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# A MULTIPLICITY RESULT OF POSITIVE SOLUTIONS FOR THIRD-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

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**Abstract.** By using fixed point theorem, multiple positive solutions for third-order multi-point boundary value problem with nonlinearity depending on all order derivative are established. The associated Green's function is also given.

**Key Words and Phrases**: Multi-point boundary value problem, positive solution, cone, fixed point.
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## 1. Introduction

Third order nonlinear differential equations arise in a variety of different areas of applied mathematics and physics [1]. Recently, many authors studied the existence and multiplicity of positive solutions for two-point or three-point boundary value problem for nonlinear third-order ordinary differential equations, see in [2-10]. For examples, Anderson [2] established the existence of at least three positive solution of problem

$$x'''(t) = f(x(t)),\ t \in (0,1)$$
$$x(0) = x'(t_2) = x''(1) = 0, \quad t_2 \in (0,1),$$

where $f : R \to [0, +\infty)$ is continuous and $1/2 \leq t_2 < 1$. By using Guo-Krasnoselski$\breve{i}$ fixed point theorem, Palamides and Smyrlis [4] obtained the existence of positive solution for third-order three-point boundary value problem

$$x'''(t) = a(t)f(t,\ x(t)),\ t \in (0,1)$$
$$x''(\eta) = 0,\ x(0) = x(1) = 0,\ \eta \in (0,1).$$

But in all these work ([2-10]), the first and second order derivative are not involved in the nonlinear term. Furthermore, all these work are concentrated on the two or three point boundary conditions. Few paper deals with the existence of positive solutions to m-point problems for third-order differential equations which the first and second order derivative are involved in the nonlinear term explicitly.

In this paper, we consider the positive solutions for multi-point third-order boundary value problem

$$\begin{cases} x'''(t) = f(t,\ x(t),\ x'(t),\ x''(t)),\ t \in [0,1] \\ x''(1) = 0,\ x'(1) = 0,\ x(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1.1}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \beta_i < 1,\ i = 1, 2, \cdots, m-2,\ \sum_{i=1}^{m-2} \beta_i < 1$ and $f \in C([0,1] \times R^3, [0, +\infty))$. Positive solutions for this problem have not been considered before. By using Avery and Peterson fixed point [11], existence of at least three concave positive solutions for problem (1.1) are established. The results established in this paper are general than the papers before.

## 2. Definitions and Lemmas

**Definition 2.1.** The map $\alpha$ is said to be a nonnegative continuous convex functional on cone $P$ of a real Banach space $E$ provided that $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y),\ \text{for all}\ x,\ y \in P\ \text{and}\ t \in [0,1].$$

**Definition 2.2.** The map $\beta$ is said to be a nonnegative continuous concave functional on cone $P$ of a real Banach space $E$ provided that $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \ge t\beta(x) + (1-t)\beta(y),\ \text{for all}\ x,\ y \in P\ \text{and}\ t \in [0,1].$$

Let $\gamma, \theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$ and $\psi$ be a nonnegative continuous functional on P. Then for positive numbers $a,\ b,\ c$ and $d$, we define the following convex sets:

$$P(\gamma,\ d) = \{x \in P | \gamma(x) < d\},$$

$$P(\gamma,\ \alpha,\ b,\ d) = \{x \in P | b \le \alpha(x),\ \gamma(x) \le d\},$$

$$P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d) = \{x \in P | b \le \alpha(x),\ \theta(x) \le c,\ \gamma(x) \le d\}$$

and a closed set

$$R(\gamma,\ \psi,\ a,\ d) = \{x \in P | a \le \psi(x),\ \gamma(x) \le d\}.$$

**Lemma 2.1.** Let $P$ be a cone in Banach space E. Let $\gamma,\ \theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$ and $\psi$ be a nonnegative continuous functional on $P$ satisfying:

$$\psi(\lambda x) \le \lambda \psi(x),\ for\ 0 \le \lambda \le 1, \tag{2.1}$$

such that for some positive numbers $l$ and d,

$$\alpha(x) \le \psi(x),\ \|x\| \le l\gamma(x) \tag{2.2}$$

for all $x \in \overline{P(\gamma,\ d)}$. Suppose $T : \overline{P(\gamma,\ d)} \to \overline{P(\gamma,\ d)}$ is completely continuous and there exist positive numbers $a,\ b,\ c$ with $a < b$ such that
$(S_1)$ $\{x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d) | \alpha(x) > b\} \ne \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$;
$(S_2)$ $\alpha(Tx) > b$ for $x \in P(\gamma,\ \alpha,\ b,\ d)$ with $\theta(Tx) > c$;

