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THE FIXED POINT PROPERTY OF THE CARTESIAN PRODUCT OF ROBERTS SPACES

NGUYEN HOANG THANH

University of Da Nang, College of Education, Faculty of Mathematics
459 Ton Duc Thang, Da Nang, Viet Nam
E-mail: nh_thanh@ud.edu.vn

Abstract. We know all Roberts spaces have the fixed point property (see [6]) and it is not certain that the Cartesian product of fixed point spaces is a fixed point space. The aim of this paper is to show that the product of Roberts spaces has the fixed point property. This is an extension of the result in [6].
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1. Introduction

Throughout most of this paper, by a linear metric space we mean a topological linear space X which is metrizable. By Kakutani's theorem (see [11]) there is an invariant metric ρ on X. We denote $\|x-y\| = \rho(x,y)$. Observe that $\|.\|$ is not a norm, in particular $\|\lambda x\| \neq |\lambda|\|x\|$.

However we assume that $\|.\|$ is monotonous, that is $\|\lambda x\| \leqslant \|x\|$ for every $x \in X$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant 1$.

A topological space X is called to have the fixed point property if for every continuous map $f : X \to X$, there exists a point $x_0 \in X$ such that $f(x_0) = x_0$.

In 1935 Schauder proved that (see [3])

Theorem 1.1. *In a locally convex linear metric space every compact convex set has the fixed point property.*

Schauder conjectured that his theorem holds true for non-locally convex spaces as well.

In 1977 Roberts constructed the compact convex sets without any extreme points. Roberts' example setted a long standing open problem in mathematics and has also revealed a new class of linear metric spaces, called needle point spaces. Using his needle point spaces Roberts established a general method for constructing compact convex sets with no extreme point, see [5, 8, 9]. We call all the compact convex sets with no extreme points constructed by Roberts's method of needle point spaces Roberts spaces. We know that a compact convex set which can be affinely embedded in a locally convex linear space has the fixed point property. Roberts in [10] has proved that

Theorem 1.2. *If K is a compact convex set then K is affinely embeddable in a locally convex linear space iff K is locally convex.*

And Roberts spaces are the first known examples of compact convex sets which are not locally convex. Because of Roberts spaces are not locally convex, every one thinks that Roberts spaces could yeild a counter example to Schauder's conjecture. But N.T. Nhu and L.H.Tri proved that

Theorem 1.3. (see [6]) *All Roberts spaces have the fixed point property.*

It is not certain that the Cartesian product of fixed point spaces is a fixed point space. So we will consider the fixed point property of the Cartesian product of Roberts spaces in this note.

Now we're going to construct Roberts spaces following Roberts' method.

Let X be a linear metric space. We say that X is a needle point space if and only if X is a complete linear metric space and for every $a \in X\backslash\{0\}$, for every $\epsilon > 0$ there exists a finite set $A(a, \epsilon) = \{a_1, a_2, \dots, a_m\}$ satisfying the following conditions

(1) $\|a_i\| < \epsilon$ for every $i = 1, \dots, m$;

(2) For each $b \in A^+(a, \epsilon)$ there is an $\alpha \in [0, 1]$ such that $\|b - \alpha a\| < \epsilon$;

(3) $a = \frac{1}{m}(a_1 + a_2 + \ldots + a_m)$,

where $A^+ = \mathrm{conv}(A \cup \{0\})$.

Some of examples of needle point spaces were given in [5, 6, 9].

Let a_0 be a non-zero point of a needle point space X. We choose by induction a sequence $\{A_n\}$ of finite subsets of X where $A_0 = \{a_0\}$ with the following properties

(4) $\|a\| < \epsilon_n$ for every $a \in A_n$;

(5) $\epsilon_n = [m(n-1)]^{-1}2^{-n}$, where $m(n) = \mathrm{card}A_n$;

(6) If $A_n = \{a_1^n, \dots, a_{m(n)}^n\}$ then A_{n+1} defined by the formula:

$$A_{n+1} = \bigcup_{i=1}^{m(n)} A(a_i^n, \epsilon_{n+1}),$$

where $A(a_i^n, \epsilon_{n+1})$, $i = 1, \dots, m(n)$, satisfy conditions (1), (2), (3) with $a = a_i^n$ and $\epsilon_{n+1} = (m(n))^{-1}2^{-n-1}$.

Denote that $\widehat{A} = \mathrm{conv}(A^+ \cup (-A^+))$. From (6) it follows that

(7) $\widehat{A}_n \subset \widehat{A}_{n+1}$ for every $n \in \mathbb{N}$.

