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ZERO POINT THEOREMS OF m-ACCRETIVE OPERATORS IN A BANACH SPACE

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Abstract. The purpose of this work is to modify the Mann iteration for the case of m-accretive operators. Two strong convergence theorems to the zero points are established in a real Banach space.

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1. Introduction and Preliminaries

Let C be a nonempty closed convex subset of a Banach space E and E^* be the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The *normalized duality mapping* $j : E \to 2^{E^*}$ is defined by

$$j(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *smooth* or said to be have *a Gâteaux differentiable norm* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. E is said to be *uniformly smooth* or said to have a *uniformly Fréchet differentiable norm* if the limit is attained uniformly for $x, y \in U_E$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping j is single valued and uniformly norm to weak* continuous on each bounded subset of E.

Let C be a nonempty closed convex subset of a Banach space E, $T : C \to C$ be a mapping and F be a subset of C. We use $F(T)$ to denote the set of fixed points of T. Recall the following definitions.

(1) The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C;$$

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(2) The mapping T is said to be *pseudocontractive* if there exists some $j(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \quad \forall x, y \in C;$$

(3) A mapping Q of C onto F is said to be *sunny* if $Q(Q(x)+t(x-Q(x))) = Q(x)$ for any $x \in C$ and $t \ge 0$ with $Q(x)+t(x-Q(x)) \in C$;

(4) F is called a *nonexpansive retract* of C if there exists a nonexpansive retraction of C onto F.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 1.1 ([11]) *Let E be a smooth Banach space and C be a nonempty subset of E. Let $Q : E \to C$ be a retraction. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive*;

(b) $\|Q(x)-Q(y)\|^2 \le \langle x-y, j(Q(x)-Q(y))\rangle$, $\forall x, y \in E$;

(c) $\langle x-Q(x), j(y-Q(x))\rangle \le 0$, $\forall x \in E, y \in C$.

Regularization methods is efficient way to study the class of nonexpansive mappings. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. That is,

$$x_t = tu + (1-t)Tx_t. \tag{1.1}$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [1] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [11] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \ne \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists some $j(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \ge 0.$$

An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A.

For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A.

Recently, many authors studied m-accretive operators by iterative methods, see [4-10,17-19]. For approximating zero points of m-accretive operators, Kim and Xu [8] studied the following iterative process

$$x_0 = x \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad \forall n \ge 0, \tag{1.2}$$

where u is a fixed element in C, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $J_{r_n} = (I+r_nA)^{-1}x_n$. They proved that the sequence $\{x_n\}$ generated by the above iterative process converges strongly to some point in $A^{-1}(0)$.

Recently, Qin and Su [10] further studied the problem of finding zero points of m-accretive operators by considering the following iterative process

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1-\beta_n)J_{r_n}x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where u is a fixed element in C and $J_{r_n} = (I+r_nA)^{-1}$. To be more precise, they proved the following theorem.

Theorem QS. *Assume that E is a uniformly smooth Banach space and A is an m-accretive operator in E such that $A^{-1}(0) \neq \emptyset$. Given a point $u \in C$ and sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the conditions*

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $r_n \geq \epsilon$ *for each* $n \geq 0$ *and* $\beta_n \in [0,a)$ *for some* $a \in (0,1)$;

(c) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty$.

Let $\{x_n\}$ be a sequence defined by the above iterative process. Then the sequence $\{x_n\}$ converges strongly to a zero point of A.

In this paper, motivated by the recent research work, we continue to study the problem of finding zero points of m-accretive operators by the iterative process (1.3) in a more general framework of Banach spaces. Strong convergence theorems of zero points of m-accretive operators are established.

In order to prove our main results, we need the lemmas as follows. The following lemma is the well known subdifferential inequality.

