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THE EXISTENCE OF MULTIPLE FIXED POINTS FOR THE SUM OF TWO OPERATORS AND APPLICATIONS

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Abstract. In this paper, by means of τ-φ-concave (convex) operators, the existence of two positive fixed points for some nonlinear operators is considered. In particular, the fixed point theorems of the sum of a φ_1-concave operator and a φ_2-convex operator are obtained, our tools are based on the properties of cones and the fixed point theorem of cone expansion and compression. Our abstract results are applied to superlinear second-order multi-point boundary value problems.
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1. Introduction

The fixed point theory for nonlinear operators with convexity and concavity has been investigated extensively in the past several decades and is applied to the study of various nonlinear differential equations (see [1-11] and the references therein). Krasnoselskii [1] studied the definitions and properties of e-concave operators and e-convex operators. In [2], Potter introduced the definitions of α-concave operators and α-convex operators. We note that Zhao [3] considered the existence of multiple positive fixed points for some nonlinear operators, a particular case of the operators is the sum of α-concave operators and β-convex operators. Paper [4] is the continuation of paper [3], the author further discussed the existence of multiple positive fixed points for the sum of two operators, in particular, Corollary 3.1 of [4] showed that the sum of an e-concave operator and an e-convex operator has at least two positive fixed points

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under reasonable conditions. Very recently, Zhai and Cao [5] defined τ-φ-concave operator, which is essentially sublinear, α-concave operator ($0 < \alpha < 1$) is its particular case. Motivated by paper [5], Zhao [6] introduced τ-φ-convex operator, which is essentially superlinear, β-convex operator ($\beta > 1$) is its particular case. Under some conditions, the author obtained the existence of fixed points for the class of operators. As corollaries, some fixed point theorems for e-convex operators and α-convex operators were also given.

In [10], the two point expansion fixed point theorem for increasing operator was discussed by using fixed point index theory, moreover, fixed point theorems for superlinear operator, convex operator and sum of convex operator and concave operator were established. In [11], Cabada and Cid presented sufficient conditions for a nondecreasing operator defined on an ordered Banach space to have at least a non-zero fixed point. Their main results combined the monotone iterative technique with the expansion fixed point theorem of Krasnoselskii.

In this paper, we will combine τ-φ-concave operators with τ-φ-convex operators, the existence of two positive fixed points for some nonlinear operators is considered. As corollaries, we also obtain some fixed point theorems for the sum of a φ_1-concave operator and a φ_2-convex operator. In particular, the assumption of the monotonicity of the operator is not required. Our results generalize and improve the corresponding ones in [3, 4]. Moreover, as a sample of application, we apply our fixed point theorem to a class of multi-point boundary value problems for second-order differential equations.

Throughout this paper, E is a real Banach space with norm $\|\cdot\|$, θ is the zero element of E, and P is a cone in E. So, a partially ordered relation in E is given by $x \le y$ iff $y - x \in P$. A cone $P \subset E$ is said to be normal if there exists a constant N, such that $\theta \le x \le y \Longrightarrow \|x\| \le N\|y\|$, the smallest N is called the normal constant of P. We write $\mathbb{R}^+ = [0, +\infty)$, $P^+ = P - \{\theta\}$ and

$$C_e = \{x \in E : \text{there exist positive numbers a, b such that } ae \le x \le be\}, \text{ for } e \in P^+.$$

Assume D is a subset of E, operator $A : D \longrightarrow E$ is continuous and bounded. If there is a constant k, $0 \le k < 1$ such that $\gamma(A(S)) \le k\gamma(S)$ for any bounded set $S \subset D$, then A is called a strict set contraction, where $\gamma(D)$ denotes the Kuratowski measure of noncompactness of bounded set S.

All the concepts discussed above can be found in [1, 2, 9, 12]. We state below some definitions and a lemma.

Definition 1.1. (See [1, 9].) Let P be a cone of real Banach space E, $e \in P^+$.
(i) $A_1 : P \longrightarrow P$ is called e-concave if and only if for any $x \in P^+$, $A_1x \in C_e$; for any $(x,t) \in C_e \times (0,1)$, there exists $\zeta_1 = \zeta_1(x,t) > 0$ such that $A_1(tx) \ge t(1+\zeta_1)A_1x$.
(ii) $A_2 : P \longrightarrow P$ is called e-convex if and only if for any $x \in P^+$, $A_2x \in C_e$; for any $(x,t) \in C_e \times (0,1)$, there exists $\zeta_2 = \zeta_2(x,t) > 0$ such that $A_2(tx) \le t(1-\zeta_2)A_2x$.

Definition 1.2. (See [2].) Let $A : P \longrightarrow P$ and $\alpha \in \mathbb{R}$. Then we say A is α-concave (β-convex) if and only if $A(tx) \ge t^\alpha Ax$ ($A(tx) \le t^\beta Ax$) for all $(x,t) \in P \times (0,1)$.