$(S_3)$ $0 \notin R(\gamma,\ \psi,\ a,\ d)$ and $\psi(Tx) < a$ for $x \in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(x) = a$.
Then $T$ has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\gamma,\ d)}$ such that:

$$\gamma(x_i) \le d,\ i = 1,\ 2,\ 3;\ b < \alpha(x_1); a < \psi(x_2),\ \alpha(x_2) < b;\ \psi(x_3) < a. \tag{2.3}$$

## 3. Main results

Consider the third-order m-point boundary value problem

$$x'''(t) = y(t),\ t \in [0,1] \tag{3.1}$$

$$x''(1) = 0,\ x'(1) = 0,\ x(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \tag{3.2}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \beta_i < 1,\ i = 1, 2, \cdots, m-2$, and

$$\sum_{i=1}^{m-2} \beta_i < 1.$$

**Lemma 3.1.** Denote $\xi_0 = 0,\ \xi_{m-1} = 1,\ \beta_0 = \beta_{m-1} = 0,\ y(t) \in C[0,1]$, problem (3.1), (3.2) has the unique solution

$$x(t) = \int_0^1 G(t,s) y(s) ds,$$

where for $i = 1,\ 2,\ \cdots,\ m-1$,

$$G(t,s) = \frac{1}{1 - \sum\limits_{i=0}^{m-1} \beta_k} \begin{cases} (1 - \sum\limits_{i=0}^{m-1} \beta_i)(-\frac{t^2}{2} + st) \\ -\frac{1}{2} \sum\limits_{k=0}^{i-1} \beta_k \xi_k^2 + \sum\limits_{k=0}^{i-1} \beta_k \xi_k s + \frac{1}{2} \sum\limits_{k=i}^{m-1} \beta_k s^2, & \xi_{i-1} \le s \le \xi_i \\ \frac{s^2}{2} - \frac{1}{2} \sum\limits_{k=0}^{i-1} \beta_k \xi_k^2 + \sum\limits_{k=0}^{i-1} \beta_k \xi_k s - \frac{1}{2} \sum\limits_{k=0}^{i-1} \beta_k s^2, & \xi_{i-1} \le s \le \xi_i \end{cases}$$

*Proof.* Let $G(t,s)$ is the Green's function of problem $x'''(t) = 0$ with boundary condition (3.2). We can suppose

$$G(t,s) = \begin{cases} a_2 t^2 + a_1 t + a_0 & t \le s,\ \xi_{i-1} \le s \le \xi_i,\ i = 1,\ 2,\ \cdots,\ m-1 \\ b_2 t^2 + b_1 t + b_0 & t \ge s,\ \xi_{i-1} \le s \le \xi_i,\ i = 1,\ 2,\ \cdots,\ m-1 \end{cases}$$

Considering the definition and properties of Green's function together with the boundary condition (3.2), we have

$$\begin{cases} a_2 s^2 + a_1 s + a_0 = b_2 s^2 + b_1 s + b_0 \\ 2a_2 s + a_1 = 2b_2 s + b_1 \\ 2a_2 - 2b_2 = -1 \\ b_2 = 0 \\ 2a_2 s + a_1 = 0 \\ a_0 = \sum\limits_{k=0}^{i-1} \beta_k (a_2 \xi_k^2 + a_1 \xi_k + a_0) + \sum\limits_{k=i}^{m-1} \beta_k (b_2 \xi_k^2 + b_1 \xi_k + b_0) \end{cases}$$

Hence

$$a_2 = -\frac{1}{2}, a_1 = s, b_2 = b_1 = 0,$$

$$a_0 = \frac{-\frac{1}{2}\sum_{k=0}^{i-1}\beta_k\xi_k^2 + \sum_{k=0}^{i-1}\beta_k\xi_k s + \frac{1}{2}\sum_{k=i}^{m-1}\beta_k s^2}{1-\sum_{i=0}^{m-1}\beta_i},$$

$$b_0 = \frac{\frac{1}{2}s^2 - \frac{1}{2}\sum_{k=0}^{i-1}\beta_k\xi_k^2 + \sum_{k=0}^{i-1}\beta_k\xi_k s - \frac{1}{2}\sum_{k=0}^{i-1}\beta_k s^2}{1-\sum_{i=0}^{m-1}\beta_i}$$

These give the explicit expression of the Green's function of $G(t,s)$. The proof of Lemma 3.1 is completed.