Lemma 1.1. ([15]) *In a Banach space E, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 1.2. ([14]) *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C be a closed convex subset of E and T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Assume that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let $u \in C$ and z_t be a unique element of C which satisfies $z_t = tu + (1-t)Tz_t$ and $0 < t < 1$. Then $\{z_t\}$ converges strongly to a fixed point of T as $t \to 0$. Further, if $Q(u) = \lim_{t\to 0} z_t$ for each $u \in C$, then $\langle u - Q(u), j(Q(u) - z)\rangle \geq 0$ for all $z \in F(T)$ and Q is a unique sunny nonexpansive retraction of C onto $F(T)$.*

Lemma 1.3. ([2]) *Let E be a Banach space and A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$ and $x \in E$, we have*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right),$$

where $J_\lambda = (I + \lambda A)^{-1}$ *and* $J_\mu = (I + \mu A)^{-1}$.

Lemma 1.4. ([13]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, $\forall n \ge 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5. ([16]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Now, we are ready to give our main results.

Theorem 2.1. *Let* E *be a real reflexive Banach space with the uniformly Gâteaux differentiable norm and* A *be an* m*-accretive operators in* E *such that* $C := \overline{D(A)}$ *is convex. Assume that every weakly compact convex subset of* E *has the fixed point property for nonexpansive mappings. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad n \ge 0, \end{cases} \tag{2.1}$$

where u *is a fixed element in* C, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$, $\{r_n\}$ *is a positive real numbers sequence and* $J_{r_n} = (I + r_n A)^{-1}$. *Assume that* $A^{-1}(0) \ne \emptyset$ *and the above control sequences satisfy the following restrictions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(c) $r_n \ge \epsilon$ *for each* $n \ge 0$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$,

Then the sequence $\{x_n\}$ *converges strongly to a zero point of* A.

Proof. First, we prove that $\{x_n\}$ is bounded. Let $p \in A^{-1}(0)$. It follows that

$$\|y_n - p\| \le \beta_n\|x_n - p\| + (1 - \beta_n)\|J_{r_n} x_n - p\| \le \|x_n - p\|.$$

This implies that

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n\|u - p\| + (1 - \alpha_n)\|y_n - p\| \\ &\le \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|. \end{aligned}$$

By simple inductions, we arrive at

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \|u - p\|\}.$$

This completes the boundedness of the sequence $\{x_n\}$, from which it follows that $\{y_n\}$ is bounded. Put $r_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. This implies that

$$x_{n+1} = (1-\beta_n)r_n + \beta_n x_n, \quad n \ge 0. \tag{2.2}$$

Now, we compute $\|r_{n+1} - r_n\|$. Note that

$$\begin{aligned} & r_{n+1} - r_n \\ &= \frac{\alpha_{n+1}u + (1-\alpha_{n+1})y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\alpha_n)y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}(u - y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(u-y_n)}{1-\beta_n} + J_{r_{n+1}}x_{n+1} - J_{r_n}x_n. \end{aligned}$$

It follows that

$$\begin{aligned} \|r_{n+1} - r_n\| \le & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| \\ & + \|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\|. \end{aligned} \tag{2.3}$$

If $r_{n+1} \ge r_n$, we see from Lemma 1.3 that

$$\begin{aligned} \|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| &\le \left\| \frac{r_n}{r_{n+1}}x_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}x_{n+1} - x_n \right\| \\ &\le \frac{r_n}{r_{n+1}}\|x_{n+1} - x_n\| + \left(1 - \frac{r_n}{r_{n+1}}\right)M \\ &\le \|x_{n+1} - x_n\| + \frac{r_{n+1} - r_n}{r_{n+1}}M, \end{aligned} \tag{2.4}$$

where M is an appropriate constant such that $M \ge \sup_{n\ge 1}\{\|J_{r_{n+1}}x_{n+1} - x_n\|\}$. Combining (2.3) with (2.4) yields that

$$\begin{aligned} \|r_{n+1} - r_n\| - \|x_{n+1} - x_n\| \le & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| \\ & + \frac{r_{n+1} - r_n}{\epsilon}M. \end{aligned}$$