Definition 1.3. Assume $P \subset E$ is a cone. We say an operator $A_1 : P \longrightarrow P$ is φ-concave if there exists a functional $\varphi : P \times (0,1) \longrightarrow \mathbb{R}^+$ with $\varphi(x,t) > t$, $\forall\, t \in (0,1)$

such that

$$A_1(tx) \geq \varphi(x,t)A_1x, \quad \forall\, t \in (0,1), \quad x \in P.$$

We say an operator $A_2 : P \longrightarrow P$ is φ-convex if there exists a functional $\varphi : P \times (0,1) \longrightarrow \mathbb{R}^+$ with $\varphi(x,t) < t,\ \forall\, t \in (0,1)$ such that

$$A_2(tx) \leq \varphi(x,t)A_2x, \quad \forall\, t \in (0,1), \quad x \in P.$$

Definition 1.4. Assume $P \subset E$ is a cone. We say an operator $A_1 : P \longrightarrow P$ is τ-φ-concave if there exist a function $\tau : (a,b) \longrightarrow (0,1)$ and a functional $\varphi : P \times (a,b) \longrightarrow \mathbb{R}^+$ with $\varphi(x,t) > \tau(t),\ \forall\, t \in (a,b)$ such that

$$A_1(\tau(t)x) \geq \varphi(x,t)A_1x, \quad \forall\, t \in (a,b), \quad x \in P.$$

We say an operator $A_2 : P \longrightarrow P$ is τ-φ-convex if there exist a function $\tau : (a,b) \longrightarrow (0,1)$ and a functional $\varphi : P \times (a,b) \longrightarrow \mathbb{R}^+$ with $\varphi(x,t) < \tau(t),\ \forall\, t \in (a,b)$ such that

$$A_2(\tau(t)x) \leq \varphi(x,t)A_2x, \quad \forall\, t \in (a,b), \quad x \in P.$$

Lemma 1.1. (See [12].) *Let $P_{r,s} = \{x \in P :\ r \leq \|x\| \leq s\}$ with $s > r > 0$. Suppose that $A : P_{r,s} \longrightarrow P$ is a strict set contraction such that*

$$Ax \ngeq x \quad for \quad x \in P, \quad \|x\| = r \quad and \quad Ax \nleq x \quad for \quad x \in P, \quad \|x\| = s.$$

Then A has a fixed point $x \in P$ such that $r < \|x\| < s$.

2. Main results

Theorem 2.1. *Suppose that the following conditions are satisfied*
(H_1) P is a normal cone of real Banach space E, N is the normal constant of P, $A : P \longrightarrow P$ is a strict set contraction, which satisfies that

$$\sup\{\|Ax\| :\ x \in P,\ \|x\| = 1\} < \frac{1}{N}; \tag{2.1}$$

(H_2) there exist operators $A_i : P \longrightarrow P$ such that

$$Ax \geq A_ix, \quad \forall\, x \in P, \quad i = 1,2. \tag{2.2}$$

(H_3) A_1 is a τ_1-φ_1-concave operator, and

$$\lim_{t\to a^+} \tau_1(t) = 0, \quad \overline{\lim_{t\to a^+}} \frac{\varphi_1(x,t)}{\tau_1(t)} > \frac{N}{m_1}, \quad uniformly\ for\ x \in P^+, \tag{2.3}$$

where

$$m_1 = \inf\{\|A_1x\| :\ x \in P,\ \|x\| = 1\} > 0. \tag{2.4}$$

If there exists a positive number c such that

$$m_2 = \inf\{\|A_2x\| :\ x \in P,\ \|x\| = c\} > 0. \tag{2.5}$$

A_2 is a τ_2-φ_2-convex operator, and

$$\lim_{t\to a^+} \tau_2(t) = 0, \quad \underline{\lim_{t\to a^+}} \frac{\varphi_2(x,t)}{\tau_2(t)} < \frac{m_2}{cN}, \quad uniformly\ for\ x \in P^+. \tag{2.6}$$

Then A has at least two fixed points $x_1^*,\ x_2^*$ in P^+, such that $\|x_1^*\| < 1 < \|x_2^*\|$.

