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INTEGRO-DIFFERENTIAL EQUATION WITH TWO TIME LAGS

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Abstract. We consider an integro-differential equation with two time lags and we prove the existence, uniqueness and convergence of the sequence of the successive approximation by using contraction principle and step method with a weaker Lipschitz condition. Also, we propose a new algorithm of successive approximation sequence generated by the step method and we give an example to illustrate the applications of the abstract results.
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1. Introduction

This paper is concerned with the following integro-differential equation

$$x'(t) = g(t, x(t), x(t-\tau)) + \int_{t-h}^{t} K(s, x(s))ds, \ t \in S, \tag{1.1}$$

where g and K are continuous functions on a Banach space and satisfy certain conditions to be specified later.

Regarding the earlier works on existence, uniqueness and convergence of the sequence of the successive approximation to integro-differential equations with delays and functional-differential equations with delays under different conditions, we refer to Guo *et al* [2], Kolmanowskii and Mishkis [5], Precup [7], Precup and Kirr [8], I.A. Rus [12] and the references therein. The related results for the existence and uniqueness, convergence of the sequence of the successive approximation, lower and upper solutions to the differential equations with delays can be found in Dobriţoiu *et al* [1], Ilea [3] and Otrocol [6].

The authors Rus, Şerban, Trif [13] have considered the following integral equation

$$x(t) = g(t, x(t-\tau)) + \int_{t-\tau}^{t} K(t, s, x(s))ds,\ t \in [a, b],\ \tau > 0$$
$$x(t) = \phi(t),\ t \in [a-\tau, a],$$

in a Banach space and proved that the sequence of the successive approximation generated by the step method converges to the solution of this integral equation using the results of Rus [9].

Sakata and Hara [14] have considered the linear differential equation with two kinds of time lags

$$x'(t) = ax(t-\tau) + b\int_{t-h}^{t} x(s)ds$$

where $\tau > 0,\ h > 0$ and a, b are both real and they have studied the dependence on delays towards stability regions.

In the present work we use the ideas of Rus, Şerban, Trif [13] to establish the convergence of the sequence of successive approximation to equation (1.1). Regarding the two delays we have the following cases: $h > 0,\ \tau > 0,\ \tau > h$ and $h > 0,\ \tau < 0,\ |\tau| > h$. Here, the authors study the first case, while the second case is studied in [4].

The aim of this paper is to obtain existence and uniqueness theorems using contraction principle and step method. Such kind of results have been proved in [13]. The approach proposed in the present paper is different to the ones in [13] and [1] and it is based on the different time lags. Also, we present here some lower and upper solution result, and a numerical example concerning equation (1.1).

We note that Sakata and Hara study in [14] the stability regions for similar integro-differential equation with two time lags.

2. Preliminaries

Let $\tau > 0,\ h > 0,\ h < \tau$ and

$$x'(t) = g(t, x(t), x(t-\tau)) + \int_{t-h}^{t} K(s, x(s))ds,\ t \in [0, T], \tag{2.1}$$

$$x(t) = \varphi(t),\ t \in [-\tau, 0]. \tag{2.2}$$

Relative to (2.1)–(2.2) we consider the following conditions:

(C_1) $(\mathbb{B}, \|\cdot\|)$ is a Banach space, $g \in C([0,T] \times \mathbb{B}^2, \mathbb{B}),\ K \in C([0,T] \times \mathbb{B}, \mathbb{B}),\ \varphi \in C([-\tau, 0], \mathbb{B})$;

(C_2) there exists $L_g > 0$ such that

$$\|g(t, u_1, v_1) - g(t, u_2, v_2)\| \le L_g(\|u_1 - u_2\| + \|v_1 - v_2\|),\ u_i, v_i \in \mathbb{B}, t \in [0, T];$$

(C_2') there exists $L_g' > 0$ such that

$$\|g(t, u_1, v) - g(t, u_2, v)\| \le L_g' \|u_1 - u_2\|,\ u_1, u_2, v \in \mathbb{B}, t \in [0, T];$$

(C_3) there exists $L_K > 0$ such that

$$\|K(t,u) - K(t,v)\| \leq L_K \|u - v\|, \ u, v \in \mathbb{B}, \ t \in [0,T];$$

(C_4) $\varphi'(0) = g(0, \varphi(0), \varphi(-\tau)) + \int_{-h}^{0} K(s, \varphi(s))ds.$

We consider the space $X = C([-\tau, T], \mathbb{B})$ endowed with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\lambda$ where

$$\|x\|_\infty := \max_{t\in[-\tau,T]} \{\|x(t)\|\}, \quad \|x\|_\lambda := \sup_{t\in[-\tau,T]} \{\|x(t)\| e^{-\lambda(t+\tau)}\}.$$

The following relation between the Cebyshev and Bielecki norms holds

$$\|x\|_\infty \leq \|x\|_\lambda \cdot e^{\lambda(t+\tau)}, \ \forall t \in [-\tau, T].$$

The problem (2.1)–(2.2) is equivalent with the following fixed point problem:

$$x(t) = \begin{cases} \varphi(t), \ t \in [-\tau, 0] \\ \varphi(0) + \int_0^t g(\xi, x(\xi), x(\xi - \tau))d\xi + \int_0^t \int_{\xi-h}^{\xi} K(s, x(s))dsd\xi, \ t \in [0,T] \end{cases}$$

3. Fibre weakly Picard operator

Let (X,d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the terminologies and notations from [13]. For the convenience of the reader we shall recall some of them.