From the conditions (a)-(c), we see that

$$\limsup_{n\to\infty}\big(\|r_{n+1} - r_n\| - \|x_n - x_{n+1}\|\big) \le 0.$$

It follows from Lemma 1.4 that

$$\lim_{n\to\infty}\|r_n - x_n\| = 0. \tag{2.5}$$

In a similar way, we can obtain (2.5) in the case that $r_n \ge r_{n+1}$. In view of (2.2), we have

$$x_{n+1} - x_n = (1-\beta_n)(r_n - x_n),$$

which combines with (2.5) gives that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.6}$$

Thanks to (2.1), we can obtain that

$$\begin{aligned}\|x_n - J_{r_n}x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - J_{r_n}x_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - y_n\| + \beta_n\|x_n - J_{r_n}x_n\|.\end{aligned}$$

This implies that $(1-\beta_n)\|x_n - J_{r_n}x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|u - y_n\|$. From the conditions (a), (b) and (2.6), we conclude that

$$\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0. \tag{2.7}$$

Take a fixed number r such that $\epsilon > r > 0$. From Lemma 1.3, we obtain that

$$\begin{aligned}\|J_{r_n}x_n - J_rx_n\| &= \left\|J_r\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}x_n\right) - J_rx_n\right\| \\ &\le \left\|\left(1-\frac{r}{r_n}\right)(J_{r_n}x_n - x_n)\right\| \\ &\le \|J_{r_n}x_n - x_n\|.\end{aligned} \tag{2.8}$$

Note that

$$\begin{aligned}\|x_n - J_rx_n\| &= \|x_n - J_{r_n}x_n + J_{r_n}x_n - J_rx_n\| \\ &\le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_rx_n\| \\ &\le 2\|x_n - J_{r_n}x_n\|,\end{aligned}$$

which combines with (2.7) and (2.8) implies that

$$\lim_{n\to\infty}\|x_n - J_rx_n\| = 0. \tag{2.9}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle u - Q(u), j(x_n - Q(u))\rangle \le 0, \tag{2.10}$$

where $Q(u) = \lim_{t\to 0} z(t,u,r), u \in C$ and $z(t,u,r)$ solves the fixed point equation

$$z(t,u,r) = tu + (1-t)J_rz(t,u,r), \quad \forall t \in (0,1),$$

from which it follows that

$$\|z(t,u,r) - x_n\| = \|(1-t)(J_rz(t,u,r) - x_n) + t(u - x_n)\|.$$

For any $t \in (0,1)$, we see that

$$\begin{aligned}&\|z(t,u,r) - x_n\|^2 \\ &= (1-t)\langle J_rz(t,u,r) - x_n, j(z(t,u,r) - x_n)\rangle + t\langle u - x_n, j(z(t,u,r) - x_n)\rangle \\ &= (1-t)\big(\langle J_rz(t,u,r) - J_rx_n, j(z(t,u,r) - x_n)\rangle \\ &\quad + \langle J_rx_n - x_n, j(z(t,u,r) - x_n)\rangle\big) + t\langle u - z(t,u,r), j(z(t,u,r) - x_n)\rangle \\ &\quad + t\langle z(t,u,r) - x_n, j(z(t,u,r) - x_n)\rangle \\ &\le (1-t)\big(\|z(t,u,r) - x_n\|^2 + \|J_rx_n - x_n\|\|z(t,u,r) - x_n\|\big) \\ &\quad + t\langle u - z(t,u,r), j(z(t,u,r) - x_n)\rangle + t\|z(t,u,r) - x_n\|^2 \\ &\le \|z(t,u,r) - x_n\|^2 + \|J_rx_n - x_n\|\|z(t,u,r) - x_n\| \\ &\quad + t\langle u - z(t,u,r), j(z(t,u,r) - x_n)\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\langle z(t,u,r)-u, j(z(t,u,r)-x_n)\rangle\\&\leq \frac{1}{t}\|J_r x_n - x_n\|\|z(t,u,r)-x_n\|, \quad \forall t\in(0,1).\end{aligned}$$

