}(X)$ and $x \in X$, we have*

$$D(x, A) = 0 \Longleftrightarrow x \in \bar{A} = A.$$

Let Φ be the set of functions $\phi : [0, +\infty) \to [0, +\infty)$ that satisfy

(1) $\phi(0) = 0$ and $\phi(t) > (1 - \frac{1}{s^2})t$ for each $t > 0$,
(2) ϕ is lower semi-continuous.

Note that if $\phi \in \Phi$, we have $\phi(t) > 0$ for all $t > 0$.
In this paper, we establish a common fixed result for single-valued and multi-valued mappings involving a weak ϕ-contraction on complete b-metric spaces.

2. MAIN RESULTS

Several papers deal with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see [3, 4, 7, 10, 11, 20]).

Our main result is the following.

Theorem 2.1. *Let (X, d) be a complete $b-$metric space and $\phi \in \Phi$. Suppose that $T : X \to X$ and $S : X \to P_{cl,b}(X)$ are such that for all $x, y \in X$*

$$H(\{Tx\}, Sy) \leq M(x, y) - \phi(M(x, y)) \tag{2.1}$$

where

$$M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Sy), \frac{1}{2s}[D(x, Sy) + D(y, Tx)]\}, \tag{2.2}$$

then T and S have a unique common fixed point in X.

Proof. It is clear that $M(x, y) = 0$ if and only if $x = y$ is a common fixed point of T and S. Thus we may assume that $M(x, y) > 0$ for all $x, y \in X$.

Let $x_0 \in X$ and $x_1 \in Sx_0$. Set $x_2 = Tx_1$. By choosing $\alpha = \frac{\phi(M(x_2,x_1))}{2} > 0$ in Lemma 1.4, there exists $x_3 \in Sx_2$ such that

$$d(x_3, x_2) \leq H(\{Tx_1\}, Sx_2) + \frac{\phi(M(x_2, x_1))}{2}.$$

We let $x_4 = Tx_3$. In analogous way, one can find $x_5 \in Sx_4$ such that

$$d(x_5, x_4) \leq H(\{Tx_3\}, Sx_4) + \frac{\phi(M(x_4, x_3))}{2}.$$

Inductively, we let $x_{2n} = Tx_{2n-1}$, and by Lemma 1.4, there exists $x_{2n+1} \in Sx_{2n}$ such that

$$d(x_{2n+1}, x_{2n}) \leq H(\{Tx_{2n-1}\}, Sx_{2n}) + \frac{\phi(M(x_{2n}, x_{2n-1}))}{2} \text{ for all } n \in \mathbb{N}^*. \tag{2.3}$$

From (2.1) and (2.3), we get that

$$d(x_{2n+1}, x_{2n}) \leq M(x_{2n}, x_{2n-1}) - \frac{\phi(M(x_{2n}, x_{2n-1}))}{2} \text{ for all } n \in \mathbb{N}^*. \tag{2.4}$$

Step 1: We claim that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.
For any $n \in \mathbb{N}^*$, we have

$$\begin{aligned}
&d(x_{2n-1}, x_{2n}) \le M(x_{2n}, x_{2n-1}) = M(x_{2n-1}, x_{2n}) \\
&= \max\Big\{d(x_{2n-1}, x_{2n}),\ D(x_{2n-1}, Tx_{2n-1}),\ D(x_{2n}, Sx_{2n}), \\
&\quad \frac{1}{2s}[D(x_{2n-1}, Sx_{2n}) + D(x_{2n}, Tx_{2n-1})]\Big\} \\
&\le \max\Big\{d(x_{2n-1}, x_{2n}),\ d(x_{2n-1}, x_{2n}),\ d(x_{2n}, x_{2n+1}), \frac{1}{2s} d(x_{2n-1}, x_{2n+1})\Big\} \\
&\le \max\Big\{d(x_{2n-1}, x_{2n}),\ d(x_{2n}, x_{2n+1}), \frac{1}{2s}[s(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}))]\Big\} \\
&= \max\Big\{d(x_{2n-1}, x_{2n}),\ d(x_{2n}, x_{2n+1})\Big\}.
\end{aligned}$$