Denote by $A_0 := 1_X, \ A^1 := A, \ A^{n+1} := A \circ A^n, \ n \in \mathbb{N}$, the iterate operators of the operator A. Also

$$P(X) := \{Y \subseteq X | \ Y \neq \emptyset\}, \ F_A := \{x \in X | \ A(x) = x\},$$
$$I(A) := \{Y \in P(X) | A(Y) \subseteq Y\}.$$

Definition 3.1. *$A : X \to X$ is called a Picard operator (briefly PO) if:*

(i) $F_A = \{x^*\}$;

(ii) $A^n(x) \to x^*$ *as $n \to \infty$, for all $x \in X$.*

Definition 3.2. *$A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.*

If $A : X \to X$ is a WPO, then we may define the operator $A^\infty : X \to X$ by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Obviously $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

Lemma 3.3. *Let $(X, d, \leq)$ an ordered metric space and $A, B, C : X \to X$ be such that:*

(i) the operator A, B, C are WPOs;

(ii) $A \leq B \leq C$;

(iii) the operator B is increasing.

Then $x \leq y \leq z$ implies that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

Theorem 3.4. *(Fibre contraction principle, Rus* [10]*) Let (X, d) be a metric space and (Y, ρ) be a complete metric space. Let $B : X \to X$ and $C : X \times Y \to Y$ be two operators. We suppose that:*

(i) *B is a WPO;*
(ii) *$C(x, \cdot) : Y \to Y$ is α-contraction, for all $x \in X$;*
(iii) *if $(x^*, y^*) \in F_A$, where $A : X \times Y \to X \times Y$, $A(x, y) = (B(x), C(x, y))$, then $C(\cdot, y^*)$ is continuous in x^*.*

Then A is a WPO. Moreover, if B is PO then A is PO.

By induction, from the above result we have:

Theorem 3.5. *(Rus* [11]*) Let (X_i, d_i), $i = \overline{0, m}$, $m \geq 1$, be some metric spaces. Let*

$$A_i : X_0 \times \cdots \times X_i \to X_i,\ i = \overline{0, m}$$

be some operator. We suppose that:

(i) *(X_i, d_i), $i = \overline{1, m}$, are complete metric spaces;*
(ii) *the operator A_0 is WPO;*
(iii) *there exists $\alpha_i \in (0; 1)$ such that:*

$$A_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i,\ i = \overline{1, m}$$

are α_i-contractions;

(iv) *the operator A_i, $i = \overline{1, m}$, are continuous.*

Then the operator $A : X_0 \times \cdots \times X_m \to X_0 \times \cdots \times X_m$,

$$A(x_0, \ldots, x_m) = (A_0(x_0), A_1(x_0, x_1), \ldots, A_m(x_0, \ldots, x_m))$$

is WPO. If A_0 is PO, then A is PO.

4. Existence and uniqueness

In this section we give an existence theorem for the solution of the problem (2.1)–(2.2).

Theorem 4.1. *In the condition (C_1), (C_2), (C_3) and (C_4), the problem (2.1)–(2.2) has in $C([-\tau, T], \mathbb{B})$ a unique solution x^* and the sequence of successive approximation, $(x^n)_{n \in \mathbb{N}}$*

$$x^{n+1}(t) = \begin{cases} \varphi(t),\ t \in [-\tau, 0] \\ \varphi(0) + \int_0^t g(\xi, x^n(\xi), x^n(\xi - \tau))d\xi + \int_0^t \int_{\xi-h}^{\xi} K(s, x^n(s))dsd\xi, \\ \qquad t \in [0, T] \end{cases}$$

converges uniformly to x^*, for every $x^0 \in C([-\tau, T], \mathbb{B})$, with $x^0 \big|_{[-\tau,0]} = \varphi$.

Proof. Let $X_\varphi \subset X$, $X_\varphi = \{x \in X |\ x(t) = \varphi(t),\ t \in [-\tau, 0]\}$ and $A : X_\varphi \to X_\varphi$ defined by

$$A(x)(t) = \begin{cases} \varphi(t),\ t \in [-\tau, 0] \\ \varphi(0) + \int_0^t g(\xi, x(\xi), x(\xi - \tau))d\xi + \int_0^t \int_{\xi-h}^{\xi} K(s, x(s))dsd\xi, \\ \qquad t \in [0, T]. \end{cases}$$

Note that X_φ is a closed subset of X, so $(X_\varphi, d_{\|\cdot\|_\lambda})$ is a complete metric space.

