Fixed Point Theory, 13(2012), No. 1, 137-145
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON A HIGHER-ORDER m-POINT BOUNDARY VALUE PROBLEM

RODICA LUCA

Department of Mathematics, Gh. Asachi Technical University
11 Blvd. Carol I, Iasi 700506, Romania
E-mail: rluca@math.tuiasi.ro

Abstract. We study the existence and nonexistence of positive solutions for a nonlinear higher-order differential system subject to some m-point boundary conditions.

Key Words and Phrases: Higher-order differential system, boundary conditions, positive solutions, fixed point theorem.
2010 Mathematics Subject Classification: 34B10, 34B18, 47H10.

1. Introduction

We consider the n-th order nonlinear differential system

$$(S)\qquad \begin{cases} u^{(n)}(t) + b(t)f(v(t)) = 0, & t \in (0,T) \\ v^{(n)}(t) + c(t)g(u(t)) = 0, & t \in (0,T), \quad n \geq 2, \end{cases}$$

with the m-point boundary conditions

$$(BC)\qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) + b_0 \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \quad v(T) = \displaystyle\sum_{i=1}^{m-2} a_i v(\xi_i) + b_0, \end{cases}$$

where $m \in \mathbb{N}$, $m \geq 3$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$ and $a_i > 0$, $i = \overline{1, m-2}$.

The system (S) with $b(t) = \lambda \widetilde{b}(t)$, $c(t) = \mu \widetilde{c}(t)$ (denoted by $(\widetilde{S})$), $T = 1$ and the three-point nonlocal boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$, $v(1) = \alpha v(\eta)$, where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$, has been investigated in [2]. By using the Guo-Krasnoselskii fixed point theorem, the authors give sufficient conditions for λ and μ such positive solutions of the above problem exist. In the paper [5] the authors studied the existence of positive solutions to the n-th order m-point boundary value problem

$$u^{(n)}(t) + f(t, u, u') = 0, \ t \in (0,1),$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i),$$

by using the extension of Krasnoselkii's fixed point theorem in a cone. In [8] we give sufficient conditions for λ and μ such that the system $(\widetilde{S})$ with $n=2$ and the boundary conditions

$$(BC_0)\qquad \begin{cases} \beta u(0)-\gamma u'(0)=0,\;\; u(T)=\displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i)+b_0 \\ \beta v(0)-\gamma v'(0)=0,\;\; v(T)=\displaystyle\sum_{i=1}^{m-2} a_i v(\xi_i)+b_0,\;\; m\in\mathbb{N},\; m\geq 3, \end{cases}$$

for $b_0=0$, has positive solutions. In [9] we investigate the existence and nonexistence of positive solutions of the system (S) with $n=2$ and the boundary conditions (BC_0) with $b_0>0$. The discrete case of the $(\widetilde{S})$ for $n=2$, namely the system

$$\begin{cases} \Delta^2 u_{n-1}+\lambda b_n f(v_n)=0,\;\; n=\overline{1,N-1} \\ \Delta^2 v_{n-1}+\mu c_n g(u_n)=0,\;\; n=\overline{1,N-1},\;\; N\geq 2, \end{cases}$$

with the $m+1$ - point boundary conditions

$$\begin{cases} \beta u_0-\gamma\Delta u_0=0,\;\;\; u_N-\displaystyle\sum_{i=1}^{m-2} a_i u_{\xi_i}=0, \\ \beta v_0-\gamma\Delta v_0=0,\;\;\; v_N-\displaystyle\sum_{i=1}^{m-2} a_i v_{\xi_i}=0,\;\; m\geq 3, \end{cases}$$

where Δ is the forward difference operator with stepsize 1, $\Delta u_n=u_{n+1}-u_n$, and $\overline{k,m}\overset{def}{=}\{k,k+1,\ldots,m\}$ for $k,m\in\mathbb{N}$, has been studied in [7]. We also mention the paper [6] where the authors investigated the existence and nonexistence of positive solutions for the m-point boundary value problem on time scales

$$u^{\Delta\nabla}(t)+a(t)f(u(t))=0,\;\; t\in(0,T),$$

$$\beta u(0)-\gamma u^{\Delta}(0)=0,\;\; u(T)-\sum_{i=1}^{m-2} a_i u(\xi_i)=b,\;\; m\geq 3,\;\; b>0.$$

The multi-point boundary value problems for ordinary differential or difference equations have applications in a variety of different areas of applied mathematics and physics. For example the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see [12]); also many problems in the theory of elastic stability can be handled as multi-point problems (see [14]). The study of multi-point boundary value problems for second order differential equations was initiated by Il'in and Moiseev (see [3]-[4]). Since then such multi-point boundary value problems (continuous or discrete cases) have been studied by many authors (see for example [1], [10]-[11], [13], [15]-[16]), by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

Inspired by the work [6], in this paper we shall prove an existence result for the positive solutions of problem (S), (BC), by using the Schauder fixed point theorem.