If for some $n \ge 1$, $d(x_{2n-1}, x_{2n}) < d(x_{2n}, x_{2n+1})$, then

$$d(x_{2n-1}, x_{2n}) \le M(x_{2n-1}, x_{2n}) \le d(x_{2n}, x_{2n+1}).$$

Thus, by (2.4) we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n}) &\le M(x_{2n}, x_{2n-1}) - \frac{\phi(M(x_{2n-1}, x_{2n}))}{2} \\
&\le d(x_{2n}, x_{2n+1}) - \frac{\phi(M(x_{2n-1}, x_{2n}))}{2},
\end{aligned}$$

so $\frac{\phi(M(x_{2n-1}, x_{2n}))}{2} = 0$, that is, $M(x_{2n-1}, x_{2n}) = 0$, which is a contradiction. Thus, for each $n \ge 1$, we get that

$$d(x_{2n-1}, x_{2n}) \ge d(x_{2n}, x_{2n+1}). \tag{2.5}$$

Therefore,

$$M(x_{2n-1}, x_{2n}) = d(x_{2n-1}, x_{2n}) \quad \text{for each } n \ge 1. \tag{2.6}$$

Also using (2.1), we have

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}) = D(x_{2n+1}, \{Tx_{2n+1}\}) \le H(Sx_{2n}, \{Tx_{2n+1}\}) \\
&= H(\{Tx_{2n+1}\}, Sx_{2n}) \le M(x_{2n+1}, x_{2n}) - \phi(M(x_{2n+1}, x_{2n})) \\
&< M(x_{2n+1}, x_{2n}) = \max\Big\{d(x_{2n+1}, x_{2n}),\ D(x_{2n+1}, Tx_{2n+1}),\ D(x_{2n}, Sx_{2n}), \\
&\quad \frac{1}{2s}[D(x_{2n+1}, Sx_{2n}) + D(x_{2n}, Tx_{2n+1})]\Big\} \\
&\le \max\Big\{d(x_{2n+1}, x_{2n}),\ d(x_{2n+1}, x_{2n+2}), \frac{1}{2s}[s(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))]\Big\} \\
&= \max\Big\{d(x_{2n+1}, x_{2n}),\ d(x_{2n+1}, x_{2n+2})\Big\}.
\end{aligned}$$

This inequality shows that

$$M(x_{2n+1}, x_{2n+2}) = d(x_{2n+1}, x_{2n+2}) \quad \text{for each } n \geq 0, \tag{2.7}$$

and

$$d(x_{2n+1}, x_{2n+2}) \geq d(x_{2n+2}, x_{2n+3}) \quad \text{for each } n \geq 0. \tag{2.8}$$

We deduce from (2.5) and (2.8)

$$d(x_{n+1}, x_{n+2}) \geq d(x_n, x_{n+1}) \quad \text{for each } n \geq 0, \tag{2.9}$$

We have that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence and bounded below, so

$$\lim_{n\to+\infty} M(x_{n+1}, x_n) = \lim_{n\to+\infty} d(x_{n+1}, x_n) = L \geq 0. \tag{2.10}$$

Assume that $L > 0$, so by a property of ϕ we have $\phi(L) > 0$. Taking the upper limit as $n \to \infty$ in (2.4) and using the fact that ϕ is lower semi-continuous, we obtain that

$$L \leq L - \frac{\phi(L)}{2} < L$$

a contradiction. Hence $L = 0$, that is,

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

Step 2: The sequence $\{x_n\}$ is bounded.
If $\{x_n\}$ were unbounded, then by step 1, $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are unbounded. We choose the sequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 1$, $n(2) > n(1)$ is even and minimal in the sense that $d(x_{n(2)}, x_{n(1)}) > 1$, and similarly $n(3) > n(2)$ is odd and minimal in the sense that $d(x_{n(3)}, x_{n(2)}) > 1$, ...,$n(2k) > n(2k-1)$ is even and minimal in the sense that $d(x_{n(2k)}, x_{n(2k-1)}) > 1$, and $n(2k+1) > n(2k)$ is odd and minimal in the sense that $d(x_{n(2k+1)}, x_{n(2k)}) > 1$.
Obviously $n(k) \geq k$ for every $k \in \mathbb{N}$. By Step 1 there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$ we have $d(x_{k+1}, x_k) < \frac{1}{4s}$. So for every $k \geq N_0$ we have $n(k+1) - n(k) \geq 2$ and