In a standard way we have

$$\|A(x) - A(y)\|_\lambda \leq \frac{1}{\lambda}(L_g + L_K h)\,\|x - y\|_\lambda\,,\ \text{for all } x, y \in X_\varphi,$$

which proves that A is Lipschitz with $L_A = \frac{1}{\lambda}(L_g + L_K h)$. We can choose λ sufficiently large such that $L_A = \frac{1}{\lambda}(L_g + L_K h) < 1$, so A is contraction. Applying the contraction principle we get the conclusion. □

Remark 4.2. *From the proof of Theorem 4.1, it follows that the operator A is PO.*

5. Step method

Using step method and contraction principle on each step for the problem (2.1)–(2.2), *in this section we obtain a better result by replacing the condition ($C2$) from Theorem 4.1 with ($C2'$).*

Let $m \in \mathbb{N}^*$ such that $(m-1)h \leq T,\ mh > T$. To simplify our presentation we suppose that $h < \tau \leq 2h$. In the conditions $(C_1), (C_2'), (C_3)$ and (C_4) the step method for (2.1)–(2.2) consist in the following:

$$(p_0)\quad x_0(t) = \varphi(t),\quad t \in [-\tau, 0]$$

$$(p_1)\quad x_1(t) = \varphi(0) + \int_0^t g(\xi, x_1(\xi), \varphi(\xi-\tau))d\xi + \int_0^t \int_{\xi-h}^0 K(s, \varphi(s))dsd\xi + \\ + \int_0^t \int_0^\xi K(s, x_1(s))dsd\xi,\ t \in [0, h]$$

$$(p_2)\quad x_2(t) = x_1^*(h) + \int_h^\tau g(\xi, x_2(\xi), \varphi(\xi-\tau))d\xi + \int_\tau^t g(\xi, x_2(\xi), x_1^*(\xi-\tau))d\xi + \\ + \int_h^t \int_{\xi-h}^h K(s, x_1^*(s))dsd\xi + \int_h^t \int_h^\xi K(s, x_2(s))dsd\xi,\ t \in [h, 2h]$$

....

$$(p_i)\quad x_i(t) = x_{i-1}^*((i-1)\,h) + \int_{(i-1)h}^{(i-2)h+\tau} g(\xi, x_i(\xi), x_{i-2}^*(\xi-\tau))d\xi + \\ + \int_{(i-2)h+\tau}^t g(\xi, x_i(\xi), x_{i-1}^*(\xi-\tau))d\xi + \\ + \int_{(i-1)h}^t \int_{\xi-h}^{(i-1)h} K(s, x_{i-1}^*(s))dsd\xi + \\ + \int_{(i-1)h}^t \int_{(i-1)h}^\xi K(s, x_i(s))dsd\xi,\ t \in [(i-1)\,h, ih]$$

....

$$(p_{m-1})\quad x_{m-1}(t) = x_{m-2}^*((m-2)h) + \int_{(m-2)h}^{(m-3)h+\tau} g(\xi, x_{m-1}(\xi), x_{m-3}^*(\xi-\tau))d\xi + \\ + \int_{(m-3)h+\tau}^t g(\xi, x_{m-1}(\xi), x_{m-2}^*(\xi-\tau))d\xi + \\ + \int_{(m-2)h}^t \int_{\xi-h}^{(m-2)h} K(s, x_{m-2}^*(s))dsd\xi + \\ + \int_{(m-2)h}^t \int_{(m-2)h}^\xi K(s, x_{m-1}(s))dsd\xi,\ t \in [(m-2)h, (m-1)h]$$

$$(p_m)\quad x_m(t) = x_{m-1}^*((m-1)h) + \int_{(m-1)h}^{(m-2)h+\tau} g(\xi, x_m(\xi), x_{m-2}^*(\xi-\tau))d\xi + \\ + \int_{(m-2)h+\tau}^t g(\xi, x_m(\xi), x_{m-1}^*(\xi-\tau))d\xi + \\ + \int_{(m-1)h}^t \int_{\xi-h}^{(m-1)h} K(s, x_{m-1}^*(s))dsd\xi + \\ + \int_{(m-1)h}^t \int_{(m-1)h}^\xi K(s, x_m(s))dsd\xi,\ t \in [(m-1)h, T]$$

where x_i^* is the unique solution of (p_i), $i = \overline{1, m}$.

So, we have the following result:

Theorem 5.1. *In the conditions $(C_1), (C_2'), (C_3)$ and (C_4) we have:*

a) *the problem (2.1)–(2.2) has in* $C([-\tau,T],\mathbb{B})$ *a unique solution* x^*,

$$x^*(t)=\begin{cases}\varphi(t),\ t\in[-\tau,0]\\ x_1^*(t),\ t\in[0,h]\\ \dots\\ x_m^*(t),\ t\in[(m-1)h,T]\end{cases}$$

b) *for each* $x_i^0\in C([(i-1)\,h,ih],\mathbb{B})$, $i=\overline{1,m-1}$, $x_m^0\in C([(m-1)\,h,T],\mathbb{B})$, *the sequence defined by:*

$$\begin{aligned}x_1^{n+1}(t)= {} & \varphi(0)+\int_0^t g(\xi,x_1^n(\xi),\varphi(\xi-\tau))d\xi+\int_0^t\int_{\xi-h}^0 K(s,\varphi(s))dsd\xi+\\ & +\int_0^t\int_0^\xi K(s,x_1^n(s))dsd\xi,\ t\in[0,h],\end{aligned}$$

$$\begin{aligned}x_2^{n+1}(t)= {} & x_1^*(h)+\int_h^\tau g(\xi,x_2^n(\xi),\varphi(\xi-\tau))d\xi+\int_\tau^t g(\xi,x_2^n(\xi),x_1^*(\xi-\tau))d\xi+\\ & +\int_h^t\int_{\xi-h}^h K(s,x_1^*(s))dsd\xi+\int_h^t\int_h^\xi K(s,x_2^n(s))dsd\xi,\ t\in[h,2h],\end{aligned}$$