Proof. Let $T_r = \{x \in E : \ \|x\| = r\}, \ r > 0$. We prove below that there exist real numbers $r_1, \ r_2$ such that $0 < r < 1 < r_2$, and

$$Ax \not\leq x, \quad \forall\, x \in T_{r_1} \cap P, \tag{2.7}$$

$$Ax \not\leq x, \quad \forall\, x \in T_{r_2} \cap P, \tag{2.8}$$

$$Ax \not\geq x, \quad \forall\, x \in T_1 \cap P. \tag{2.9}$$

It follows from (2.3) that there exists $t_1 \in (a, b)$ such that

$$0 < \tau_1(t_1) < 1, \quad \varphi_1(x, t_1) > \frac{N\tau_1(t_1)}{m_1}, \quad \forall\, x \in P^+. \tag{2.10}$$

Setting $r_1 = \tau_1(t_1)$. Assume that there exists $x_1 \in T_{r_1} \cap P$ such that $Ax_1 \leq x_1$. By the definition of A_1 and (2.2), we have

$$x_1 \geq A(x_1) \geq A_1(x_1) = A_1\left(\tau_1(t_1)\frac{x_1}{\tau_1(t_1)}\right) \geq \varphi_1\left(\frac{x_1}{\tau_1(t_1)}, t_1\right) A_1\left(\frac{x_1}{\tau_1(t_1)}\right),$$

which together with the normality of P and (2.10) implies

$$\|x_1\| \geq \frac{1}{N}\varphi_1\left(\frac{x_1}{\tau_1(t_1)}, t_1\right)\left\|A_1\left(\frac{x_1}{\tau_1(t_1)}\right)\right\| > \frac{1}{N}\frac{N\tau_1(t_1)}{m_1}m_1 = r_1,$$

which contradicts $x_1 \in T_{r_1} \cap P$, and so (2.7) holds.

In view of (2.6), we know that there exists $t_2 \in (a, b)$ such that

$$0 < \tau_2(t_2) < \min\{1, c\}, \quad \varphi_2(x, t_2) < \frac{m_2\tau_2(t_2)}{cN}, \quad \forall\, x \in P^+. \tag{2.11}$$

We take $r_2 = \frac{c}{\tau_2(t_2)}$, then $r_2 > 1$. Moreover, assume that there exists $x_2 \in T_{r_2} \cap P$ such that $Ax_2 \leq x_2$, thus, $\|\tau_2(t_2)x_2\| = c$. By the definition of A_2, we have $A_2(\tau_2(t_2)x_2) \leq \varphi_2(x_2, t_2)A_2(x_2)$. Therefore

$$A_2(x_2) \geq \frac{1}{\varphi_2(x_2, t_2)} A_2(\tau_2(t_2)x_2). \tag{2.12}$$

It follows from (2.2), (2.12) and the normality of P that

$$\|x_2\| \geq \frac{1}{N}\frac{1}{\varphi_2(x_2, t_2)}\|A_2(\tau_2(t_2)x_2)\| > \frac{1}{N}\frac{cN}{m_2}\frac{1}{\tau_2(t_2)}m_2 = r_2,$$

which contradicts $x_2 \in T_{r_2} \cap P$, and so (2.8) holds.

Assume that there exists $x_3 \in T_1 \cap P$ such that $Ax_3 \geq x_3$. By (2.1), we can know that $1 = \|x_3\| \leq N\|Ax_3\| < 1$, which is a contradiction, hence (2.9) holds.

By (2.7), (2.8) and (2.9), applying Lemma 1.1, we assert that A has at least two fixed points $x_1^*, \ x_2^*$ in P^+, such that $r_1 < \|x_1^*\| < 1 < \|x_2^*\| < r_2$.

3. Some corollaries

By taking $(a,b)=(0,1)$ and $\tau_1(t)=\tau_2(t)=t$ in Theorem 2.1, we can obtain the following corollary.

Corollary 3.1. *Suppose* (H_1) *in Theorem* 2.1 *is satisfied. The operator* A *can be written as* $A=A_1+A_2$, *where* $A_1:P\longrightarrow P$ *is* φ_1*-concave, and*

$$\overline{\lim_{t\to 0^+}}\frac{\varphi_1(x,t)}{t}>\frac{N}{m_1},\quad uniformly\ for\ x\in P^+,$$

$A_2:P\longrightarrow P$ *is* φ_2*-convex, and*

$$\underline{\lim_{t\to 0^+}}\frac{\varphi_2(x,t)}{t}<\frac{m_2}{cN},\quad uniformly\ for\ x\in P^+,$$

where N,m_1,m_2,c *as in Theorem* 2.1. *Then* A *has at least two fixed points* $x_1^*,\ x_2^*$ *in* P^+, *such that* $\|x_1^*\|<1<\|x_2^*\|$.

