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FIXED POINT PROPERTY FOR GENERAL TOPOLOGIES IN BANACH LATTICES

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Abstract. In this paper we use methods developed in [8] to extend fixed point results for Banach lattices obtained for the weak topology in [3] and [4] to other topologies. We obtain conditions which guarantee that Banach lattices which have topology τ have τ-normal structure and τ-fixed point property.

Key Words and Phrases: order uniform noncreasiness, τ-fixed point property, Banach lattice.
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1. Introduction

The main goal of this paper is to study conditions which imply existence of fixed points of nonexpansive mappings in Banach lattices. For mappings on weakly compact convex sets such problems were studied in [16], [3] and [4]. However in function and sequence lattices it is also natural to consider other topologies, in particular the topology of convergence in measure in case of function lattices or the topology of coordinatewise convergence in case of sequence lattices. Those topologies were successfully applied to the fixed point theory in [8].

In this paper we use methods developed in [8] to extend fixed point results obtained for the weak topology in [3] and [4] to other topologies. In our results we consider properties related to the lattice order. However, some of them are also related to a topology τ. This is in particular the case of the so-called τ-orthogonality. In some lattices this property does not hold for the weak topology but it holds for a different one. The basic example is the space L_1 with the topology of convergence in measure (see Example 3.1). This is the main motivation for considering topologies τ different from the weak topology.

In this paper we consider Banach lattices equipped with linear topologies τ. First we prove that every τ-orthogonal uniformly monotone Banach lattice such that its every τ-sequentially compact subset is separable has normal structure with respect to τ. Then we show that a Banach space has the fixed point property with respect to a topology τ if it admits an equivalent norm such that with this new norm it is a τ-orthogonal order uniformly noncreasy Banach lattice. This extends the Browein and

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Sims result from [5]. Sims considered also a stronger version of weak orthogonality and showed that every such Banach lattice has the weak fixed point property (see [16]). We extend his result by showing that a Banach lattice has the fixed point property with respect to τ if it is strong τ-orthogonal.

2. Basic notation and some geometric properties of Banach lattices

Let X be a Banach space. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of X, respectively. For notation and terminology concerning lattices we refer the reader to [12]. Let us recall that if an inequality involves lattice and algebraic operations, then it is enough to check its validity for real numbers to be sure that it holds for vectors in an arbitrary Banach lattice (see [12, p. 1]). In the next lemma we collect some inequalities and equality which will be used in the sequel.

Lemma 2.1. *Let X be a Banach lattice. Then*

(i) $|x|+|y| = 2(|x|\wedge|y|) + ||x|-|y|| \le 2(|x|\wedge|y|)+|x-y|$,
(ii) $|x| - |x|\wedge|y| \le |x-y|$,
(iii) $|z| \le |x-z|\vee|y-z| + |x|\wedge|y|$

for all $x,y,z\in X$.

Let X be σ-complete Banach lattice. Then for each $x\in X$ we can define the principle band projection P_x by

$$P_x(y) = \bigvee_{n=1}^{\infty}\left[(n|x|)\wedge y^+\right] - \bigvee_{n=1}^{\infty}\left[(n|x|)\wedge y^-\right].$$

For all $x,y\in X$ the projection has the following properties $P_x(x)=x$, $\|P_x\| = \|I-P_x\| = 1$ and $P_x(y)=0$, $\|P_x+P_y\|=1$ whenever $|y|\wedge|x|=0$. Here I denotes the identity operator.

The projection can be used to prove the following lemma that will be needed later.

Lemma 2.2. *Let $(X,\|\cdot\|)$ be a Banach lattice and $a_1,a_2\in X$ be such that $\||a_1|\wedge|a_2|\|=0$ and $\||a_i|\wedge|x|\|=0$ for $i=1,2$ and every $x\in X$. Then*

$$\|u\| \le \frac{1}{2}\left(\|u-a_1\|+\|u-a_2\|+\|u-x\|\right)$$

for all $u,x\in X$.