....

$$\begin{aligned}x_m^{n+1}(t)= {} & x_{m-1}^*((m-1)h)+\int_{(m-1)h}^{(m-2)h+\tau} g(\xi,x_m^n(\xi),x_{m-2}^*(\xi-\tau))d\xi+\\ & +\int_{(m-2)h+\tau}^t g(\xi,x_m^n(\xi),x_{m-1}^*(\xi-\tau))d\xi+\\ & +\int_{(m-1)h}^t\int_{\xi-h}^{(m-1)h} K(s,x_{m-1}^*(s))dsd\xi+\\ & +\int_{(m-1)h}^t\int_{(m-1)h}^\xi K(s,x_m^n(s))dsd\xi,\ t\in[(m-1)h,T]\end{aligned}$$

converge and $\lim\limits_{n\to\infty}x_i^n=x_i^*,\ i=\overline{1,m}$.

Proof. In order to proof this theorem we apply the contraction principle for each step: $[(i-1)h,ih]$, $[(m-1)h,T]$, where $i=\overline{1,m-1}$.

For the first step we consider the Banach space $X_1:=(C([0,h],\mathbb{B}),\|\cdot\|_{\lambda_1})$, where $\|x\|_{\lambda_1}=\max\limits_{t\in[0,h]}\{\|x(t)\|\,e^{-\lambda_1 t}\}$ and the operator $A_1:X_1\to X_1$ defined by

$$\begin{aligned}A_1(x)(t)=\varphi(0)+&\int_0^t g(\xi,x(\xi),\varphi(\xi-\tau))d\xi+\\ &+\int_0^t\int_{\xi-h}^0 K(s,\varphi(s))dsd\xi+\int_0^t\int_0^\xi K(s,x(s))dsd\xi.\end{aligned}$$

For $x,y\in X_1$, we have

$$\|A_1(x)-A_1(y)\|_{\lambda_1}\le\frac{1}{\lambda_1}(L_g'+L_Kh)\,\|x-y\|_{\lambda_1}.$$

We can choose a $\lambda_1>0$ such that $\frac{1}{\lambda_1}(L_g'+L_Kh)<1$, so A_1 is a contraction, therefore $F_{A_1}=\{x_1^*\}$.

For the next steps let us consider the following Banach spaces: for $i=\overline{2,m-1}$ given by

$$X_i:=\left(C([(i-1)h,ih];\mathbb{B}),\|\cdot\|_{\lambda_i}\right),\ \text{with}\ \|x\|_{\lambda_i}:=\max_{t\in[(i-1)h,ih]}\left\{\|x(t)\|\,e^{-\lambda_i(t-(i-1)h)}\right\}$$

and

$$X_m:=(C([(m-1)h,T];\mathbb{B}),\|\cdot\|_{\lambda_m}),\ \text{with}\ \|x\|_{\lambda_m}:=\max_{t\in[(m-1)h,T]}\left\{\|x(t)\|\,e^{-\lambda_m(t-(m-1)h)}\right\}$$

and the operators $A_i : X_i \to X_i,\ i = \overline{2,m}$ defined by

$$A_i(x)(t) := x^*_{i-1}((i-1)\,h) + \int_{(i-1)h}^{(i-2)h+\tau} g(\xi, x(\xi), x^*_{i-2}(\xi-\tau))d\xi +$$
$$+ \int_{(i-2)h+\tau}^{t} g(\xi, x(\xi), x^*_{i-1}(\xi-\tau))d\xi + \int_{(i-1)h}^{t}\int_{\xi-h}^{(i-1)h} K(s, x^*_{i-1}(s))dsd\xi +$$
$$+ \int_{(i-1)h}^{t}\int_{(i-1)h}^{\xi} K(s, x(s))dsd\xi,$$

For $x, y \in X_i$ we have $\|A_i(x) - A_i(y)\|_{\lambda_i} \leq \frac{1}{\lambda_i}(L'_g + L_K h)\,\|x-y\|_{\lambda_i}$, so A_i is a contraction for a suitable choice of λ_i such that $\frac{1}{\lambda_i}(L'_g + L_K h) < 1$. Therefore, we get that $F_{A_i} = \{x^*_i\}$, $i = \overline{2,m}$.