$$\begin{aligned}
1 <& d(x_{n(k+1)}, x_{n(k)}) \\
\leq& s[d(x_{n(k+1)}, x_{n(k+1)-2}) + d(x_{n(k+1)-2}, x_{n(k)})] \\
\leq& s[sd(x_{n(k+1)}, x_{n(k+1)-1}) + sd(x_{n(k+1)-1}, x_{n(k+1)-2}) + d(x_{n(k+1)-2}, x_{n(k)})] \\
\leq& s[sd(x_{n(k+1)}, x_{n(k+1)-1}) + sd(x_{n(k+1)-1}, x_{n(k+1)-2}) + 1].
\end{aligned} \tag{2.12}$$

Hence $1 \leq \limsup_{k\to\infty}[d(x_{n(k+1)}, x_{n(k)})] = \alpha \leq s < +\infty$. Also, by the triangular inequality (bm-3), we can write

$$\begin{aligned}
d(x_{n(k+1)}, x_{n(k)}) &\leq s[d(x_{n(k+1)}, x_{n(k+1)+1}) + d(x_{n(k+1)+1}, x_{n(k)})] \\
&\leq s[d(x_{n(k+1)}, x_{n(k+1)+1}) + sd(x_{n(k+1)+1}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)})].
\end{aligned} \tag{2.13}$$

Take $\limsup_{k\to\infty}[d(x_{n(k+1)+1}, x_{n(k)+1})] = \beta$. Letting $k \to \infty$ in (2.13) and using Step 1, we have

$$\alpha \leq s^2\beta.$$

If $n(k+1)$ is odd, so $n(k)$ is even. By (2.1), for all $k \geq N_0$, we have

$$\begin{aligned} d(x_{n(k)+1}, x_{n(k+1)+1}) &\leq H(\{Tx_{n(k+1)}\}, Sx_{n(k)}) \\ &\leq M(x_{n(k+1)}, x_{n(k)}) - \phi(M(x_{n(k+1)}, x_{n(k)})), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} d(x_{n(k+1)}, x_{n(k)}) &\leq M(x_{n(k+1)}, x_{n(k)}) \\ &\leq \max\{d(x_{n(k+1)}, x_{n(k)}), d(x_{n(k+1)}, x_{n(k+1)+1}), d(x_{n(k)}, x_{n(k)+1}), \\ &\frac{1}{2}[2d(x_{n(k+1)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k+1)}, x_{n(k+1)+1})]\}. \end{aligned} \tag{2.15}$$

Letting $k \to \infty$ in (2.15), we get that

$$\lim_{k\to\infty} M(x_{n(k+1)}, x_{n(k)}) = \alpha.$$

Since ϕ is lower semi-continuous, so letting $k \to \infty$ in (2.14), we have $\beta \leq \alpha - \phi(\alpha)$. Having in mind that $\alpha \leq s^2\beta$, we conclude that

$$\phi(\alpha) \leq (1 - \frac{1}{s^2})\alpha,$$

which is a contradiction because of $\alpha \geq 1 > 0$ and a property of ϕ.
Step 3: $\{x_n\}$ is a Cauchy sequence.
Now, we show that $\{x_n\}$ is a Cauchy sequence in the b-metric space (X, d). For this purpose, define

$$t_n = \sup\{d(x_i, x_j),\ i, j \geq n\}.$$

If $\lim_{n\to\infty} t_n = 0$, then $\{x_n\}$ is a Cauchy sequence.
From $d(x_i, x_j) \leq s[d(x_i, x_{i+1}) + d(x_{i+1}, x_j)]$ and Step 1, it is enough to show that $\lim_{n\to\infty} a_n = 0$, where