Corollary 3.2. *Suppose* (H_1) *in Theorem* 2.1 *is satisfied. The operator* A *can be written as* $A=A_1+B_1+A_2+B_2$, *where* $A_1:P\longrightarrow P$ *is* φ_1*-concave,* $A_2:P\longrightarrow P$ *is* φ_2*-convex, and* $B_i:P\longrightarrow P$ *are homogeneous* $(i=1,2)$. *If there exist two positive numbers* q_i $(i=1,2)$ *such that*

$$\overline{\lim_{t\to 0^+}}\frac{\varphi_1(x,t)}{t}>\frac{N-m_1(1-q_1)}{m_1q_1},$$

$$\underline{\lim_{t\to 0^+}}\frac{\varphi_2(x,t)}{t}<\frac{m_2-cN(1-q_2)}{cNq_2},\quad uniformly\ for\ x\in P^+,\tag{3.1}$$

$$A_ix\geq q_i(A_ix+B_ix)(i=1,2),\quad\forall\ x\in P,\tag{3.2}$$

where m_1,m_2,N,c *as in Theorem* 2.1. *Then* A *has at least two fixed points* $x_1^*,\ x_2^*$ *in* P^+, *such that* $\|x_1^*\|<1<\|x_2^*\|$.

Proof. By the definitions A_1 and B_1, we can know that for any $t\in(0,1)$ and $x\in P$, we have

$$\begin{aligned}A_1(tx)+B_1(tx)&=A_1(tx)+tB_1x\\&\geq\varphi_1(x,t)A_1x+t(A_1x+B_1x-A_1x)\\&=[\varphi_1(x,t)-t]A_1x+t(A_1x+B_1x)\\&\geq[\varphi_1(x,t)-t]q_1(A_1x+B_1x)+t(A_1x+B_1x)\\&=[(\varphi_1(x,t)-t)q_1+t](A_1x+B_1x).\end{aligned}\tag{3.3}$$

In (3.3), we have used (3.2).

It follows from (3.3) and (3.1) that

$$\overline{\lim_{t\to 0^+}}\frac{[\varphi_1(x,t)-t]q_1+t}{t}=1+\left(\overline{\lim_{t\to 0^+}}\frac{\varphi_1(x,t)}{t}-1\right)q_1>\frac{N}{m_1}.$$

In a similar way, we have

$$\begin{aligned} A_2(tx) + B_2(tx) &= A_2(tx) + tB_2x \\ &\leq \varphi_2(x,t)A_2x + t(A_2x + B_2x - A_2x) \\ &= [\varphi_2(x,t) - t]A_2x + t(A_2x + B_2x) \\ &\leq [\varphi_2(x,t) - t]q_2(A_2x + B_2x) + t(A_2x + B_2x) \\ &= [(\varphi_2(x,t) - t)q_2 + t](A_2x + B_2x), \end{aligned}$$

which together with (3.1) implies

$$\underline{\lim}_{t\to 0^+} \frac{[\varphi_2(x,t) - t]q_2 + t}{t} = 1 + \left(\underline{\lim}_{t\to 0^+} \frac{\varphi_2(x,t)}{t} - 1\right) q_2 < \frac{m_2}{cN}.$$

By Corollary 3.1, we can deduce the conclusion of Corollary 3.2.

Combining the proof of Corollary 3.1 in [4] with Corollary 3.1 in this paper, we can obtain the following two corollaries.

Corollary 3.3. (See [4].) *Suppose* (H_1) *in Theorem* 2.1 *is satisfied. The operator* A *can be written as* $A = A_1 + A_2$, *where* $A_1 : P \longrightarrow P$ *is increasing e-concave,* $A_2 : P \longrightarrow P$ *is increasing e-convex. If there exist* $\epsilon_i > 0$ $(i = 1, 2)$ *such that*

$$A_ix \geq \epsilon_i\|A_ix\|e, \quad \forall\ x \in P^+,$$

$$\overline{\lim}_{t\to 0^+} \zeta_1(x,t) > \frac{N^2}{\epsilon_1\|A_1(\epsilon_0 e)\|\|e\|} - 1, \quad uniformly\ for\ x \in\ C_e, \tag{3.4}$$

$$\underline{\lim}_{t\to 0^+} \zeta_2(x,t) > 1 - \frac{1}{N^2}\epsilon_2\|A_2(\epsilon_0 e)\|\|e\|,\ uniformly\ for\ x \in C_e, \tag{3.5}$$

then A *has at least two fixed points* $x_1^*,\ x_2^*$ *in* P^+, *such that*

$$\|x_1^*\| < 1 < \|x_2^*\|, \quad \min\{\epsilon_1, \epsilon_2\}\|x_i^*\|e \leq x_i^* \leq M_ie, \quad \exists\ M_i > 0,\ i = 1, 2. \tag{3.6}$$

Corollary 3.4. (See [4].) *Suppose* (H_1) *in Theorem* 2.1 *is satisfied. The operator* A *can be written as* $A = A_1 + A_2 + A_3$, *where* $A_i : P \longrightarrow P$ $(i = 1, 2, 3)$, A_1 *is* α*-concave* $(0 < \alpha < 1)$, A_2 *is* β*-convex* $(\beta > 1)$. *If there exist positive numbers* c_i $(i = 1, 2)$ *such that*

$$\overline{m}_i = \inf\{\|A_ix\| :\ x \in P,\ \|x\| = c_i\} > 0, \quad i = 1, 2,$$

then A *has at least two fixed points* $x_1^*,\ x_2^*$ *in* P^+, *such that* $\|x_1^*\| < 1 < \|x_2^*\|$.