We define

$$C = \overline{\bigcup_{n=1}^{\infty} \widehat{A}_n} \subset X.$$

Roberts show in [9] that C is a compact convex subset with no extreme points. We call C a Roberts space.

2. Main results

In this section we extend the result of N.T.Nhu and L.H.Tri in [6] for the Cartesian product of Roberts spaces. First, we have the following lemmas

Lemma 2.1. (Lemma 2 in [6]) *Let X be an infinite dimensional needle point space and let $A = \{a_1, \ldots, a_n\}$ be a finite subset of X and $\epsilon > 0$. Then for every $i = 1, 2, \ldots, 2n$ there exists $b_i = b(a_i) \in X$, where $a_{n+i} = -a_i$ for $i = 1, 2, \ldots, n$, with the following properties:*

(i) $\|a_i - b_i\| < (2n)^{-1}\epsilon$ *for every* $i = 1, 2, \ldots, 2n$;

(ii) $B = \{b_1, b_2, \ldots, b_{2n}\}$ *is a linearly independent subset of* X;

(iii) *There exists a continuous map* $p : B^+ \to \widehat{A}$ *such that* $\|x - p(x)\| \leqslant \epsilon$ *for every* $x \in B^+$;

(iv) $\|x - B^+\| \leqslant \epsilon$ *for every* $x \in \widehat{A}$,
where $\|x - B^+\| = \inf\{\|x - y\| \mid y \in B^+\}$.

Lemma 2.2. (Corollary 1 in [6]) *Let* $A_n = \{a_1^n, \ldots, a_{m(n)}^n\}$,$m(n) = cardA_n$, *see* (6). *Then there exists a sequence* $\{B_n\}$ *of finite subsets of* X *with the following properties:*

(i) $B_n = \{b_1^n, \ldots, b_{2m(n)}^n\}$ *where* $b_i^n = b(a_i^n)$, $i = 1, \ldots, 2m(n)$ *and* $a_{m(n)+i}^n = -a_i^n$ *for* $i = 1, \ldots, m(n)$;

(ii) $\|a_i^n - b_i^n\| \leqslant (2m(n-1)2m(n))^{-1}2^{-n-1}$ *for every* $i = 1, \ldots, 2m(n)$;

(iii) B_n *is a linearly independent finite subset of* X;

(iv) B_{n+1} *can be written in the form* $B_{n+1} = \bigcup_{i=1}^{2m(n)} B_{n+1}(b_i^n)$, *where* $b_i^n = b(a_i^n)$, *and*

a) $B_{n+1}(b_i^n) = \{b \in B_{n+1} \mid b = b(a)$ *for some* $a \in A(a_i^n, \epsilon_{n+1})\}$;

b) $B_{n+1}(b_i^n) \cap B_{n+1}(b_j^n) = \emptyset$ *for every* $i \neq j$;

(v) *For every* $n \in \mathbb{N}$ *there exists a continuous map* $p_n : B_n^+ \to \widehat{A}_n$ *such that* $\|p_n(x) - x\| < 2^{-n-1}$ *for every* $x \in B_n^+$;

(vi) $\|x - B_n^+\| < 2^{-n-1}$ *for every* $x \in \widehat{A}_n$.

Lemma 2.3. (Corollary 2 in [6]) *For every* $n \in \mathbb{N}$, $k \in \mathbb{N}$ *there exists a continuous map* $r_{k,n} : B_{n+k}^+ \to B_n^+$ *such that* $\|x - r_{k,n}(x)\| < 2^{-n+5}$ *for every* $x \in B_{n+k}^+$.

Lemma 2.4. (Lemma 4 in [6]) *Let P be a finite dimensional compact convex polyhedron in X and let $f : P \to C$ be a continuous map. Then for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and an affine map $g : P \to B_n^+$ such that $\|f(x) - g(x)\| < \epsilon$ for every $x \in P$.*

Lemma 2.5. *Let K be a compact convex subset of X. $K_1, K_2, \ldots, K_n$ are finite dimensional compact convex subsets of X such that $K_n \subset K_{n+1}$ for every $n \in \mathbb{N}$ and $\overline{\bigcup_{n=1}^{\infty} K_n} = K$. Suppose that for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ satisfying the condition: for every $p \in \mathbb{N}$ there exists a continuous map $r : K_{n+p} \to K_n$ such that $\|x - r(x)\| < \epsilon$ for every $x \in K_{n+p}$. Then K has the fixed point property.*