$$a_n = \sup\{d(x_{2i}, x_{2j+1}),\ i, j \geq n\}.$$

From Step 2, we have (a_n) is bounded, so $a_n < +\infty$ for all $n \in \mathbb{N}$. Also, it is clear that the sequence $\{a_n\}$ is decreasing, so it converges. Then, there exists a real $a \geq 0$ such that

$$\lim_{n\to\infty} a_n = a.$$

We argue by contradiction by assuming that $a > 0$. For every $k \in \mathbb{N}$, there exist $n(k), m(k) \in \mathbb{N}$ such that $m(k) > n(k) > k$ and

$$a_k - \frac{1}{k} \leq d(x_{m(k)}, x_{n(k)}) \leq a_k. \tag{2.16}$$

By (2.16), we get that

$$\lim_{k\to+\infty} d(x_{m(k)}, x_{n(k)}) = a. \tag{2.17}$$

By the triangular inequality (bm-3), we have

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) \leq& s[d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})] \\ \leq& s[s(d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1})) + d(x_{n(k)+1}, x_{n(k)})]. \end{aligned}$$

Taking the upper limit as $k \to +\infty$ and having in mind (2.11), (2.17), we obtain that

$$\limsup_{k\to+\infty} d(x_{m(k)+1}, x_{n(k)+1}) \geq \frac{a}{s^2}. \tag{2.18}$$

Using definition of a_n, we may assume that for every $k \in \mathbb{N}$, $m(k)$ is odd and $n(k)$ is even. By (2.1), we have

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) =& D(Tx_{m(k)}, x_{n(k)+1}) = D(x_{n(k)+1}, \{Tx_{m(k)}\}) \\ \leq& H(Sx_{n(k)}, \{Tx_{m(k)}\}) = H(\{Tx_{m(k)}\}, Sx_{n(k)}) \\ \leq& M(x_{m(k)}, x_{n(k)}) - \phi(M(x_{m(k)}, x_{n(k)})), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} &d(x_{m(k)}, x_{n(k)}) \leq M(x_{m(k)}, x_{n(k)}) \\ =& \max\Big\{d(x_{m(k)}, x_{n(k)}),\ D(x_{m(k)}, Tx_{m(k)}),\ D(x_{n(k)}, Sx_{n(k)}), \\ &\frac{1}{2s}[D(x_{m(k)}, Sx_{n(k)}) + D(x_{n(k)}, Tx_{m(k)})]\Big\} \\ \leq& \max\Big\{d(x_{m(k)}, x_{n(k)}),\ d(x_{m(k)}, x_{m(k)+1}),\ d(x_{n(k)}, x_{n(k)+1}), \\ &\frac{1}{2}[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1})]\Big\}. \end{aligned}$$

By (2.11) and (2.17), we deduce that

$$\lim_{k\to+\infty} M(x_{m(k)}, x_{n(k)}) = a. \tag{2.20}$$

Taking the upper limit in (2.19) and combining (2.18) and (2.20), we find that

$$\frac{a}{s^2} \leq a - \phi(a),$$

which contradicts a property of ϕ since $a > 0$. Thus, $a = 0$, so the sequence $\{x_n\}$ is Cauchy. Since the b-metric space (X, d) is complete there exists $u \in X$ such that

$$\lim_{n\to+\infty} d(x_n, u) = 0. \tag{2.21}$$

We claim that $u = Tu \in Su$. From (2.1), we have

$$\begin{aligned} D(x_{2n+2}, Su) = D(Tx_{2n+1}, Su) \leq& H(\{Tx_{2n+1}\}, Su) \\ \leq& M(x_{2n+1}, u) - \phi(M(x_{2n+1}, u)) \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} &D(u, Su) \leq M(x_{2n+1}, u) \\ =& \max\Big\{d(x_{2n+1}, u),\ d(x_{2n+1}, x_{2n+2}),\ D(u, Su), \frac{1}{2s}[D(x_{2n+1}, Su) + d(u, x_{2n+2})]\Big\}. \end{aligned} \tag{2.23}$$