Proof. Since $X\subset X^{**}$, we can replace X by X^{**} which is σ-order complete (see [12]). Now we can consider the principle band projection

$$P_{a_i}y = \bigvee_{n=1}^{\infty}\left[(n|a_i|)\wedge y^+\right] - \bigvee_{n=1}^{\infty}\left[(n|a_i|)\wedge y^-\right]$$

for $i=1,2$. Following [16], we have

$$\begin{aligned}\|u\| &= \frac{1}{2}\|(P_{a_1}+P_{a_2})(u-x)+(I-P_{a_1})(u-a_1)+(I-P_{a_2})(u-a_2)\|\\ &\le \frac{1}{2}(\|u-x\|+\|u-a_1\|+\|u-a_2\|).\end{aligned}$$

□

A Banach lattice X is said to be strictly monotone if $\|x\| < \|y\|$ whenever $x, y \in X$, $0 \le x \le y$ and $x \ne y$. Given a Banach lattice X and $\varepsilon \in [0,1]$, we put

$$\delta_{m,X}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \le y \le x,\ \|x\| \le 1,\ \|y\| \ge \varepsilon\}.$$

We say that X is uniformly monotone if $\delta_{m,X}(\varepsilon) > 0$ for all $\varepsilon \in (0,1]$. Clearly, uniformly monotone lattices are strictly monotone. The coefficient

$$\varepsilon_{0,m}(X) = \sup\{\varepsilon \in [0,1) : \delta_{m,X}(\varepsilon) = 0\}$$

is called the characteristic of monotonicity of the lattice X. Since $\delta_{m,X}(0) = 0$ and $\delta_{m,X}$ is a nondecreasing function, the condition $\varepsilon_{0,m}(X) < 1$ is equivalent to $\lim_{\varepsilon\to 1} \delta_{m,X}(\varepsilon) > 0$. In [11], the modulus of order smoothness of a Banach lattice X was introduced. We set

$$\rho_{m,X}(t) = \sup\{\|x \vee ty\| - 1 : x, y \in B(X),\ x, y \ge 0\}$$

where $t \in [0,1]$. A Banach lattice X is called order uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_{m,X}(t)}{t} = 0.$$

Observe that

$$\rho_{m,X}(1) + 1 = \sup\{\|x \vee y\| : x, y \in B(X),\ x, y \ge 0\}$$

is called the Riesz angle of X and denoted by $\alpha(X)$ (see [5]). Kurc [11] proved the duality formulae

$$\rho_{m,X^*}(t) = \sup_{0\le\varepsilon\le 1} (\varepsilon t - \delta_{m,X}(\varepsilon))$$
$$\delta_{m,X}(\varepsilon) = \sup_{0\le t\le 1} (\varepsilon t - \rho_{m,X^*}(t))$$

for all $\varepsilon, t \in [0,1]$. They show that a Banach lattice X is uniformly monotone (resp. order uniformly smooth) if and only if the dual lattice X^* is order uniformly smooth (resp. uniformly monotone). Using the above formulae it is also easy to see that $\rho_{m,X^*}(1) < 1$ if and only if $\varepsilon_{0,m}(X) < 1$.

The following definition which combines uniform monotonicity and uniform order smoothness was introduced in [3]

Definition. Let $r \in (0,1]$. A Banach lattice X is said to be r-order uniformly noncreasy (r-OUNC) if for all $u, v \in \frac{1}{2}B(X)$ we have either

$$\||u| \vee |v|\| \le r$$

or for every $y \in X$ the conditions $|y| \le |u - v|$, $\|y\| \ge r$ imply that $\||u - v| - |y|\| \le r$.

A Banach lattice X is order uniformly noncreasy (OUNC) if it is r-OUNC for some $r \in (0,1)$.