4. Applications to a multi-point boundary value problem

Multi-point boundary value problems arise in many applied sciences. For example, the vibrations of a guy wire composed of N parts with a uniform cross-section throughout but different densities in different parts can be set up as multi-point boundary value problems (see [13]). Many problems in the theory of elastic stability can be modelled by multi-point boundary value problems (see [14]). In recent years, there has been a large amount of attention paid to multi-point boundary value problems for second-order differential equations, see [15-21] and the references therein.

In this section, we apply Theorem 2.1 to the following boundary value problem

$$\begin{cases} -u'' + k^2 u = g(t,u), \quad a < t < b, \\ u'(a) = 0, \quad u(b) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases} \tag{4.1}$$

where $k > 0,\ m > 2,\ \eta_i \in (a,b),\ \alpha_i \in \mathbb{R}^+ (i = 1, 2, \cdots, m-2)$ are given numbers, $g : (a,b) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous. In order to obtain our result, we need the following lemmas.

Lemma 4.1. (See [6, 22].) *Suppose the function $f(t)$ is continuous on $[a,b]$ and in addition assume $k > 0$, $\cosh(k(b-a)) \neq \sum\limits_{i=1}^{m-2} \alpha_i \cosh(k(\eta_i - a))$. Then the linear boundary value problem*

$$\begin{cases} -u'' + k^2 u = f(t), \quad a \le t \le b, \\ u'(a) = 0, \quad u(b) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i) \end{cases}$$

has a unique solution

$$u(t) = \int_a^b K(t,s) f(s) ds,$$

where the Green's function

$$K(t,s) = G(t,s) + \frac{\cosh(k(t-a))}{\cosh(k(b-a)) - \sum\limits_{i=1}^{m-2} \alpha_i \cosh(k(\eta_i - a))} \sum_{i=1}^{m-2} \alpha_i G(\eta_i, s), \tag{4.2}$$

with

$$G(t,s) = \begin{cases} \dfrac{\cosh(k(s-a))\sinh(k(b-t))}{k\cosh(k(b-a))}, & a \le s \le t, \\ \dfrac{\cosh(k(t-a))\sinh(k(b-s))}{k\cosh(k(b-a))}, & t \le s \le b. \end{cases} \tag{4.3}$$

Lemma 4.2. (See [6].) *For any $t,\ s \in [a,b]$, the Green's function $K(t,s)$ satisfies*

$$M_1 \frac{b-s}{\cosh(k(b-a))} \le M_1 G(s,s) \le K(t,s) \le M_2 G(s,s) \le M_2 \frac{\sinh(k(b-a))}{k(b-a)}(b-s), \tag{4.4}$$

where G is defined in (4.3),

$$M_1 = \frac{k \sum\limits_{i=1}^{m-2} \alpha_i G(\eta_i, \eta_i)}{\sinh(k(b-a)) \left[\cosh(k(b-a)) - \sum\limits_{i=1}^{m-2} \alpha_i \cosh(k(\eta_i - a))\right]}, \tag{4.5}$$

$$M_2 = 1 + \frac{\cosh(k(b-a)) \sum\limits_{i=1}^{m-2} \alpha_i}{\cosh(k(b-a)) - \sum\limits_{i=1}^{m-2} \alpha_i \cosh(k(\eta_i - a))}. \tag{4.6}$$

Our main result is the following theorem.

Theorem 4.1. *Suppose that* $\cosh(k(b-a)) > \sum\limits_{i=1}^{m-2} \alpha_i \cosh(k(\eta_i - a))$*, there exists* $\overline{\beta} > 1$ *such that for any* $0 < r < 1$*, we have*

$$r^{\overline{\beta}} g(t,u) \leq g(t,ru), \quad \forall\ (t,u) \in (a,b) \times \mathbb{R}^+. \tag{4.7}$$

Furthermore, g *can be expressed as* $g = g_1 + g_2$*, for fixed* $t \in [a,b]$*,* $g_i(t,u)$ $(i = 1,2)$ *are both increasing in* u*, and*

$$\int_a^b (b-s) g_i(s,1) ds > 0 \ (i = 1,2), \qquad \int_a^b (b-s) g(s,1) ds < \frac{k(b-a)}{M_2 \sinh(k(b-a))}, \tag{4.8}$$

where M_2 *as in* (4.6)*. In addition, there exist a function* $\tau_1 : (a,b) \longrightarrow (0,1)$ *and a function* $\varphi_1 : (a,b) \longrightarrow \mathbb{R}^+$ *with* $\varphi_1(t) > \tau_1(t)$*,* $\forall\ t \in (a,b)$ *such that*

$$g_1(t, \tau_1(\lambda) u) \geq \varphi_1(\lambda) g_1(t,u), \quad \forall\ t \in (a,b),\ u \in \mathbb{R}^+. \tag{4.9}$$