From condition (C_4) we get $\varphi(0) = x^*_1(0)$ and from definition of $A_i,\ i = \overline{1,m}$, we have

$$x^*_{i-1}((i-1)h) = x^*_i((i-1)h),\ i = \overline{1,m},$$

therefore

$$x^*(t) = \begin{cases} \varphi(t),\ t \in [-\tau, 0] \\ x^*_1(t),\ t \in [0,h] \\ \cdots \\ x^*_m(t),\ t \in [(m-1)h, T] \end{cases}$$

is the unique solution in $C([-h,T],\mathbb{B})$. □

Now the question is: *Can we put an approximation of* x^n_i, $i = \overline{1,m}$ *instead of* x^*_i, $i = \overline{1,m}$*?*

The answer of this question is given by the following theorem:

Theorem 5.2. *In the condition of Theorem 5.1, for each* $x^0_i \in C([(i-1)\,h, ih], \mathbb{B})$, $i = \overline{1,m-1}$, $x^0_m \in C([(m-1)\,h, T], \mathbb{B})$, *the sequences defined by:*

$$\begin{aligned} x^{n+1}_1(t) = \;& \varphi(0) + \textstyle\int_0^t g(\xi, x^n_1(\xi), \varphi(\xi-\tau))d\xi + \int_0^t\int_{\xi-h}^0 K(s,\varphi(s))dsd\xi + \\ & + \textstyle\int_0^t\int_0^\xi K(s, x^n_1(s))dsd\xi,\ t \in [0,h], \end{aligned}$$

$$\begin{aligned} x^{n+1}_2(t) = \;& x^n_1(h) + \textstyle\int_h^\tau g(\xi, x^n_2(\xi), \varphi(\xi-\tau))d\xi + \int_\tau^t g(\xi, x^n_2(\xi), x^n_1(\xi-\tau))d\xi + \\ & + \textstyle\int_h^t\int_{\xi-h}^h K(s, x^n_1(s))dsd\xi + \int_h^t\int_h^\xi K(s, x^n_2(s))dsd\xi,\ t \in [h, 2h], \end{aligned}$$

....

$$\begin{aligned} x^{n+1}_m(t) = \;& x^n_{m-1}((m-1)h) + \textstyle\int_{(m-1)h}^{(m-2)h+\tau} g(\xi, x^n_m(\xi), x^n_{m-2}(\xi-\tau))d\xi + \\ & + \textstyle\int_{(m-2)h+\tau}^t g(\xi, x^n_m(\xi), x^n_{m-1}(\xi-\tau))d\xi + \\ & + \textstyle\int_{(m-1)h}^t\int_{\xi-h}^{(m-1)h} K(s, x^n_{m-1}(s))dsd\xi + \\ & + \textstyle\int_{(m-1)h}^t\int_{(m-1)h}^\xi K(s, x^n_m(s))dsd\xi,\ t \in [(m-1)h, T] \end{aligned} \tag{5.1}$$

converge and $\lim\limits_{n\to\infty} x^n_i = x^*_i,\ i = \overline{1,m}$.

Proof. We consider the following Banach spaces (with $\lambda > 0$):

$$\begin{aligned}
X_0 &= \left(C([-\tau,0],\mathbb{B}), \|\cdot\|_{\lambda_0}\right),\ \|\cdot\|_{\lambda_0} \\
&= \max_{t\in[-\tau,0]}\{\|x(t)\|\, e^{-\lambda_0(t+\tau)}\}, \\
X_i &= \left(C([(i-1)h,ih],\mathbb{B}), \|\cdot\|_{\lambda_i}\right),\ \|\cdot\|_{\lambda_i} \\
&= \max_{t\in[(i-1)h,ih]}\{\|x(t)\|\, e^{-\lambda_i(t-(i-1)h)}\},\ i=\overline{1,m-1}, \\
X_m &= \left(C([(m-1)h,T],\mathbb{B}), \|\cdot\|_{\lambda_m}\right),\ \|\cdot\|_{\lambda_m} \\
&= \max_{t\in[(m-1)h,T]}\{\|x(t)\|\, e^{-\lambda_m(t-(m-1)h)}\},
\end{aligned}$$

and the operators:

$$A_0 : X_0 \to X_0,\ A_0(x_0)(t) = \varphi(t),\ t\in[-\tau,0],$$

$$A_1 : X_0\times X_1 \to X_1,$$

$$\begin{aligned}
A_1(x_0,x_1)(t) = \varphi(0) &+ \int_0^t g(\xi,x_1(\xi),x_0(\xi-\tau))d\xi + \int_0^t\int_{\xi-h}^0 K(s,x_0(s))dsd\xi + \\
&+ \int_0^t\int_0^\xi K(s,x_1(s))dsd\xi,\ t\in[0,h],
\end{aligned}$$

$$A_i : X_{i-2}\times X_{i-1}\times X_i \to X_i,\ i=\overline{2,m-1}$$

$$\begin{aligned}
A_i\left(x_{i-2},x_{i-1},x_i\right)(t) = x_{i-1}((i-1)h) &+ \int_{(i-1)h}^{(i-2)h+\tau} g(\xi,x_i(\xi),x_{i-2}(\xi-\tau))d\xi + \\
&+ \int_{(i-2)h+\tau}^{t} g(\xi,x_i(\xi),x_{i-1}(\xi-\tau))d\xi + \\
&+ \int_{(i-1)h}^{t}\int_{\xi-h}^{(i-1)h} K(s,x_{i-1}(s))dsd\xi + \\
&+ \int_{(i-1)h}^{t}\int_{(i-1)h}^{\xi} K(s,x_i(s))dsd\xi,\ t\in[(i-1)h,ih]
\end{aligned}$$