We shall also give sufficient conditions for the nonexistence of the solutions for our problem.

We shall suppose that the following conditions are verified

(H1) $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < T$, $a_i > 0$ for $i = \overline{1, m-2}$,

$$d = T^{n-1} - \sum_{i=1}^{m-2} a_i \xi_i^{n-1} > 0, \ b_0 > 0.$$

(H2) The functions b, $c : [0,T] \to [0,\infty)$ are continuous and there exist t_0, $\widetilde{t}_0 \in [\xi_{m-2}, T)$ such that $b(t_0) > 0$, $c(\widetilde{t}_0) > 0$.

(H3) The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and satisfy the conditions

a) There exists $c_0 > 0$ such that $f(u) < \dfrac{c_0}{L}$, $g(u) < \dfrac{c_0}{L}$, for all $u \in [0, c_0]$.

b) $\lim\limits_{u\to\infty} \dfrac{f(u)}{u} = \infty$, $\lim\limits_{u\to\infty} \dfrac{g(u)}{u} = \infty$,

where

$$L = \max\left\{ \frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s)\, ds, \ \frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)\, ds \right\}.$$

2. Preliminary results

In this section we shall present some auxiliary results from [5] related to the following n-th order differential equation with boundary conditions

$$u^{(n)}(t) + y(t) = 0, \ \ 0 < t < T, \tag{1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{2}$$

Lemma 2.1. ([5]) *If* $d = T^{n-1} - \sum_{i=1}^{m-2} a_i \xi_i^{n-1} \neq 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$ *and* $y \in C([0,T])$ *then the solution of* (1), (2) *is given by*

$$\begin{aligned} u(t) \ &= \ \frac{t^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} y(s)\, ds \\ &- \ \frac{t^{n-1}}{d(n-1)!} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\, ds \\ &- \ \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s)\, ds, \ \ 0 \le t \le T. \end{aligned} \tag{3}$$

Lemma 2.2. ([5]) *Under the assumptions of Lemma 2.1, the Green function for the boundary value problem* (1), (2) *is given by*

$$G(t,s)=\begin{cases} \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1}-\displaystyle\sum_{i=j+1}^{m-2}a_i(\xi_i-s)^{n-1}\right]-\dfrac{1}{(n-1)!}(t-s)^{n-1}, \\ \qquad if\ \ \xi_j\le s<\xi_{j+1},\ \ s\le t, \\ \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1}-\displaystyle\sum_{i=j+1}^{m-2}a_i(\xi_i-s)^{n-1}\right], \\ \qquad if\ \ \xi_j\le s<\xi_{j+1},\ \ s\ge t,\ \ j=\overline{0,m-3}, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1}-\dfrac{1}{(n-1)!}(t-s)^{n-1},\ \ if\ \ \xi_{m-2}\le s\le T,\ \ s\le t, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1},\ \ if\ \ \xi_{m-2}\le s\le T,\ \ s\ge t,\ \ (\xi_0=0). \end{cases}$$

Using the Green function, the solution of problem (1),(2) is given by

$$u(t)=\int_0^T G(t,s)y(s)\,ds.$$

Lemma 2.3. ([5]) *If* $a_i>0$ *for all* $i=\overline{1,m-2}$, $0<\xi_1<\cdots<\xi_{m-2}<T$ *and* $d>0$, *then* $G(t,s)\ge 0$ *for all* t, $s\in[0,T]$.

Lemma 2.4. ([5]) *If* $a_i>0$ *for all* $i=\overline{1,m-2}$, $0<\xi_1<\cdots<\xi_{m-2}<T$, $d>0$ *and* $y\in C([0,T])$, $y(t)\ge 0$ *for all* $t\in[0,T]$, *then the unique solution* u *of problem* (1), (2) *satisfies* $u(t)\ge 0$ *for all* $t\in[0,T]$.