Clearly, each Banach lattice X is r-OUNC with $r = \frac{1}{2}(\rho_{m,X}(1) + 1) = \frac{1}{2}\alpha(X)$. Therefore, if $\rho_{m,X}(1) < 1$, then X is OUNC. It is also easy to see that X is r-OUNC where

$$r = \max\{\varepsilon, 1 - \delta_{m,X}(\varepsilon)\}$$

for any $\varepsilon \in (0,1)$. It follows that if $\varepsilon_{0,m}(X) < 1$, then X is OUNC. The class of all OUNC Banach lattices contains therefore all order uniformly smooth lattices and all uniformly monotone lattices. Moreover, there is a Banach lattice which is OUNC but neither uniformly monotone nor order uniformly smooth (see [3]).

Let X be a Banach space and τ be a linear topology on X. In this paper we will need also the definitions of a τ-sequentially lower semi continuous function and a τ-null type. We shall say that a function $f : X \to \mathbb{R}$ is τ-sequentially lower semicontinuous (τ-slsc) if for every sequence (x_n) in X which converges to $x \in X$ with respect to τ we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

If X is a Banach space and w is the weak topology on X it is known that the norm is a w-slsc function. If X is a dual space and w^* is the weak star topology on X the norm is also a w^*-slsc function. A τ-null type on X is a function of the form

$$\Gamma_{(x_n)}(x) = \limsup_{n\to\infty} \|x_n - x\|$$

where $x \in X$ and (x_n) is a τ-null sequence in X. It is know that the w-null types are w-slsc. Notice that if X is a Banach space and τ is a linear topology on X such that τ-null types are τ-slsc functions, then the norm is a τ-slsc function. In addition, the condition that the norm is a τ-slsc function is equivalent to the τ-sequential closedness of the closed unit ball in X.

Finally, for the sake of simplicity let us define additional notations. Let $(X, \|\cdot\|)$ be a Banach space and τ be a linear topology on X. We shall say that space X has property τ-(N) if all τ-null types are τ-slsc functions and τ-sequentially compact sets are τ-compact.

3. Applications to fixed point theory

A self-mapping T of a subset C of a Banach space X is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all x, y in C. Let τ be an arbitrary topology on X. We say that X has the fixed point property with respect to τ (τ-FPP) if every nonexpansive mapping defined on a nonempty τ-sequentially compact convex bounded subset of X has a fixed point. If the norm on X is additionally a τ-slsc function then this is in particular the case if X has normal structure with respect to τ (see [8], Theorem 1). Recall that a Banach spaces X is said to have normal structure with respect to τ (τ-NS) if for each τ-sequentially compact norm bounded convex subset D of X with $\operatorname{diam} D > 0$ there exists a point $x \in D$ such that

$$\sup\{\|x - y\| : y \in D\} < \operatorname{diam} D.$$

In the case when τ is the weak topology in the above definitions sequential compactness can be replaced by compactness.

In [5], Borwein and Sims introduced the definition of a weakly orthogonal Banach lattice X. We extend his definition to an arbitrary topology on X.

Definition. Let X be a Banach lattice and τ be a linear topology on X.

(1) X is said to be τ-orthogonal if

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\||x_n|\wedge|x_m|\|=0$$

whenever (x_n) is a sequence in X which converges to 0 with respect to τ.

(2) X is said to be strong τ-orthogonal if

$$\liminf_{n\to\infty}\||x_n|\wedge|x|\|=0$$

for every $x\in X$ and every sequence (x_n) in X which converges to 0 with respect to τ.

Clearly, if X is strong τ-orthogonal, then X is τ-orthogonal.

Example 3.1. The space $L_1([0,1])$ is strong τ-orthogonal where τ is the topology of convergence in measure (clm) and it is not strong weak-orthogonal.