In view of (2.9), we obtain that

$$\limsup_{n\to\infty}\langle z(t,u,r)-u, j(z(t,u,r)-x_n)\rangle \leq 0. \tag{2.11}$$

Since $z(t,u,r)\to Q(u)$ as $t\to 0$ and the fact that j is strong to weak* uniformly continuous on bounded subsets of E, we see that

$$\begin{aligned}&|\langle u-Q(u), j(x_n-Q(u))\rangle - \langle z(t,u,r)-u, j(z(t,u,r)-x_n)\rangle|\\&\leq |\langle u-Q(u), j(x_n-Q(u))\rangle - \langle u-Q(u), j(x_n-z(t,u,r))\rangle|\\&\quad+|\langle u-Q(u), j(x_n-z(t,u,r))\rangle - \langle z(t,u,r)-u, j(z(t,u,r)-x_n)\rangle|\\&\leq |\langle u-Q(u), j(x_n-Q(u))-j(x_n-z(t,u,r))\rangle|\\&\quad+|\langle z(t,u,r)-Q(u), J(x_n-z(t,u,r))\rangle|\\&\leq \|u-Q(u)\|\|j(x_n-Q(u))-j(x_n-z(t,u,r))\|\\&\quad+\|z(t,u,r)-Q(u)\|\|x_n-z(t,u,r)\|\to 0\end{aligned}$$

as $n\to\infty$. Hence, for any $\sigma>0$, there exists $\delta>0$ such that $\forall t\in(0,\delta)$

$$\langle u-Q(u), j(x_n-Q(u))\rangle \leq \langle z(t,u,r)-u, j(z(t,u,r)-x_n)\rangle + \sigma.$$

It follows that

$$\begin{aligned}&\limsup_{n\to\infty}\langle u-Q(u), j(x_n-Q(u))\rangle\\&\leq \limsup_{n\to\infty}\langle z(t,u,r)-u, j(z(t,u,r)-x_n)\rangle + \sigma.\end{aligned}$$

Since σ is arbitrary and (2.11), we obtain that (2.10) holds.

Finally, we prove that $x_n\to Q(u)$ as $n\to\infty$. It follows from Lemma 1.1 that

$$\begin{aligned}&\|x_{n+1}-Q(u)\|^2\\&=\|(1-\alpha_n)(y_n-Q(u))+\alpha_n(u-Q(u))\|^2\\&\leq (1-\alpha_n)^2\|y_n-Q(u)\|^2+2\alpha_n\langle u-Q(u), j(x_{n+1}-Q(u))\rangle\\&\leq (1-\alpha_n)^2\big(\beta_n\|x_n-Q(u)\|+(1-\beta)\|J_{r_n}x_n-Q(u)\|\big)^2\\&\quad+2\alpha_n\langle u-Q(u), j(x_{n+1}-Q(u))\rangle\\&\leq (1-\alpha_n)\|x_n-Q(u)\|^2+2\alpha_n\langle u-Q(u), j(x_{n+1}-Q(u))\rangle.\end{aligned}$$

In view of Lemma 1.5, we can obtain the desired conclusion easily. □

Remark 2.2. The convergence of the iterative process (2.1) is guaranteed by the contraction. Therefore, it is interesting to focus on the restrictions on the anchor $\{\alpha_n\}$. The strong restrictions imposed on the anchor $\{\alpha_n\}$ are relaxed comparing with Theorem QS based on Suzuki's lemma. Since uniformly smooth Banach spaces are reflexive and admit uniformly Gâteaux differentiable norms, we see that Theorem 2.1 is still valid in uniformly smooth Banach spaces. It deserves to mention that the proof is different from the one in Theorem QS on the step $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Next, we give another convergence theorem in the framework of reflexive Banach spaces which admits a weakly continuous duality mapping. Before giving the convergence, we need the following definitions and lemmas.

Let $\varphi : [0, \infty] := R^+ \to R^+$ be a continuous and strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a *gauge function.* The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \quad \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the case that $\varphi(t) = t$, we write J for J_φ and call J the normalized duality mapping.