$$A_m : X_{m-2}\times X_{m-1}\times X_m \to X_m$$

$$\begin{aligned}
A_m\left(x_{m-2},x_{m-1},x_m\right)(t) = x_{m-1}((m-1)h) &+ \int_{(m-1)h}^{(m-2)h+\tau} g(\xi,x_m(\xi),x_{m-2}(\xi-\tau))d\xi + \\
&+ \int_{(m-2)h+\tau}^{t} g(\xi,x_m(\xi),x_{m-1}(\xi-\tau))d\xi + \\
&+ \int_{(m-1)h}^{t}\int_{\xi-h}^{(m-1)h} K(s,x_{m-1}(s))dsd\xi + \\
&+ \int_{(m-1)h}^{t}\int_{(m-1)h}^{\xi} K(s,x_m(s))dsd\xi,\ t\in[(m-1)h,T]
\end{aligned}$$

and

$$A : X_0 \times \ldots \times X_m \to X_0 \times \ldots \times X_m,$$
$$A(x_0, \ldots, x_m) = (A_0(x_0), A_1(x_0, x_1), A_2(x_0, x_1, x_2), \ldots, A_m(x_{m-2}, x_{m-1}, x_m)).$$

It is easy to see that for fixed $(x_0, \ldots, x_m) \in X_0 \times \ldots \times X_m$ the sequence defined by (5.1) means

$$(x_0^n, \ldots, x_m^n) = A^n(x_0, \ldots, x_m).$$

To prove the conclusion we need to prove that the operator A is PO and for this we apply Theorem 3.5.

Since $A_0 : X_0 \to X_0$ is a constant operator then A_0 is α_0-contraction with $\alpha_0 = 0$, so A_0 is PO and $F_{A_0} = \{x_0^*\}$, where $x_0^* = \varphi$. We have the inequalities:

$$\|A_1(x_0, x_1) - A_1(x_0, y_1)\|_{\lambda_1} \leq \frac{1}{\lambda_1}(L_g' + L_K h)\, \|x_1 - y_1\|_{\lambda_1}$$

for all $x_0 \in X_0$, $x_1, y_1 \in X_1$, and

$$\|A_i(x_{i-2}, x_{i-1}, x_i) - A_i(x_{i-2}, x_{i-1}, y_i)\|_{\lambda_i} \leq \frac{1}{\lambda_i}(L_g' + L_K h)\, \|x_i - y_i\|_{\lambda_i}$$

for all $x_{i-2} \in X_{i-2}$, $x_{i-1} \in X_{i-1}$, $x_i, y_i \in X_i$, $i = \overline{2, m}$. For λ_i sufficiently large, ($\lambda_i > L_g' + L_K h$), we get that $A_1(x_0, \cdot) : X_1 \to X_1$ is α_1-contraction and $A_i(x_{i-2}, x_{i-1}, \cdot) : X_i \to X_i$ are α_i-contractions with $\alpha_i = \frac{1}{\lambda}(L_g' + L_K h)$, $i = \overline{1, m}$, so we are in the conditions of Theorem 3.5, therefore A is PO and $F_A = \{(x_0^*, \ldots, x_m^*)\}$, thus

$$(x_0^n, \ldots, x_m^n) = A^n(x_0, \ldots, x_m) \to (x_0^*, \ldots, x_m^*),$$

with $x_0^n = \varphi$ and $x_1^n, \ldots, x_m^n$, for all $n \in \mathbb{N}$, are defined by (5.1). From condition (C_4) and from the definitions of A_i, $i = \overline{1, m}$, we have

$$x_{i-1}^*((i-1)h) = x_i^*((i-1)h),\ i = \overline{1, m}$$

therefore

$$x^*(t) = \begin{cases} \varphi(t),\ t \in [-\tau, 0] \\ x_1^*(t),\ t \in [0, h] \\ \cdots \\ x_m^*(t),\ t \in [(m-1)h, T] \end{cases}$$

is the unique solution in $C([-\tau, T], \mathbb{B})$. □

6. Lower solutions, upper solutions and the solution

In this section we shall prove that the solution of the equation (1.1) is an upper bound of the lower solutions set and a lower bound of the upper solutions set.

Let the integro-differential equation (1.1) under the conditions (C_1), (C_2), (C_3), (C_4) and we denote by $x_A^* \in (C[0, T], \mathbb{B})$ the unique fixed point of the operator A. In addition, we suppose that:

(C_5) $g(t, \cdot, \cdot) : \mathbb{B}^2 \to \mathbb{B}$ is increasing, for every $t \in [0, T]$;
(C_6) $K(t, \cdot) : \mathbb{B} \to \mathbb{B}$ is increasing, for every $t \in [0, T]$.

We have

Theorem 6.1. *We suppose that the conditions* $(C_1)-(C_6)$ *are satisfied. The following implications hold:*

(a) *If* $x(t) \leq g(t, x(t), x(t-\tau)) + \int_{t-h}^{t} K(s, x(s))ds, x \in \mathbb{B}$ *then* $x \leq x_A^*$.

(b) *If* $x(t) \geq g(t, x(t), x(t-\tau)) + \int_{t-h}^{t} K(s, x(s))ds, x \in \mathbb{B}$ *then* $x \geq x_A^*$.