To see that the space $L_1([0,1])$ is strong τ-orthogonal fix $\varepsilon>0$. For any $f\in L_1([0,1])$ there exists $\eta>0$ such that for any $B\subset[0,1]$ such that $\mu(B)<\eta$ we have $\int_B|f|d\mu<\frac{\varepsilon}{2}$, where μ is the Lebesgue measure. Let (f_n) converge to 0 with respect to μ. By the Riesz theorem there exists a subsequence (f_{n_k}) of (f_n) which converges to 0 almost everywhere. From the Jegorow theorem there exists $F\subset[0,1]$ such that $\mu([0,1]\setminus F)<\eta$ and f_{n_k} is uniformly convergent to 0 on F. Hence there exists $N\in\mathbb{N}$ such that $|f_{n_k}(x)|\le\frac{\varepsilon}{2\mu([0,1])}$ for all $k\ge N$ and $x\in F$. For $k\ge N$ we have

$$\||f_{n_k}|\wedge|f|\|_{L_1}\le\left(\int_{[0,1]\setminus F}|f|d\mu+\int_F|f_{n_k}|d\mu\right)\le\varepsilon.$$

To see that the space $L_1([0,1])$ is not strong weak-orthogonal it is enough to consider the sequence of Rademacher functions.

In [3] it was proven that a weakly orthogonal Banach lattice X with $\varepsilon_0(X)<1$ has weak normal structure. Using the next theorem and following the proof from [3], we can generalize the result to an arbitrary topology on X.

Theorem 3.1 ([10])**.** *Let X be a Banach space and τ be a linear topology on X such that every τ-sequentially compact subset A of X is separable. If X does not have τ-NS, then there is a bounded sequence (x_n) in X that converges to x with respect to τ and*

$$\lim_{n\to\infty}\|x_n-x\|=\lim_{n\to\infty}\|x_n-x_m\|>0$$

for every $m\in\mathbb{N}$. In case τ is the weak topology the separability assumption is not necessary.

Theorem 3.2. *Let X be a Banach lattice and τ be a linear topology on X such that every τ-sequentially compact subset of X is separable. If X is τ-orthogonal and does not have τ-NS, then for every $\eta>0$ there exist $x,y\in S(X)$ such that $x,y\ge0$ and*

$$(1-\eta)\max\{|a|,|b|\}\le\|ax+by\|\le(1+\eta)\max\{|a|,|b|\}$$

for all $a,b\in\mathbb{R}$. If τ is the weak topology, then the separability assumption is not necessary.

Proof. Assume that X fails to have τ-normal structure. By Theorem 3.1 there exists a sequence (x_n) in X which converges to x with respect to τ and

$$\lim_{n\to\infty}\|x_n - x\| = \lim_{n\to\infty}\|x_n - x_m\| > 0$$

for all $m \in \mathbb{N}$. Put $y_n = x_n - x$. Then (y_n) converges to 0 with respect to τ and

$$\lim_{n\to\infty}\|y_n\| = \lim_{n\to\infty}\|y_n - y_m\| > 0$$

for every $m \in \mathbb{N}$. We can assume that $\|y_n\| > c > 0$ for all $n \in \mathbb{N}$. If X is τ-orthogonal, then

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\||y_n| \wedge |y_m|\| = 0$$

so we can find subsequences (y_{n_k}) and (y_{m_k}) of (y_n) such that

$$\lim_{k\to\infty}\||y_{n_k}| \wedge |y_{m_k}|\| = 0.$$

We put $u_k = \frac{|y_{n_k}|}{\|y_{n_k}\|}$ and $v_k = \frac{|y_{m_k}|}{\|y_{m_k}\|}$. Then $u_k, v_k \in S(X)$ and $u_k, v_k \geq 0$ for every $k \in \mathbb{N}$. Using the equality in Lemma 2.1 (i) we get