Lemma 2.5. *If* $a_i>0$ *for all* $i=\overline{1,m-2}$, $0<\xi_1<\cdots<\xi_{m-2}<T$, $d>0$, $y\in C([0,T])$, $y(t)\ge 0$ *for all* $t\in[0,T]$, *then the solution of problem* (1), (2) *satisfies*

$$\begin{cases} u(t)\le \dfrac{T^{n-1}}{d(n-1)!}\displaystyle\int_0^T (T-s)^{n-1}y(s)\,ds,\ \ \forall t\in[0,T], \\ u(\xi_j)\ge \dfrac{\xi_j^{n-1}}{d(n-1)!}\displaystyle\int_{\xi_{m-2}}^T (T-s)^{n-1}y(s)\,ds,\ \ \forall\, j=\overline{1,m-2}. \end{cases}$$

Proof. By (3) we have

$$u(t)\le \frac{t^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds\le \frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds,$$

for all $t\in[0,T]$.

Then by using Lemma 2.2 and Lemma 2.3 we obtain

$$\begin{aligned} u(\xi_j) &= \int_0^T G(\xi_j,s)y(s)\,ds\ge \int_{\xi_{m-2}}^T G(\xi_j,s)y(s)\,ds \\ &= \frac{\xi_j^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1}y(s)\,ds, \end{aligned}$$

for all $j=\overline{1,m-2}$. □

Lemma 2.6. ([5]) *We assume that* $0 < \xi_1 < \cdots < \xi_{m-2} < T$*,* $a_i > 0$ *for all* $i = \overline{1, m-2}$*,* $d > 0$ *and* $y \in C([0,T])$*,* $y(t) \geq 0$ *for all* $t \in [0,T]$*. Then the solution of problem* (1), (2) *verifies* $\inf_{t \in [\xi_{m-2}, T]} u(t) \geq \gamma \|u\|$*, where*

$$\gamma = \begin{cases} \min\left\{ \dfrac{a_{m-2}(T-\xi_{m-2})}{T - a_{m-2}\xi_{m-2}}, \dfrac{a_{m-2}\xi_{m-2}^{n-1}}{T^{n-1}} \right\}, & if \ \sum_{i=1}^{m-2} a_i < 1, \\ \min\left\{ \dfrac{a_1 \xi_1^{n-1}}{T^{n-1}}, \dfrac{\xi_{m-2}^{n-1}}{T^{n-1}} \right\}, & if \ \sum_{i=1}^{m-2} a_i \geq 1 \end{cases}$$

and $\|u\| = \sup_{t \in [0,T]} |u(t)|$.

3. Main results

First we shall present an existence result for the positive solutions of (S), (BC).

Theorem 3.1. *Assume that the assumptions (H1), (H2), (H3)a hold. Then the problem* (S), (BC) *has at least one positive solution for* $b_0 > 0$ *sufficiently small.*

Proof. We consider the problem

$$\begin{cases} h^{(n)}(t) = 0, \quad t \in (0,T) \\ h(0) = h'(0) = \cdots = h^{(n-2)}(0) = 0, \quad h(T) = \sum_{i=1}^{n-2} a_i h(\xi_i) + 1. \end{cases} \tag{4}$$

The solution $h(t), \ t \in (0,T)$ of equation $(4)_1$ is

$$h(t) = \frac{C_1 t^{n-1}}{(n-1)!} + \frac{C_2 t^{n-2}}{(n-2)!} + \cdots + C_{n-1} t + C_n.$$

Because $h(0) = \cdots = h^{(n-2)}(0) = 0$ we obtain $C_2 = \cdots = C_n = 0$, so $h(t) = C_1 t^{n-1}/(n-1)!$. By the condition $h(T) = \sum_{i=1}^{m-2} a_i h(\xi_i) + 1$ we obtain

$$\frac{C_1 T^{n-1}}{(n-1)!} = \sum_{i=1}^{m-2} a_i \frac{C_1 \xi_i^{n-1}}{(n-1)!} + 1 \ \text{ or } \ C_1 \left(T^{n-1} - \sum_{i=1}^{m-2} a_i \xi_i^{n-1} \right) = (n-1)!.$$