Proof. (a) We consider the operator A defined by

$$A(x)(t) = \int_0^t g(\xi, x(\xi), x(\xi-\tau))d\xi + \int_0^t \int_{\xi-h}^{\xi} K(s, x(s))dsd\xi.$$

Under the conditions $(C_1)-(C_4)$ the operator A is PO and by $(C_5)-(C_6)$ we have that the operator A is increasing. Since all the conditions of the Abstract Gronwall Lemma 3.3 are satisfied, we obtain $x \leq x_A^*$ and the proof is complete.

For (b) the proof is similar. $\square$

7. Numerical example

In this section we give a numerical example to illustrate the convergence of the sequence defined in theorem 5.1 to the solution. We consider the following integro-differential equation:

$$\begin{array}{lll} x'(t) & = -\left(6+\sin(t)\right)x(t) + x\left(t-\frac{\pi}{2}\right) - & \\ & -\int\limits_{t-\frac{\pi}{4}}^{t} \sin(s)\,x(s)\,ds + 5e^{\cos(t)} + e^{\cos\left(t-\frac{\pi}{4}\right)} - e^{\cos\left(t-\frac{\pi}{2}\right)} & ,\quad t \in \left[\frac{\pi}{4}; \pi\right] \\ x(t) & = e^{\cos(t)} & ,\quad t \in \left[0; \frac{\pi}{4}\right] \end{array}$$

which has the exact solution $x(t) = e^{\cos(t)}$.

Numerical method. (For more details see N.L. Trefethen [15], D. Trif [16]) We divide the working interval by the points $P_k = k$, $k = 0, 1, ..., M$, (concretely $M = 4$ and represents the number of subintervals). On each subinterval $I_k = [P_{k-1}; P_k]$, $k = 1, ..., M$, we find the numerical solution by the form

$$x_k(t) = c_{0,k}\frac{T_0}{2} + c_{1,k}T_1(\xi) + c_{2,k}T_2(\xi) + ... + c_{n-1,k}T_{n-1}(\xi),$$

where $T_i(\xi) = \cos(i \arccos(\xi))$ are Chebyshev polynomials of i degree, $i = 0, ..., n-1$, (concretely $n = 8$), and $t = \alpha\xi + \beta$ with $\alpha = (P_k - P_{k-1})/2$, respectively $\beta = (P_k + P_{k-1})/2$.

Choosing a mesh ξ_j, $j = 1, ..., n$, on interval $[-1; 1]$ consisting by the knots of Gauss quadrature formula generated by Matlab subprogram `[csi,w]=pd(n)`, the transformation $t = \alpha\xi + \beta$ corresponding to each interval $I_k = [P_{k-1}; P_k]$ construct a local mesh on that subinterval. The coefficients $c_{i,k}$ of x_k expansion after the Chebishev polynomials T_i are obtained from x_k values on the local mesh using Fast Fourier Transforms (if n is large) or using a matrix T generated by the subprogram `T=x2t(n,csi)`

(for n small)

$$\begin{pmatrix} c_{0,k} \\ c_{1,k} \\ \vdots \\ c_{n-2,k} \\ c_{n-1,k} \end{pmatrix} = \left(T'\right)^{-1} \cdot \begin{pmatrix} x_k\left(t_1\right) \\ x_k\left(t_2\right) \\ \vdots \\ x_k\left(t_{n-1}\right) \\ x_k\left(t_n\right) \end{pmatrix}.$$

The same formula allows the quick pass from the local coefficients to the values on local mesh.

The formulae

$$\int\limits_{\xi-h}^{\xi} T_i\left(s\right) ds = \frac{T_{i+1}\left(\xi\right)}{2\left(i+1\right)} - \frac{T_{i-1}\left(\xi\right)}{2\left(i-1\right)}$$

allow to obtain the coefficients C_i of a primitive F for a function f given by its coefficients c_i, from multiplication of them with a sparse matrix J generated by the subprogram `J=tchebj(n)`

$$\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-2} \\ C_{n-1} \end{pmatrix} = J \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix}.$$

Of course, if the primitive is calculated for other interval $[P_{k-1}; P_k]$ instead of $[-1; 1]$, the matrix J is replaced by αJ, where $\alpha = \left(P_k - P_{k-1}\right)/2$.

The algorithm from Theorem 5.1 is implemented in the following way in program `[X,sol]=step_meth2`, which can be obtained from the authors (`mserban@math.ubbcluj.ro`):

Step 0. We generate a global mesh X on $[0; \pi]$ by the union of all local meshes on which we also add the points P_k of subintervals. We calculate the values of $x^{(0)}$ on the global mesh from the values of the function φ on the local mesh of the first interval $\left[0; \frac{\pi}{4}\right]$ and from the constant value $\varphi\left(\frac{\pi}{4}\right)$ on the other knots.

Step k. Taking the values of $x^{(k)}$ on the global mesh, we obtain the values of $\sin\left(X\right) \cdot x^{(k)}$ on the local mesh, we calculate the coefficients of $\sin\left(X\right) \cdot x^{(k)}$ on each subinterval, then we get the coefficients of a primitive for $\sin\left(X\right) \cdot x^{(k)}$ on each subinterval and finally we obtain the values of that primitive on the local mesh. We add the contribution of nonintegrated part (where it is used the history from the previous intervals with two steps). The implementation of the formulae from Theorem 5.1 is now immediately, getting the values of the new iteration $x^{(k+1)}$ on the global mesh by a new integration: we pass from the values on the mesh to coefficients, then we use the integration matrix J and finally we return to the values in order to find $x^{(k+1)}$.