$$\begin{aligned}
\|u_k + v_k\| &\leq \left\|\frac{|y_{n_k}| + |y_{m_k}|}{\|y_{n_k}\|}\right\| + \left\||y_{m_k}|\left(\frac{1}{\|y_{m_k}\|} - \frac{1}{\|y_{n_k}\|}\right)\right\| \\
&\leq \frac{1}{\|y_{n_k}\|}\left(2\||y_{n_k}| \wedge |y_{m_k}|\| + \||y_{n_k}| - |y_{m_k}|\|\right) + \left|1 - \frac{\|y_{m_k}\|}{\|y_{n_k}\|}\right| \\
&\leq \frac{1}{\|y_{n_k}\|}\left(2\||y_{n_k}| \wedge |y_{m_k}|\| + \|y_{n_k} - y_{m_k}\|\right) + \left|1 - \frac{\|y_{m_k}\|}{\|y_{n_k}\|}\right|
\end{aligned}$$

Since the expression on the right hand side tends to 1 as $k \to \infty$, given $\eta > 0$ we can find $k_0 \in \mathbb{N}$ so that

$$\|u_k + v_k\| \leq 1 + \eta$$

for every $k \geq k_0$. Let $a, b \in \mathbb{R}$. We can assume that $|a| = \max\{|a|, |b|\}$. Denote $t = \frac{b}{|a|}$, then $|t| \leq 1$. We have

$$\begin{aligned}
\|u_k + tv_k\| &\leq \|u_k + |t|v_k\| \\
&\leq |t|\|u_k + v_k\| + (1 - |t|)\|u_k\| \leq 1 + \eta.
\end{aligned}$$

Since $\|u_k \wedge v_k\| = \left\|\frac{|y_{n_k}|}{\|y_{n_k}\|} \wedge \frac{|y_{m_k}|}{\|y_{m_k}\|}\right\| \leq \frac{1}{c}\||y_{n_k}| \wedge |y_{m_k}|\|$ there exists $k_1 \in \mathbb{N}$ so that $\|u_k \wedge v_k\| \leq \eta$ for every $k \geq k_1$ and hence

$$\begin{aligned}
\|u_k + tv_k\| &\geq \|u_k - |t|v_k\| \\
&= \|u_k \vee |t|v_k - u_k \wedge |t|v_k\| \\
&\geq \|u_k \vee |t|v_k\| - \|u_k \wedge |t|v_k\| \\
&\geq \|u_k\| - \|u_k \wedge v_k\| = 1 - \|u_k \wedge v_k\| \geq 1 - \eta
\end{aligned}$$

for every $k \geq k_1$. So for $k \geq \max\{k_0, k_1\}$ we have

$$(1 - \eta)|a| \leq \|au_k + bv_k\| \leq (1 + \eta)|a|.$$

□

Corollary 3.3. *Let X be a Banach lattice and τ be a linear topology on X such that every τ-sequentially compact subset of X is separable. If X is a τ-orthogonal and $\varepsilon_0(X) < 1$, then X has τ-normal structure.*

Proof. Suppose that X fails to have τ-normal structure. Let $\eta > 0$. Then by Theorem 3.2 there exist $x, y \in S(X)$ such that $x, y \geq 0$ and $\|x + y\| \leq 1 + \eta$. Denote $z = x + y$, then $0 \leq x \leq z$ and $\|z\| \leq 1 + \eta$. Let $z_1 = \frac{z}{1+\eta}$ and $v_1 = \frac{x}{1+\eta}$. We have $\|z_1\| \leq 1$, $\|v_1\| = \frac{1}{1+\eta}$ and $1 - \|z_1 - v_1\| = \frac{\eta}{1+\eta}$. Hence $\delta_{m,X}\left(\frac{1}{1+\eta}\right) \leq \frac{\eta}{1+\eta}$, so $\lim_{\varepsilon\to 1} \delta_{m,X}(\varepsilon) = 0$ which means that $\varepsilon_{0,m}(X) = 1$. □

Corollary 3.4. *Let X be a Banach lattice and τ be a linear topology on X such that every τ-sequentially compact subset of X is separable. If X is τ-orthogonal, uniformly monotone and such that $\|\cdot\|$ is τ-slsc function, then X has τ-FPP.*