Proof. Suppose the assertion of the the lemma is false. Then there exists a continuous map $f : K \to K$ such that $f(x) \neq x$ for every $x \in K$. Because of the compactness of K there exists $\epsilon_0 > 0$ such that

$$\|f(x) - x\| \geqslant \epsilon_0 \text{ for every } x \in K(*)$$

Choose $\epsilon = \epsilon_0 > 0$ then there exists $n \in \mathbb{N}$ sastifying the condition: for every $p \in \mathbb{N}$ there exist a continuous map $s_{n,p} : K_{n+p} \to K_n$ such that $\|s_{n,p}(x) - x\| < \frac{\epsilon_0}{4}$ for every $x \in K_{n+p}$.

As the map $f|_{K_n} : K_n \to K$ is continuous and K_n is a finite dimensional compact convex subset in X, there exist $p_0 \in \mathbb{N}$ and a continuous map $g : K_n \to K_{n+p_0}$ such that $\|f(x) - g(x)\| < \frac{\epsilon_0}{4}$.

Consider the continuous map $s_{n,p} \circ g : K_n \to K_n$. Because K_n is a finite dimensional compact convex subset in X, there exists $x_0 \in K_n$ such that $s_{n,p_0} \circ g(x_0) = x_0$.

Since $x_0 \in K_n$, $g(x_0) \in K_{n+p_0}$ we have $\|f(x_0) - g(x_0)\| < \frac{\epsilon_0}{4}$ and $\|s_{n,p_0} \circ g(x_0) - g(x_0)\| < \frac{\epsilon_0}{4}$. Therefore $\|f(x_0) - x_0\| < \frac{\epsilon_0}{2}$. This contradicts $(*)$. So K has the fixed point property. □

Lemma 2.6. *For every $n \in \mathbb{N}$, $p \in \mathbb{N}$ there exists a continuous map $h_{p,n} : \widehat{A}_{n+p} \to \widehat{A}_n$ such that $\|x - h_{p,n}(x)\| < 2^{-n+7}$ for every $x \in \widehat{A}_{n+p}$.*

Proof. For every $n, p \in \mathbb{N}$, by Lemma 2.4 there exist $q \in \mathbb{N}$ and a continuous map $g : \widehat{A}_{n+p} \to B^+_{n+q}$ such that $\|x - g(x)\| < 2^{-n+5}$ for every $x \in \widehat{A}_{n+p}$.

For every $n \in \mathbb{N}$, by Lemma 2.3 there exist a continuous map $r_{q,n} : B^+_{n+q} \to B^+_n$ such that $\|x - r_{q,n}(x)\| < 2^{-n+5}$ for every $x \in B^+_{n+q}$.

For every $n \in \mathbb{N}$, by Lemma 2.2 there exist a continuous map $p_n : B^+_n \to \widehat{A}_n$ such that $\|p_n(x) - x\| < 2^{-n-1}$ for every $x \in B^+_n$.

For every $n \in \mathbb{N}$, denote that $h_{p,n} = p_n \circ r_{q,n} \circ g : \widehat{A}_{n+p} \to \widehat{A}_n$. We have

$$\|h_{p,n}(x) - x\| = \|p_n(r_{q,n}(g(x))) - x\| \leqslant \|p_n(r_{q,n}(g(x))) - r_{q,n}(g(x))\|$$
$$+\|r_{q,n}(g(x)) - g(x)\| + \|g(x) - x\| < 2^{-n+5} + 2^{-n+5} + 2^{-n-1} < 2^{-n+7}$$

for every $x \in \widehat{A}_{n+p}$. □

Theorem 2.7. *Let X, X' be needle point spaces. $C \subset X, C' \subset X'$ are Roberts spaces. Then $C \times C'$ has the fixed point property.*

Proof. Suppose $\{A_n\}$ is a sequence of finite subset of X constructed by Roberts' method such that $C = \overline{\bigcup_{n=1}^{\infty} \widehat{A}_n}$ and $\{A'_n\}$ is a sequence of finite subset of X constructed by Roberts' method such that $C' = \overline{\bigcup_{n=1}^{\infty} \widehat{A'}_n}$.

It's easy to verify that $C \times C' = \overline{\bigcup_{n=1}^{\infty} (\widehat{A}_n \times \widehat{A'}_n)}$.