**Lemma 3.2.** The Green's function $G(t,\ s)$ satisfies that $G(t,s) \geq 0,\ t,\ s \in [0,1]$.

*Proof.* For $\xi_{i-1} \leq s \leq \xi_i,\ i = 1,\ 2,\ \cdots,\ m-1$ and $t \leq s$,

$$(1-\sum_{i=0}^{m-1}\beta_k)G(t,s) \geq (1-\sum_{i=0}^{m-1}\beta_k)G(0,s) = -\frac{1}{2}\sum_{k=0}^{i-1}\beta_k\xi_k^2 + \sum_{k=0}^{i-1}\beta_k\xi_k s + \frac{1}{2}\sum_{k=i}^{m-1}\beta_k s^2$$

$$\geq \sum_{k=0}^{i-1}\beta_k\xi_k(s-\xi_k) + \frac{1}{2}\sum_{k=i}^{m-1}\beta_k s^2 \geq 0.$$

For $\xi_{i-1} \leq s \leq \xi_i,\ i = 1,\ 2,\ \cdots,\ m-1$ and $t \geq s$,

$$(1-\sum_{i=0}^{m-1}\beta_k)G(t,s) = \frac{s^2}{2} - \frac{1}{2}\sum_{k=0}^{i-1}\beta_k\xi_k^2 + \sum_{k=0}^{i-1}\beta_k\xi_k s - \frac{1}{2}\sum_{k=0}^{i-1}\beta_k s^2$$

$$\geq \frac{1}{2}\sum_{k=0}^{i-1}\beta_k[s^2-(\xi_k-s)^2] \geq 0$$

These gives that $G(t,s) \geq 0,\ t,\ s \in [0,1]$.

**Lemma 3.3.** If $y(t) \geq 0, t \in [0,1]$, $x(t)$ is the solution of problem (3.1), (3.2), we claim that

(1) $\min_{0\leq t\leq 1}|x(t)| \geq \delta \max_{0\leq t\leq 1}|x(t)|$,

(2) $\max_{0\leq t\leq 1}|x(t)| \leq \gamma \max_{0\leq t\leq 1}|x'(t)|$,

where $\delta = \sum_{i=1}^{m-2}\beta_i\xi_i/(1-\sum_{i=1}^{m-2}\beta_i(1-\xi_i)), \gamma = (1-\sum_{i=1}^{m-2}\beta_i(1-\xi_i))/(1-\sum_{i=1}^{m-2}\beta_i)$ are positive constants.

*Proof.* (1) For $x'''(t) = y(t) \geq 0,\ t \in [0,1]$, we see that $x''(t)$ is increasing on [0,1]. Considering $x''(1) = 0$, we have $x''(t) \leq 0,\ t \in (0,1)$. This together with $x'(1) = 0$,

we get that $\max\limits_{0\le t\le 1} x(t) = x(1)$ and $\min\limits_{0\le t\le 1} x(t) = x(0)$.
From the concavity of $x(t)$, we have

$$\xi_i(x(1) - x(0)) \le x(\xi_i) - x(0).$$

Multiplying both sides with $\beta_i$ and considering the boundary condition, we have

$$\sum_{i=1}^{m-2} \beta_i \xi_i x(1) \le (1 - \sum_{i=1}^{m-2} \beta_i(1-\xi_i))x(0). \tag{3.3}$$

(2) Considering the mean-value theorem we get

$$x(\xi_i) - x(0) = \xi_i x'(\eta_i),\ \eta \in (\xi_i, 1).$$

From the concavity of $x$ similarly with above we know

$$(1 - \sum_{i=1}^{m-2} \beta_i)x(0) < \sum_{i=1}^{m-2} \beta_i \xi_i x'(0). \tag{3.4}$$