We will also consider lattices which do not have normal structure with respect to a topology τ. In this case we will use the ultrapower method. Given a Banach space X, by $l_\infty(X)$ we denote the space of all bounded sequences in X with the supremum norm. Let U be a free ultrafilter on $\mathbb{N}$. The ultrapower $(X)_U$ is the quotient space $l_\infty(X)/N$ where

$$N = \left\{(x_n) \in l_\infty(X) : \lim_U \|x_n\| = 0\right\}.$$

Here $\lim_U \|x_n\|$ stands for the limit over the ultrafilter U (see [1, p. 13]). The equivalence class of the sequence $(x_n) \in l_\infty(X)$ will be denoted by $(x_n)_U$. It is easy to see that the quotient norm is given by the formula (see [1, p. 23])

$$\|(x_n)_U\| = \lim_U \|x_n\|.$$

In the sequel we will need the following result (see [3]).

Theorem 3.5. *Let X be a Banach lattice and U be a free ultrafilter on $\mathbb{N}$. If X is OUNC, then $(X)_U$ is OUNC.*

The same tools as for weak topology can be use to study τ-FPP. The following theorem can be found in [8].

Theorem 3.6 (Generalized Goebel-Karlovitz Lemma)**.** *Let a Banach space X have the property τ-(N). Let T be a fixed point free nonexpansive mapping defined from a τ-sequentially compact bounded convex subset C of X into itself. Assume that C is minimal and let (x_n) be an approximated fixed point sequence for T in C. Then*

$$\lim_{n\to 0} \|x_n - x\| = \operatorname{diam} C$$

for every $x \in C$.

Let τ be an arbitrary topology on X. If C is a τ-sequentially compact convex subset of X, C is not necessarily norm closed. However, as every nonexpansive mapping T can be extended, in a continuous way, to the set $\overline{C}^{\|\cdot\|}$ and this extension is also a nonexpansive mapping. Banach Contraction Principle lets us find an approximate fixed point sequence for the extension of T in $\overline{C}^{\|\cdot\|}$. Finally, by an approximation argument, we can construct an approximate fixed point sequence for T in C. Moreover, since C is τ sequentially compact, taking a subsequence if necessary, we can assume that this sequence is τ-convergent.

This argument allows us to generalize the well-known Maurey result (see [1]).

Lemma 3.7 (Generalized Maurey Lemma)**.** *Let C be a convex bounded subset of a Banach space $(X, \|\cdot\|)$ and let $(x_n), (y_n)$ be approximated fixed point sequences for nonexpansive mapping T defined from C onto itself. Then there exists an approximation fixed point sequence (z_n) in C such that*

$$\limsup_{n\to\infty} \|x_n - z_n\| \leq \frac{1}{2} \limsup_{n\to\infty} \|x_n - y_n\|, \quad \limsup_{n\to\infty} \|y_n - z_n\| \leq \frac{1}{2} \limsup_{n\to\infty} \|x_n - y_n\|.$$

Using a reasoning from [8], we obtain the following result.

Theorem 3.8. *Let Banach space X have a property τ-(N). If X fails to have the τ-FPP, there exists a fixed point free nonexpansive mapping defined from a subset C of X into itself, which is τ-sequentially compact bounded convex and minimal. Moreover, under these conditions, C is also a diametral set.*

The Banach–Mazur distance of two isomorphic Banach spaces X and Y is defined by the formula

$$d(X,Y) = \inf \|S\|\|S^{-1}\|$$

where the infimum is taken over all linear isomorphisms S of X onto Y. Borwein and Sims [5] proved that a Banach space X has the weak fixed point property if there exists a weakly orthogonal Banach lattice Y such that

$$d(X,Y)\alpha(Y) < 2.$$

As a consequence every weakly orthogonal Banach lattice X with $\alpha(X) < 2$ has the weak fixed point property. In [3] the following theorem is proven.