By lemma 2.6, let $h_{p,n} : \widehat{A}_{n+p} \to \widehat{A}_n$, $h'_{p,n} : \widehat{A'}_{n+p} \to \widehat{A'}_n$ be continuous maps such that

$$\|h_{p,n}(x) - x\| < 2^{-n+7} \text{ for every } x \in \widehat{A}_{n+p} \text{ and}$$
$$\|h'_{p,n}(x) - x\| < 2^{-n+7} \text{ for every } x \in \widehat{A'}_{n+p}.$$

Let $H_{p,n} : \widehat{A}_{n+p} \times \widehat{A'}_{n+p} \to \widehat{A}_n \times \widehat{A'}_n$ be a map defined by

$$H_{p,n}(x, y) = (h_{p,n}(x), h'_{p,n}(y)) \text{ for every } (x, y) \in \widehat{A}_{n+p} \times \widehat{A'}_{n+p}.$$

We have

$$\|H_{p,n}(x,y)-(x,y)\| = \|h_{p,n}(x)-x\| + \|h'_{p,n}(y)-y\| < 2^{-n+7}+2^{-n+7} = 2^{-n+8}$$

for every $(x,y) \in \widehat{A}_{n+p} \times \widehat{A'}_{n+p}$.

By lemma 2.5 we have $C \times C'$ has the fixed point property. □

Similarly, we have

Theorem 2.8. *Let $X_1, X_2, \ldots, X_n$ be needle point spaces. $C_1 \subset X_1, C'_2 \subset X_2, \ldots, X_n$ are Roberts spaces. Then $C_1 \times C_2 \times \ldots \times C_n$ has the fixed point property.*

Theorem 2.9. *Suppose that $\{C_\alpha\}_{\alpha\in\Lambda}$ be a family of Roberts spaces. Then $\prod\limits_{\alpha\in\Lambda} C_\alpha$ has the fixed point property.*

Proof. Denote that $C = \prod\limits_{\alpha\in\Lambda} C_\alpha$

For every $\alpha \in \Lambda$, let $P_\alpha : C \to C_\alpha$ be a projection from C to C_α. By theorem of Tykhonov about the compactness of the product, C is a compact set.

Suppose that $f : C \to C$ is a continuous map, $\wp$ is a family of all the finite nonempty subset of Λ. For each $P \in \wp$, set $F_P = \{x \in C | P_\alpha(x) = P_\alpha(f(x)) \text{ for each } \alpha \in P\}$. We have F_P is a closed subset of C.

For every $P \in \wp$, set $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. For each $\alpha \in \Lambda\backslash P$ choose $x^0_\alpha \in C_\alpha$.

We define $f_P : C_{\alpha_1} \times C_{\alpha_2} \times \ldots \times C_{\alpha_n} \to C_{\alpha_1} \times C_{\alpha_2} \times \ldots \times C_{\alpha_n}$ by

$$f_P(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}) = (P_{\alpha_1}(f(x)), P_{\alpha_2}(f(x)), \ldots, P_{\alpha_n}(f(x)))$$

where $x = (x_\alpha)_{\alpha\in\Lambda}$,

$$x_\alpha = \begin{cases} x_{\alpha_i} & \alpha = \alpha_i \text{ for every } i \in \{1,2,\ldots,n\} \\ x^0_\alpha & \text{for every } \alpha \in \Lambda\backslash P \end{cases}$$

By Theorem 2.8, let $(x^0_{\alpha_1}, \ldots, x^0_{\alpha_n})$ be a fixed point of f_P. We have $(x^0_\alpha)_{\alpha\in\Lambda} \in F_P$ and $F_P \neq \emptyset$.

For every $m \in \mathbb{N}$, for every $P_1, P_2, \ldots, P_m \in \wp$, $F_{\bigcup\limits_{i=1}^m P_i} \neq \emptyset$ and $F_{\bigcup\limits_{i=1}^m P_i} \subset \bigcap\limits_{i=1}^m F_{P_i}$. Because of the compactness of C, $\bigcap\limits_{P\in\wp} F_P \neq \emptyset$. Choose $x_1 = (x^1_\alpha)_{\alpha\in\Lambda} \in \bigcap\limits_{P\in\wp} F_P$. For every $\alpha \in \Lambda$, $x_1 \in F_{\{\alpha\}}$. So $P_\alpha(x_1) = P_\alpha(f(x_1))$ for every $\alpha \in \Lambda$. It implies that $f(x_1) = x_1$. □

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