Considering (3.3) together with (3.4) we have $x(0) \le \gamma |x'(0)|$. These give the proof of Lemma 3.3.
Furthermore, for $x'(t) = x'(1) - \displaystyle\int_t^1 x''(s)ds$ and $x'(1) = 0$, we get

$$|x'(t)| = |\int_t^1 x''(s)ds| \le \int_0^1 |x''(s)|ds.$$

Thus

$$\max_{0\le t\le 1} |x'(t)| \le \max_{0\le t\le 1} |x''(t)|.$$

**Remark.** Above conclusion with Lemma 3.3 ensure that

$$\max\{\max_{0\le t\le 1} |x(t)|,\ \max_{0\le t\le 1} |x'(t)|,\ \max_{0\le t\le 1} |x''(t)|\} \le \gamma \max_{0\le t\le 1} |x''(t)|.$$

Let Banach space $E = C^2[0,\ 1]$ be endowed with the norm

$$\|x\| = \max\{\max_{0\le t\le 1} |x(t)|,\ \max_{0\le t\le 1} |x'(t)|,\ \max_{0\le t\le 1} |x''(t)|\},\ x \in E.$$

Define the cone $P \subset E$ by

$$P = \{x \in E | x(t) \ge 0,\ x''(1) = 0,\ x'(1) = 0,$$
$$x(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i),\ x(t) \text{ is concave on } [0,1]\}.$$

Let the nonnegative continuous concave functional $\alpha$, the nonnegative continuous convex functionals $\gamma,\ \theta$ and the nonnegative continuous functional $\psi$ be defined on the cone by

$$\gamma(x) = \max_{0\le t\le 1} |x''(t)|,\ \theta(x) = \psi(x) = \max_{0\le t\le 1} |x(t)|,\ \alpha(x) = \min_{0\le t\le 1} |x(t)|.$$

By Lemma 3.3, the functionals defined above satisfy:

$$\delta\theta(x) \le \alpha(x) \le \theta(x) = \psi(x),\ \|x\| \le \gamma\gamma(x). \tag{3.5}$$

Let

$$m = \int_0^1 G(0,s)ds,\ N = \int_0^1 G(1,s)ds,\ \lambda = \min\{m,\ \delta\gamma\}.$$

Assume that there exist constants $0 < a,\ b,\ d$ with $a < b < \lambda d$ such that

$$A_1)\ f(t,\ u,\ v,\ w) \leq d,\quad (t,\ u,\ v) \in [0,1] \times [0,\ \gamma d] \times [-d,\ 0] \times [0,\ d],$$

$$A_2)\ f(t,\ u,\ v,\ w) > b/m,\ (t,\ u,\ v) \in [0,\ 1] \times [b,\ b/\delta] \times [-d,\ 0] \times [0,\ d],$$

$$A_3)\ f(t,\ u,\ v,\ w) < a/N,\ (t,\ u,\ v) \in [0,\ 1] \times [0,\ a] \times [-d,\ 0] \times [0,\ d].$$

**Theorem.** Under assumptions $A_1) - A_3)$, problem (1.1) has at least three positive solutions $x_1,\ x_2,\ x_3$ satisfying

$$\max_{0\leq t\leq 1} |x_i''(t)| \leq d,\ i = 1,\ 2,\ 3;\ b < \min_{0\leq t\leq 1} |x_1(t)|;$$

$$a < \max_{0\leq t\leq 1} |x_2(t)|, \min_{0\leq t\leq 1} |x_2(t)| < b; \max_{0\leq t\leq 1} |x_3(t)| \leq a. \tag{3.6}$$

*Proof.* Problem (1.1) has a solution $x = x(t)$ if and only if $x$ solves the operator equation

$$x(t) = \int_0^1 G(t,s) f(s,\ x(s),\ x'(s),\ x''(s))ds = (Tx)(t).$$

By a simple computation, we have

$$(Tx)''(t) = -\int_t^1 f(s,\ x,\ x',\ x'')ds$$

For $x \in \overline{P(\gamma,\ d)}$, considering Lemma 3.3 and assumption $A_1)$, we have

$$f(t,\ x(t),\ x'(t),\ x''(\tau)) \leq d.$$

Thus

$$\gamma(Tx) = |(Tx)''(0)| = |-\int_0^1 f(s,\ x,\ x',\ x'')ds| = \int_0^1 |f(s,\ x,\ x',\ x'')|ds \leq d.$$