Hence $C_1 = (n-1)!/d$. So

$$h(t) = \frac{t^{n-1}}{d}, \quad t \in [0,T]. \tag{5}$$

We define the functions $x(t), \ y(t), \ t \in [0,T]$ by

$$x(t) = u(t) - b_0 h(t), \quad y(t) = v(t) - b_0 h(t), \quad t \in [0,T].$$

Then (S), (BC) can be equivalently written as

$$\begin{cases} x^{(n)}(t) + b(t) f(y(t) + b_0 h(t)) = 0 \\ y^{(n)}(t) + c(t) g(x(t) + b_0 h(t)) = 0, \quad t \in (0,T) \end{cases} \tag{6}$$

with the boundary conditions

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \ \ x(T) = \displaystyle\sum_{i=1}^{m-2} a_i x(\xi_i) \\ y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \ \ y(T) = \displaystyle\sum_{i=1}^{m-2} a_i y(\xi_i). \end{cases} \tag{7}$$

Using the Green function given in Lemma 2.2, a pair $(x(t), y(t))$ is a solution of problem (6), (7) if and only if

$$\begin{cases} x(t) = \displaystyle\int_0^T G(t,s)b(s)f\left(\int_0^T G(s,\tau)c(\tau)g(x(\tau)+b_0h(\tau))\,d\tau + b_0h(s)\right)ds, \\ y(t) = \displaystyle\int_0^T G(t,s)c(s)g(x(s)+b_0h(s))\,ds, \ \ 0 \le t \le T, \end{cases} \tag{8}$$

where $h(t),\ t \in [0,T]$ is given by (5).

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$ and we define the set

$$K = \{x \in C([0,T]), \ \ 0 \le x(t) \le c_0, \ \ \forall\, t \in [0,T]\} \subset X.$$

We also define the operator $\Lambda : K \to X$ by

$$\Lambda(x)(t) = \int_0^T G(t,s)b(s)f\left(\int_0^T G(s,\tau)c(\tau)g(x(\tau)+b_0h(\tau))d\tau + b_0h(s)\right)ds, \ 0 \le t \le T.$$

For sufficiently small $b_0 > 0$, by $(H3)a$ we deduce

$$f(y(t)+b_0h(t)) \le \frac{c_0}{L}, \ \ g(x(t)+b_0h(t)) \le \frac{c_0}{L}, \ \ \forall\, x,\, y \in K, \ \ \forall\, t \in [0,T].$$

Then for any $x \in K$ we have, by using Lemma 2.4, that $\Lambda(x)(t) \ge 0, \ \forall\, t \in [0,T]$. By Lemma 2.5 we also have

$$\begin{aligned} y(s) &\le \frac{T^{n-1}}{d(n-1)!}\int_0^T (T-\tau)^{n-1}c(\tau)g(x(\tau)+b_0h(\tau))\,d\tau \\ &\le \frac{c_0}{L}\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-\tau)^{n-1}c(\tau)\,d\tau \le \frac{c_0}{L}L = c_0, \ \ \forall\, s \in [0,T] \end{aligned}$$

and

$$\begin{aligned} \Lambda(x)(t) &\le \frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}b(s)f(y(s)+b_0h(s))\,ds \\ &\le \frac{c_0}{L}\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}b(s)ds \le \frac{c_0}{L}L = c_0, \ \ \forall\, t \in [0,T]. \end{aligned}$$

Therefore $\Lambda(K) \subset K$.

Using standard arguments we deduce that Λ is completely continuous (continuous and compact). By the Schauder fixed point theorem, we conclude that Λ has a fixed point $x \in K$. This element together with y given by (8) represent a solution for (6) and (7). This shows that our problem (S), (BC) has a positive solution $u = x + b_0 h,\ v = y + b_0 h$ for sufficiently small b_0. □

In what follows we shall present sufficient conditions for nonexistence of positive solutions of (S), (BC).

Theorem 3.2. *Assume that the assumptions (H1),(H2), $(H3)b$ hold. Then the problem (S), (BC) has no positive solution for b_0 sufficiently large.*

Proof. We suppose that (u, v) is a positive solution of (S), (BC) . Then

$$x = u - b_0 h,\ y = v - b_0 h$$

is solution for (6), (7), where h is the solution of problem (4). By Lemma 2.4 we have $x(t) \geq 0,\ y(t) \geq 0,\ \forall\, t \in [0, T]$, and by $(H2)$ we deduce that $\|x\| > 0,\ \|y\| > 0$. Using Lemma 2.6 we also have

$$\inf_{t\in[\xi_{m-2},T]} x(t) \geq \gamma\|x\| \ \text{ and } \ \inf_{t\in[\xi_{m-2},T]} y(t) \geq \gamma\|y\|,$$

where γ is defined in Lemma 2.6.