There exists $\lambda_1 \in (a,b)$ *such that* $\tau_1(\lambda_1) = \frac{M_1}{M_2}$*, and*

$$\lim_{t \to a^+} \tau_1(t) = 0, \quad \overline{\lim_{t \to a^+}} \frac{\varphi_1(t)}{\tau_1(t)} > \frac{\cosh(k(b-a))}{\varphi_1(\lambda_1) M_1 \displaystyle\int_a^b (b-s) g_1(s,1) ds}. \tag{4.10}$$

There exist a function $\tau_2 : (a,b) \longrightarrow (0,1)$ *and a function* $\varphi_2 : (a,b) \longrightarrow \mathbb{R}^+$ *with* $\varphi_2(t) < \tau_2(t)$*,* $\forall\ t \in (a,b)$ *such that*

$$g_2(t, \tau_2(\lambda) u) \leq \varphi_2(\lambda) g_2(t,u), \quad \forall\ t \in (a,b),\ u \in \mathbb{R}^+. \tag{4.11}$$

There exists $\lambda_2 \in (a,b)$ *such that* $\tau_2(\lambda_2) = \frac{M_2}{M_1 c}$*, and*

$$\lim_{t \to a^+} \tau_2(t) = 0, \quad \underline{\lim_{t \to a^+}} \frac{\varphi_2(t)}{\tau_2(t)} < \frac{M_1 \displaystyle\int_a^b (b-s) g_2(s,1) ds}{c \varphi_2(\lambda_2) \cosh(k(b-a))}, \tag{4.12}$$

where $c > \frac{M_2}{M_1}$*. Then the boundary value problem* (4.1) *has at least two nontrivial nonnegative solutions* $u_1(t)$ *and* $u_2(t)$ *which satisfy*

$$\max_{t \in [a,b]} u_1(t) < 1 < \max_{t \in [a,b]} u_2(t), \quad u_i(t) \geq \frac{M_1}{M_2} \|u_i\|, \quad i = 1,2,$$

where M_1 *and* M_2 *are defined in* (4.5) *and* (4.6)*, respectively.*

Proof. Let $E = C[a,b]$, $\|\cdot\|$ denote the sup norm of E,

$$P = \left\{ u(t) \in E : u(t) \geq \frac{M_1}{M_2} \|u\| \right\}.$$

Then P is a normal cone of E, the normal constant $N = 1$.

We define an operator $A : P \longrightarrow E$ by setting

$$Au(t) = \int_a^b K(t,s)g(s,u(s))ds, \quad \forall\ u \in P,$$

where $K(t,s)$ as in (4.2). By Lemma 4.1, it is easy to check that u is a solution of the problem (4.1) if and only if $u = Au$.

According to Lemma 4.2, we can know that $A : P \longrightarrow P$. It is easy to prove that A is a completely continuous and increasing operator. It follows from the monotonicity of A, (4.4) and (4.7) that

$$\begin{aligned} Au(t) &\leq \int_a^b K(t,s)g(s,\|u\|)ds \\ &\leq \int_a^b K(t,s)g\left(s,(\|u\|+1)^{\overline{\beta}}\right)ds \\ &\leq M_2(1+\|u\|)^{\overline{\beta}}\frac{\sinh(k(b-a))}{k(b-a)}\int_a^b (b-s)g(s,1)ds, \quad \forall\ u \in P. \end{aligned}$$

Therefore, in view of (4.8), $Au(t)$ is defined well.

It can be obtained by (4.4) and (4.8) that

$$Au(t) \leq M_2\frac{\sinh(k(b-a))}{k(b-a)}\int_a^b (b-s)g(s,1)ds < 1, \quad \forall\ 0 \leq u \leq 1,$$

thus, we have

$$\|Au\| < 1 = \frac{1}{N}, \quad \forall\ u \in P, \quad \|u\| = 1,$$

which implies (2.1) is satisfied.

Let

$$A_1u(t) = \int_a^b K(t,s)g_1(s,u(s))ds, \quad A_2u(t) = \int_a^b K(t,s)g_2(s,u(s))ds, \quad \forall\ u \in P.$$

From (4.9) and (4.11), we can know that A_1 is a τ_1-φ_1-concave operator and A_2 is a τ_2-φ_2-convex operator.