The condition (bm-3) yields $d(x_{2n+1},u) \leq s(d(x_{2n+1},x_{2n})+d(x_{2n},u))$, so from (2.11) and (2.21)

$$\lim_{n\to+\infty} d(x_{2n+1},u) = 0. \tag{2.24}$$

Again, by Lemma 1.3, $D(x_{2n+1},Su) \leq s(d(x_{2n+1},u) + D(u,Su)$, then letting $n \to +\infty$, we get

$$\limsup_{n\to+\infty} D(x_{2n+1},Su) \leq s\, D(u,Su), \tag{2.25}$$

Using (2.11), (2.21), (2.24), (2.25) and letting $n \to +\infty$ in (2.23), we get

$$\lim_{n\to+\infty} M(x_{2n+1},u) = D(u,Su). \tag{2.26}$$

On the other hand,

$$D(u,Su) \leq s[d(u,x_{2n+1}) + D(x_{2n+1},Su)],$$

so $\limsup\limits_{n\to+\infty} D(x_{2n+1},Su)) \geq \dfrac{D(u,Su)}{s}$. Combining this and (2.26) in (2.22), we get that

$$\frac{D(u,Su)}{s} \leq D(u,Su) - \phi(D(u,Su)).$$

Assume that $D(u,Su) > 0$, so $\phi(D(u,Su)) \leq (1-\frac{1}{s})D(u,Su) \leq (1-\frac{1}{s^2})D(u,Su)$, which is a contradiction with a property of ϕ, hence $D(u,Su) = 0$, so $u \in Su$ since Su is a closed subset in X. Moreover, from (2.1)

$$D(Tu,u) \leq H(\{Tu\},Su) \leq M(u,u) - \phi(M(u,u)),$$

where

$$M(u,u) = \max\{d(u,u), D(u,Tu), D(u,Su), \frac{1}{2s}[D(u,Su)+D(u,Tu)]\} = D(u,Tu).$$

Thus, $D(u,Tu) \leq D(u,Tu) - \phi(D(u,Tu))$, which is possible only if $D(u,Tu) = 0$, so $u = Tu$. We deduce that

$$u = Tu \in Su.$$

So u is a common fixed point.
Uniqueness of the common fixed point follows from (2.1) and this completes the proof. □

We illustrate Theorem 2.1 with the following examples.

Example 2.2. *Let $X = [0,1]$ be equipped with the b-metric $d(x,y) = |x-y|^2$ for all $x,y \in X$, ($s = 2$) and let $T : X \to X$ and $S : X \to \mathcal{CB}(X)$ defined by*

$$Tx = 0 \quad \textit{and} \quad Sx = [0, \frac{x}{5}].$$

Let $\phi(t) = \frac{4}{5}t$ for all $t \geq 0$ Then $u = 0$ is the unique common fixed of T and S.

Example 2.3. *Let $X = [0,\infty)$ be equipped with the b-metric $d(x,y) = |x-y|^2$ and let $T : X \to X$ and $S : X \to \mathcal{CB}(X)$ defined by*

$$Tx = \frac{x}{3} \quad \textit{and} \quad Sx = \{\frac{x}{3}\}.$$

Take $\phi(t) = \frac{8}{9}t$ for all $t \geq 0$. Then $u = 0$ is the unique common fixed of T and S.

We state next some remarks which follow from our main result.

Remark 2.4. *Taking S as a singlevalued operator in Theorem 2.1 we obtain that T and S have a unique common fixed point in X.*

Remark 2.5. *Taking $S = T$ in Theorem 2.1 we obtain that T has a unique fixed point in X.*

Remark 2.6. *If we take in Theorem 2.1 $\varphi(t) = (1-k)t$ with $k < \frac{1}{s^2}$ we obtain that T and S have a unique common fixed point in X.*

If we take $S = T$ then we get that T has a unique fixed point in X.

Remark 2.7. *Our results generalize some results given by Zhang and Song* [15], *Rhoades* [18], *Ćirić* [8], *Rouhani and Moradi* [19] *and Daffer and Kaneko* [13].

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