Using now (5) - the expression for h, we deduce that

$$\inf_{t\in[\xi_{m-2},T]} h(t) = \frac{\xi_{m-2}^{n-1}}{d} = \frac{\xi_{m-2}^{n-1}}{T^{n-1}} \cdot \frac{T^{n-1}}{d}.$$

So

$$\inf_{t\in[\xi_{m-2},T]} h(t) = \frac{\xi_{m-2}^{n-1}}{T^{n-1}}\|h\| \geq \gamma\|h\|.$$

Then

$$\begin{aligned}\inf_{t\in[\xi_{m-2},T]}(x(t) + b_0 h(t)) &\geq \inf_{t\in[\xi_{m-2},T]} x(t) + b_0 \inf_{t\in[\xi_{m-2},T]} h(t)\\ &\geq \gamma(\|x\| + b_0\|h\|) \geq \gamma\|x + b_0 h\|\end{aligned}$$

and

$$\begin{aligned}\inf_{t\in[\xi_{m-2},T]}(y(t) + b_0 h(t)) &\geq \inf_{t\in[\xi_{m-2},T]} y(t) + b_0 \inf_{t\in[\xi_{m-2},T]} h(t)\\ &\geq \gamma(\|y\| + b_0\|h\|) \geq \gamma\|y + b_0 h\|.\end{aligned}$$

We now consider

$$R = \frac{d(n-1)!}{\gamma\xi_{m-2}^{n-1}}\left(\min\left\{\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)\,ds, \int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)\,ds\right\}\right)^{-1} > 0.$$

By $(H3)b$, for R defined above we deduce that there exists $M > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$, for all $u \geq M$.

We consider $b_0 > 0$ sufficiently large such that

$$\inf_{t\in[\xi_{m-2},T]}(x(t) + b_0 h(t)) \geq M \ \text{ and } \ \inf_{t\in[\xi_{m-2},T]}(y(t) + b_0 h(t)) \geq M.$$

By using Lemma 2.5 and the above considerations, we have

$$\begin{aligned}
y(\xi_{m-2}) &\geq \frac{\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)g(x(s)+b_0h(s))\,ds\\
&\geq \frac{\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)\cdot 2R(x(s)+b_0h(s))\,ds\\
&\geq \frac{\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)\cdot 2R\inf_{\tau\in[\xi_{m-2},T]}(x(\tau)+b_0h(\tau))\,ds\\
&\geq \frac{2R\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)\,ds\cdot\|x+b_0h\| \geq 2\|x+b_0h\| \geq 2\|x\|.
\end{aligned}$$

Therefore we obtain

$$\|x\| \leq \frac{1}{2}y(\xi_{m-2}) \leq \frac{1}{2}\|y\|. \tag{9}$$

In a similar manner we deduce $x(\xi_{m-2}) \geq 2\|y+b_0h\| \geq 2\|y\|$ and so

$$\|y\| \leq \frac{1}{2}x(\xi_{m-2}) \leq \frac{1}{2}\|x\|. \tag{10}$$

By (9) and (10) we obtain $\|x\| \leq \frac{1}{2}\|y\| \leq \frac{1}{4}\|x\|$, which is a contradiction, because $\|x\| > 0$. Then, when b_0 is sufficiently large, our problem (S), (BC) has no positive solution. □

4. An example

We consider $T = 1$, $b(t) = bt$, $c(t) = ct$, $t \in [0,1]$, $b, c > 0$, $n = 3$, $m = 5$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{2}{3}$, $a_1 = 1$, $a_2 = \frac{1}{2}$. Then $d = 1 - \sum_{i=1}^{2} a_i\xi_i^2 = \frac{2}{3} > 0$.