For any $u \in P \cap T_1$, we have $u(t) \geq \frac{M_1}{M_2}\|u\| = \frac{M_1}{M_2}$. Since there exists $\lambda_1 \in (a,b)$ such that $\tau_1(\lambda_1) = \frac{M_1}{M_2}$, it follows from (4.9) that

$$g_1\left(t,\frac{M_1}{M_2}\right) = g_1(t,\tau_1(\lambda_1)) \geq \varphi_1(\lambda_1)g_1(t,1), \quad \forall\ t \in (a,b),$$

which together with (4.4) and (4.8) implies

$$\begin{aligned} A_1u(t) &\geq \int_a^b K(t,s)g_1\left(s, \frac{M_1}{M_2}\right)ds \\ &\geq \varphi_1(\lambda_1)\int_a^b K(t,s)g_1(s,1)ds \\ &\geq \varphi_1(\lambda_1)\frac{M_1}{\cosh(k(b-a))}\int_a^b (b-s)g_1(s,1)ds \\ &> 0, \quad \forall\ u(t) \in P \cap T_1. \end{aligned}$$

Hence,

$$m_1 = \inf\{\|A_1x\| : x \in P, \|x\| = 1\} = \varphi_1(\lambda_1)\frac{M_1}{\cosh(k(b-a))}\int_a^b (b-s)g_1(s,1)ds.$$

Therefore, by (4.10), we have

$$\overline{\lim_{t\to a^+}}\frac{\varphi_1(t)}{\tau_1(t)} > \frac{1}{m_1}.$$

For any $u \in P \cap T_c$, we have $u(t) \geq \frac{M_1}{M_2}\|u\| = \frac{M_1}{M_2}c$. Since $c > \frac{M_2}{M_1}$, and there exists $\lambda_2 \in (a,b)$ such that $\tau_2(\lambda_2) = \frac{M_2}{M_1 c}$, according to (4.11), we obtain

$$g_2\left(t, \frac{M_1}{M_2}c\right) \geq \frac{1}{\varphi_2(\lambda_2)}g_2(t,1), \quad \forall\ t \in (a,b),$$

which together with (4.4) and (4.8) implies

$$\begin{aligned} A_2u(t) &\geq \int_a^b K(t,s)g_2\left(s, \frac{M_1}{M_2}c\right)ds \\ &\geq \frac{M_1}{\varphi_2(\lambda_2)\cosh(k(b-a))}\int_a^b (b-s)g_2(s,1)ds \\ &> 0, \quad \forall\ u(t) \in P \cap T_c. \end{aligned}$$

Hence,

$$m_2 = \inf\{\|A_2x\| : x \in P,\ \|x\| = c\} = \frac{M_1}{\varphi_2(\lambda_2)\cosh(k(b-a))}\int_a^b (b-s)g_2(s,1)ds.$$

Therefore, it follows from (4.12) that

$$\underline{\lim_{t\to a^+}}\frac{\varphi_2(t)}{\tau_2(t)} < \frac{m_2}{c}.$$

All the conditions of Theorem 2.1 are satisfied, and the conclusion of Theorem 4.1 follows from Theorem 2.1. This completes the proof of the theorem.

Example 4.1. Assume that $k - M_2 \sinh(k(b-a)) > 0$. Consider the following boundary value problem

$$\begin{cases} -u'' + k^2 u = \frac{u^{\frac{1}{3}}}{b-t} + \frac{xu^5}{b-t}, \quad a < t < b, \\ u'(a) = 0, \quad u(b) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases} \tag{4.13}$$

where $0 < x < \frac{k - M_2 \sinh(k(b-a))}{M_2 \sinh(k(b-a))}$.

In this example, set $g_1(t,u) = \frac{1}{b-t} u^{\frac{1}{3}}$, $g_2(t,u) = \frac{x}{b-t} u^5$, $\tau_1(t) = \tau_2(t) = \frac{t-a}{b-a}$, $\varphi_1(t) = [\tau_1(t)]^{\frac{1}{2}}$, $\varphi_2(t) = [\tau_2(t)]^3$. Then $\varphi_1(t) > \tau_1(t)$, $\varphi_2(t) < \tau_2(t)$, $t \in (a,b)$. For $u \geq 0$, it is easy to check that

$$g_1(t, \tau_1(\lambda)u) = \frac{1}{b-t}\left(\frac{\lambda - a}{b-a} u\right)^{\frac{1}{3}} \geq \left(\frac{\lambda - a}{b-a}\right)^{\frac{1}{2}} \frac{1}{b-t} u^{\frac{1}{3}} = \varphi_1(\lambda) g_1(t,u), \quad t \in (a,b),$$

$$\lim_{t \to a^+} \tau_1(t) = 0, \qquad \overline{\lim_{t \to a^+}} \left(\frac{t-a}{b-a}\right)^{-\frac{1}{2}} = +\infty.$$

$$g_2(t, \tau_2(\lambda)u) = \frac{x}{b-t}[\tau_2(\lambda)]^5 u^5 \leq [\tau_2(\lambda)]^3 \frac{x}{b-t} u^5 = \varphi_2(\lambda) g_2(t,u), \quad t \in (a,b),$$

$$\lim_{t \to a^+} \tau_2(t) = 0, \qquad \underline{\lim_{t \to a^+}} [\tau_2(t)]^2 = 0.$$