Hence $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ and $T$ is a completely continuous operator obviously.
The fact that the constant function $x(t) = b/\delta \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$ and $\alpha(b/\delta) > b$ implies that

$$\{x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d | \alpha(x) > b)\} \neq \emptyset.$$

This ensures that condition (S1) of Lemma 2.1 holds.
For $x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$, we have $b \leq x(t) \leq b/\delta$ and $|x''(t)| < d$. From assumption $(A_2)$, $f(t,\ x,\ x',\ x'') > b/m$.
Hence, by definition of $\alpha$ and the cone $P$, we can get

$$\alpha(Tx) = (Tx)(0) = \int_0^1 G(0,s) f(s,\ x,\ x',\ x'')ds \geq \frac{b}{m}\int_0^1 G(0,s)ds > \frac{b}{m} m = b,$$

which means $\alpha(Tx) > b,\ \forall x \in P(\gamma,\ \theta,\ \alpha,\ b,\ b/\delta,\ d)$.
Second, with (3.4) and $b < \lambda d$, we have

$$\alpha(Tx) \geq \delta\theta(Tx) > \delta \times \frac{b}{\delta} = b$$

for all $x \in P(\gamma,\ \alpha,\ b,\ d)$ with $\theta(Tx) > \dfrac{b}{\delta}$.
Thus, condition $(S_2)$ of Lemma 2.1 holds. Finally we show that $(S_3)$ also holds. We see $\psi(0) = 0 < a$ and $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(x) = a$, then by the assumption of $(A_3)$,

$$\psi(Tx) = \max_{0\le t\le 1} |(Tx)(t)| = \int_0^1 G(1,\ s) f(s,\ x,\ x',\ x'') d\tau ds < \frac{a}{N}\int_0^1 G(1, s) ds = a,$$

which ensures that condition $(S_3)$ of Lemma 2.1 is satisfied. Thus, an application of Lemma 2.1 implies that the third-order m-point boundary value problem (1.1) have at least three positive concave solutions $x_1,\ x_2,\ x_3$ satisfying the conditions that

$$\max_{0\le t\le 1} |x_i''(t)| \le d,\ i = 1,\ 2,\ 3; b < \min_{0\le t\le 1} |x_1(t)|;$$

$$a < \max_{0\le t\le 1} |x_2(t)|,\ \min_{0\le t\le 1} |x_2(t)| < b; \max_{0\le t\le 1} |x_3(t)| \le a.$$

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## References

[1] M. Gregus, *Third Order Linear Differential Equations*, in: Math. Appl., Reidel, Dordrecht, 1987.
[2] D.R. Anderson, *Multiple positive solutions for a three-point boundary value problem*, Math. Comput. Modelling, **27**(6)(1998), 49-57.
[3] A.P. Palamides, G. Smyrlis, *Positive solutions to a singular third-order three-point boundary value problem with an indefinitely signed Green's function,* Nonlinear Anal., **68**(2008), 2104-2118.
[4] L. Guo, J. Sun and Y. Zhao, *Existence of positive solutions for nonlinear third-order three-point boundary value problems*, Nonlinear Anal., **68**(2008), 3151-3158.
[5] B. Hopkins, N. Kosmatov, *Third-order boundary value problems with sign-changing solutions*, Nonlinear Anal., **67**(2007), 126-137.
[6] D.R. Anderson, *Green's function for a third-order generalized right focal problem*, J. Math. Anal. Appl., **288**(2003), 1-14.
[7] J. Chu, Z. Zhou, *Positive solutions for singular non-linear third-order periodic boundary value problems*, Nonlinear Anal., **64**(2006), 1528-1542.
[8] J.R. Graef, L. Kong, *Positive solutions for third order semi-positone boundary value problems*, Appl. Math. Lett., **22**(2009), 1154-1160.
[9] X. Lin, Z. Du and W. Liu, *Uniqueness and existence results for a third-order nonlinear multi-point boundary value problem*, Appl. Math. Comput., **205**(2008), 187-196.
[10] Q. Yao, *Positive solution for a semi-linear third-order two-point boundary value problem*, Appl. Math. Lett., **17**(2004), 1171-1175.
[11] R.I. Avery, A.C. Peterson, *Three positive fixed points of nonlinear operators on an ordered Banach space,* Comput. Math. Appl., **208**(2001), 313-322.

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