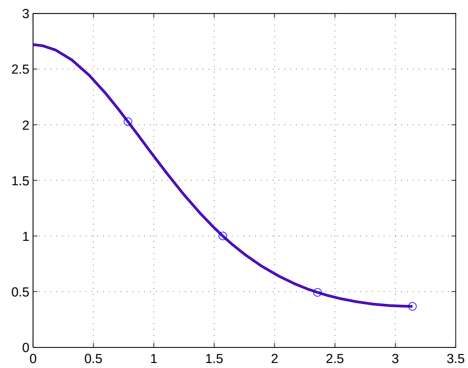
Stoping test. We evaluate the difference in norm between the values of $x^{(k)}$ and $x^{(k+1)}$ and iterations stop when this is below than a chosen value (concretely 10^{-9}). We represent the graph of solution and the norm of difference for different k.

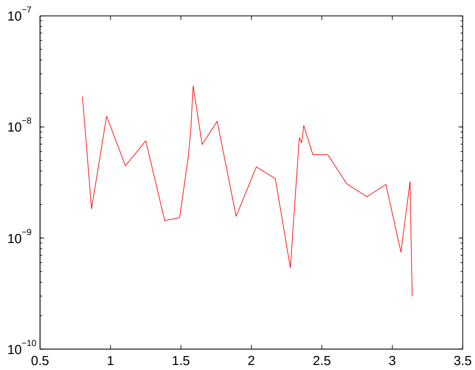
For the efficiency estimation of this algorithm, the integro-differential equation is written in the form of delay differential equation system and we use the Matlab command `dde23` to solve it. We impose the relative error to 10^{-9} and the absolute error to 10^{-12} to obtain a accuracy comparable with the step method. We display the graph of solution.

Results. Running the program we get the following results:

>>[X,sol]=step_meth2;
Step method solution
Elapsed time is 0.054235 seconds.
Matlab dde23 solution 1253 successful steps
0 failed attempts
3760 functions evaluations
Elapsed time is 0.671015 seconds.

The graph of solutions and the evolution of the differences between two successive iterations are given below:





Conclusions. For the chosen example, the step method obtains the solution in 68 iterations with an error of 10^{-10} in 0.054 CPU seconds. The Matlab program `dde23` needs 0.671 CPU seconds (12 times bigger) for a similar precision. The above comparisons validate the step method from the accuracy and efficiency point of view.

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References

[1] M. Dobriţoiu, I.A. Rus, M.A. Şerban, An integral equation arising from infectious diseases, via Picard operator, *Studia Univ. "Babes-Bolyai", Mathematica*, **52**(2007), No. 3, 81-83.

[2] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Acad. Publ., Dordrecht, 1996.

[3] V.A. Ilea, *Functional Differential Equations of First Order with Advanced and Retarded Arguments*, Cluj University Press, 2006, (in Romanian).

[4] V.A. Ilea, D. Otrocol, Integro-differential equation with two times modifications, *Carpathian J. Math.*, **27**(2011), No. 2, 209-216.

[5] V. Kolmanovskii, A. Mishkis, *Applied Theory of Functional Differential Equations*, Kluwer Acad. Publ., 1992.

[6] D. Otrocol, *Lotka-Volterra Systems with Retarded Argument*, Cluj University Press, 2007, (in Romanian).

[7] R. Precup, Positive solution of initial value problem for an integral equation modelling infectious diseases, *Seminar of Fixed Point Theory*, Cluj-Napoca, 1991, 25-30.

[8] R. Precup, E. Kirr, Analysis of nonlinear integral equation modelling infectious diseases, *Proc. Conf. West. Univ. of Timişoara*, 1997, 178-195.

[9] I.A. Rus, Abstract models of step method which imply the convergence of successive approximations, *Fixed Point Theory*, **9**(2008), No. 1, 293-307.

[10] I.A. Rus, Picard operators and applications, *Sci. Math. Jpn.*, **58**(2003), No. 1, 191-219.

[11] I.A. Rus, Picard operators and applications, *Seminar on Fixed Point Theory*, Cluj-Napoca, **2**(2001), 41-58.

[12] I.A. Rus, Some nonlinear functional differential and integral equations, via weakly Picard operator theory: a survey, *Carpathian J. Math.*, **26**(2010), No. 2, 230-258.

[13] I.A. Rus, M.A. Şerban, D. Trif, Step method for some integral equations from biomathematics, *Bull. Math. Soc. Sci. Math. Roumanie*, **54(102)**(2011), No. 2, 167-183.

[14] S. Sakata, T. Hara, Stability regions for linear differential equations with two kinds of time lags, *Funkcialaj Ekvacioj*, **47**(2004), 129-144.

[15] N.L. Trefethen, An extension of Matlab to continuous functions and operators, *SIAM J. Sci. Comput.*, **25**(2004), No. 5, 1743-1770.

[16] D. Trif, LibScEig 1.0, > Mathematics > Differential Equations > LibScEig 1.0, http://www.mathworks.com/matlabcentral/fileexchange, 2005.

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