Furthermore, we can obtain

$$\int_a^b (b-s)\left(\frac{1}{b-s} + \frac{x}{b-s}\right) ds < \frac{k(b-a)}{M_2 \sinh(k(b-a))}.$$

We choose $\overline{\beta} = 7$, for any $0 < r < 1$, we have

$$r^7 \left(\frac{1}{b-t} u^{\frac{1}{3}} + \frac{x}{b-t} u^5\right) \leq \frac{1}{b-t}(ru)^{\frac{1}{3}} + \frac{x}{b-t}(ru)^5.$$

By Theorem 4.1, we can know that the BVP (4.13) has at least two nontrivial non-negative solutions $u_1(t)$ and $u_2(t)$ which satisfy the conclusion stated in Theorem 4.1.

Remark 4.1. In the above example, the existence of two solutions of a multi-point boundary value problem is discussed by using one of our results for τ_1-φ_1-concave operators and τ_2-φ_2-convex operators, which cannot be solved by means of previously available methods [4, 15-22].

References

[1] M.A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[2] A.J.B. Potter, *Applications of Hilbert's projective metric to certain classes of non-homogeneous operators*, Quart. J. Math. Oxford, **2**(1977), 93-99.

[3] Z.Q. Zhao, *Multiple positive fixed points for the sum operator of α-concave operator and β-convex operator and their applications*, J. Systems Sci. Math. Sci., **27**(2) (2007), 177-183 (in Chinese).

[4] Z.Q. Zhao, *Multiple fixed points of a sum operator and applications*, J. Math. Anal. Appl., **360**(2009), 1-6.

[5] C.B. Zhai, X.M. Cao, *Fixed point theorems for τ-φ-concave operators and applications*, Comput. Math. Appl., **59**(2010), 532-538.

[6] Z.Q. Zhao, *Fixed points of τ-φ-convex operators and applications*, Appl. Math. Lett., **23**(2010), 561-566.

[7] K. Li, J. Liang, T.J. Xiao, *A fixed point theorem for convex and decreasing operators*, Nonlinear Anal., **63**(2005), 209-216.

[8] Y.Z. Chen, *The existence of a fixed point for the sum of two monotone operators*, Positivity, **12**(2008), 643-652.

[9] D.J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[10] F.Y. Li, *Two point expansion fixed point theorem and solutions of convex concave operator equations with applications*, Acta Math. Sinica, **40**(3)(1997), 457-464 (in Chinese).

[11] A. Cabada, J.Á. Cid, *Existence of a non-zero fixed point for nondecreasing operators proved via Krasnoselskii's fixed point theorem*, Nonlinear Anal., **71**(2009), 2114-2118.

[12] D.J. Guo, V. Lakshmikantham, X.Z. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, 1996.

[13] M. Moshinsky, *Sobre los problems de condiciones a la frontiera en una dimension de caracteristicas discontinuas*, Bol. Soc. Mat. Mexicana, **7**(1950), 1-25.

[14] S. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

[15] A.R. Khan, J.R.L. Webb, *Existence of at least three solutions of a second-order three-point boundary value problem*, Nonlinear Anal., **64**(2006), 1356-1366.

[16] R.Y. Ma, D. O'Regan, *Nodal solutions for second-order m-point boundary value problems with nonlinearities across several eigenvalues*, Nonlinear Anal., **64**(2006), 1562-1577.

[17] B.P. Rynne, *Spectral properties and nodal solutions for second-order, m-point, boundary value problems*, Nonlinear Anal., **67**(2007), 3318-3327.

[18] J.R.L. Webb, *Remarks on a non-local boundary value problem*, Nonlinear Anal., **72**(2010), 1075-1077.

[19] X. Xu, D. O'Regan, J. Sun, *Multiplicity results for three-point boundary value problems with a non-well-ordered upper and lower solution condition*, Math. Comput. Modelling, **45**(2007), 189-200.

[20] L.J. Kong, Q.K. Kong, J.S.W. Wong, *Nodal solutions of multi-point boundary value problems*, Nonlinear Anal., **72**(2010), 382-389.

[21] B.M. Liu, L.S. Liu, Y.H. Wu, *Positive solutions for singular second order three-point boundary value problems*, Nonlinear Anal., **66**(2007), 2756-2766.

[22] Z.Q. Zhao, *Exact solutions of a class of second-order nonlocal boundary value problems and applications*, Appl. Math. Comput., **215**(2009), 1926-1936.

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