We also consider the functions $f, g : [0,\infty) \to [0,\infty)$, $f(x) = \frac{\widetilde{a}x^3}{x+1}$, $g(x) = \frac{\widetilde{b}x^3}{x+1}$ with $\widetilde{a}, \widetilde{b} > 0$. We have $\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{g(x)}{x} = \infty$. The constant L from $(H3)$ is in this case

$$L = \max\left\{\frac{1}{2d}\int_0^1 (1-s)^2 bs\,ds, \frac{1}{2d}\int_0^1 (1-s)^2 cs\,ds\right\} = \frac{1}{16}\max\{b,c\}.$$

We choose $c_0 = 1$ and if we select $\widetilde{a}$ and $\widetilde{b}$ satisfying the conditions

$$\widetilde{a} < \frac{2}{L} = \frac{32}{\max\{b,c\}} = 32\min\left\{\frac{1}{b},\frac{1}{c}\right\},\ \ \widetilde{b} < \frac{2}{L} = 32\min\left\{\frac{1}{b},\frac{1}{c}\right\},$$

then we obtain $f(x) \leq \frac{\widetilde{a}}{2} < \frac{1}{L}$, $g(x) \leq \frac{\widetilde{b}}{2} < \frac{1}{L}$, for all $x \in [0,1]$.

Thus all the assumptions $(H1)-(H3)$ are verified. By Theorem 3.1 and Theorem 3.2 we deduce that the nonlinear third-order differential system

$$\begin{cases} u'''(t) + bt\dfrac{\widetilde{a}v^3(t)}{v(t)+1} = 0 \\ v'''(t) + ct\dfrac{\widetilde{b}u^3(t)}{u(t)+1} = 0, \quad t \in (0,1) \end{cases}$$

with the boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \quad u(1) = u(\frac{1}{3}) + \frac{1}{2}u(\frac{2}{3}) + b_0 \\ v(0) = v'(0) = 0, \quad v(1) = v(\frac{1}{3}) + \frac{1}{2}v(\frac{2}{3}) + b_0, \end{cases}$$

has at least one positive solution for sufficiently small $b_0 > 0$ and no positive solution for sufficiently large b_0.

References

[1] J.R. Graef, J. Henderson, B. Yang, *Positive solutions of a nonlinear higher order boundary-value problem*, Electron. J. Differ. Equ., **2007**(45)(2007), 1-10.

[2] J. Henderson, S. K. Ntouyas, *Positive solutions for systems of nth order three-point nonlocal boundary value problems*, Electron. J. Qual. Theory Differ. Equ., **(2007)**(18)(2007), 1-12.

[3] V.A. Il'in, E.I. Moiseev, *Nonlocal boundary value problems of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differ. Equ., **23**(1987), 803-810.

[4] V.A. Il'in, E.I. Moiseev, *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator*, Differ. Equ., **23**(1987), 979-987.

[5] Y. Ji, Y. Guo, C. Yu, *Positive solutions to $(n-1,n)$ m-point boundary value problemss with dependence on the first order derivative*, Appl. Math. Mech., Engl. Ed., **30**(2009), 527-536.

[6] W.T. Li, H.R. Sun, *Positive solutions for second-order m-point boundary value problems on times scales*, Acta Math. Sin., Engl. Ser., **22**(2006), 1797-1804.

[7] R. Luca, *Positive solutions for $m+1$-point discrete boundary value problems*, Libertas Math., **29**(2009), 65-82.

[8] R. Luca, *Positive solutions for a second-order m-point boundary value problems*, Dyn. Contin. Discrete Impuls. Syst., **18**(2011), 161-176.

[9] R. Luca, *On a class of m-point boundary value problems*, Math. Bohem., in press.

[10] R. Ma, *Positive solutions for second order three-point boundary value problems*, Appl. Math. Lett., **14**(2001), 1-5.

[11] R. Ma, Y. Raffoul, *Positive solutions of three-point nonlinear discrete second order boundary value problem*, J. Difference Eq. Appl., **10**(2004), 129-138.

[12] M. Moshinsky, *Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas*, Bol. Soc. Mat. Mex., **7**(1950), 1-25.

[13] S.K. Ntouyas, *Nonlocal initial and boundary value problems: a survey, Handbook of differential equations: Ordinary differential equations*, Vol. II, 461-557, Elsevier, Amsterdam, 2005.

[14] S. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

[15] J.R.L. Webb, *Positive solutions of some three point boundary value problems via fixed point index theory*, Nonlinear Anal., **47**(2001), 4319-4332.

[16] Y. Zhou, Y. Xu, *Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations*, J. Math. Anal. Appl., **320**(2006), 578-590.

Received: March 22, 2010; Accepted: October 13, 2010.