Following Browder [3], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to $J_\varphi x$). It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad \forall t \geq 0.$$

Then $J_\varphi(x) = \partial\Phi(\|x\|)$ for all $x \in E$, where ∂ denotes the subdifferential in the sense of convex analysis.

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [9].

Lemma 2.3. *Assume that a Banach space E has a weakly continuous duality mapping J_φ with a gauge φ.*

(a) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle.$$

In particular, for all $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle.$$

(b) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Lemma 2.4. ([18]) *Let E be a reflexive Banach space and has a weakly continuous duality mapping $j_\varphi(x)$ with gauge φ. Let C be closed convex subset of E and $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C to the equation* (1.1). *Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges strongly as $t \to 0^+$ to a fixed point of T.*

Lemma 2.5. ([9]) *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality mapping j_φ, C be a nonempty closed convex subset of E and $T : C \to C$ be a nonexpansive mapping with a fixed point. Then $I - T$ is* *demi-closed at zero, i.e., if $\{x_n\}$ is a sequence in C which converges weakly to x and if the sequence $\{(I-T)x_n\}$ converges strongly to zero, then $x = Tx$.*

Theorem 2.6. *Let E a real reflexive Banach space enjoying the weakly continuous duality mapping $j_\varphi(x)$ with gauge φ. Let A be an m-accretive operators in E such that $C := \overline{D(A)}$ is convex. Let $\{x_n\}$ be a sequence defined by (2.1). Assume that $A^{-1}(0) \neq \emptyset$ and the above control sequences satisfy the following restrictions:*
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $r_n \geq \epsilon$ *for each* $n \geq 0$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$,
Then the sequence $\{x_n\}$ converges strongly to a zero point of A.
Proof. We only conclude the difference.

First, we show that $\limsup_{n\to\infty}\langle u - Q(u), J_\varphi(x_n - Q(u))\rangle \leq 0$, where $Q : C \to A^{-1}(0)$ is the sunny nonexpansive retraction. To show it, we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle u - Q(u), J_\varphi(x_n - Q(u))\rangle \\ &= \lim_{i\to\infty}\langle u - Q(u), J_\varphi(x_{n_i} - Q(u))\rangle.\end{aligned} \tag{2.12}$$

Since E is reflexive, we may further assume that $x_{n_i} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. In view (2.9) and Lemma 2.5, we see that $\bar{x} \in F(J_r) = A^{-1}(0)$. Hence, we arrive at

$$\limsup_{n\to\infty}\langle u - Q(u), J_\varphi(x_n - Q(u))\rangle = \lim_{i\to\infty}\langle u - Q(u), J_\varphi(\bar{x} - Q(u))\rangle \leq 0. \tag{2.13}$$

Finally, we show that $x_n \to Q(u)$ as $n \to \infty$. Notice that

$$\begin{aligned}\Phi(\|y_n - Q(u)\|) &= \Phi(\|\beta_n(x_n - Q(u)) + (1-\beta_n)(J_{r_n}x_n - Q(u))\|) \\ &\leq \Phi(\|x_n - Q(u)\|).\end{aligned}$$

It follows from Lemma 2.3 that

$$\begin{aligned}&\Phi(\|x_{n+1} - Q(u)\|) \\ &= \Phi(\|\alpha_n(u - Q(u)) + (1-\alpha_n)(y_n - Q(u))\|) \\ &\leq \Phi((1-\alpha_n)\|y_n - Q(u)\|) + \alpha_n\langle u - Q(u), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\leq (1-\alpha_n)\Phi(\|x_n - Q(u)\|) + \alpha_n\langle u - Q(u), J_\varphi(x_{n+1} - Q(u))\rangle\end{aligned}$$

From Lemma 1.5, we see that $\Phi(\|x_{n+1} - Q(u)\|) \to 0$ as $n \to \infty$. That is,

$$\|x_n - Q(u)\| \to 0 \quad \text{as } n \to \infty.$$

This completes the proof. □

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