Theorem 3.9. *A Banach space X has the weak fixed point property if there exists a weakly orthogonal r-OUNC Banach lattice Y such that*

$$d(X,Y)r < 1.$$

We will show how to modify the proof to establish the following generalization.

Theorem 3.10. *Let $(X, \|\cdot\|)$ be a Banach lattice with property τ-(N). Then X has the τ-FPP, if there exist a norm $\|\cdot\|_1$ on X and $d > 0$ such that $\|x\|_1 \leq \|x\| \leq d\|x\|_1$ for every $x \in X$ and $Y = (X, \|\cdot\|_1)$ is τ-orthogonal r-OUNC Banach lattice that satisfies condition $dr < 1$.*

Proof. Assume that X fails to have τ-FPP. Then there exist a nonempty τ-sequentially compact convex bounded $C \subset X$ and nonexpansive mapping $T : C \to C$ which has no fixed point. Moreover, there exist a nonempty τ-sequentially compact convex minimal and T-invariant subset $D \subset C$ and an approximate fixed point sequence (x_n) for T in D. We may assume that $\operatorname{diam} D = 1$ and (x_n) converges to 0 with respect to τ (in particular $0 \in D$).

Let $Y = (X, \|\cdot\|_1)$ be a τ-orthogonal r-OUNC Banach lattice such that $\|x\|_1 \leq \|x\| \leq d\|x\|_1$ for every $x \in X$ and $dr < 1$. We choose $r' > 0$ so that $r' > r$ and $dr' < 1$. By τ-orthogonality of Y, there exists an increasing sequence (n_k) such that

$$\liminf_{m\to\infty} \||x_{n_k}| \wedge |x_m|\|_1 < \frac{1}{k}$$

and by Lemma 3.6 we have

$$\lim_{m\to\infty} \|x_{n_k} - x_m\| = 1$$

for all $k \in \mathbb{N}$. Hence for every $k \in \mathbb{N}$ there exists $i_k \in \mathbb{N}$ such that

$$\|x_{n_k} - x_m\| > 1 - \frac{1}{k}$$

for all $m \geq i_k$ and there exists $m_k \geq i_k$ such that

$$\||x_{n_k}| \wedge |x_{m_k}|\|_1 < \frac{1}{k} \text{ and } \|x_{n_k} - x_{m_k}\| > 1 - \frac{1}{k}.$$

We can assume that (m_k) is an increasing sequence. We have

$$\lim_{k\to\infty} \||x_{n_k}| \wedge |x_{m_k}|\|_1 = 0 \text{ and } \lim_{k\to\infty} \|x_{n_k} - x_{m_k}\| = 1.$$

Using Lemma 3.7, we obtain an approximate fixed point sequence (z_k) in C for which

$$\limsup_{k\to\infty} \|z_k - x_{n_k}\| \leq \frac{1}{2} \limsup_{k\to\infty} \|x_{m_k} - x_{n_k}\|, \ \lim_{k\to\infty} \|z_k - x_{m_k}\| \leq \frac{1}{2} \limsup_{k\to\infty} \|x_{m_k} - x_{n_k}\|$$

and $\lim_{k\to\infty} \|z_k\| = \lim_{k\to\infty} \|x_{n_k}\| = 1$. We can further assume, taking a subsequence if necessary, that the following limits exist

$$\lim_{k\to\infty} \|z_k - x_{n_k}\| \leq \frac{1}{2}, \quad \lim_{k\to\infty} \|z_k - x_{m_k}\| \leq \frac{1}{2}.$$

Let U be a free ultrafilter on $\mathbb{N}$ and

$$u = (z_k - x_{n_k})_U\,,$$

$$v = (z_k - x_{m_k})_U\,,$$

$$y = (|x_{n_k}| - |x_{n_k}| \wedge |x_{m_k}|)_U\,.$$

Then

$$\|u\|_{\tilde{Y}} = \lim_U \|z_k - x_{n_k}\|_1 \leq \lim_U \|z_k - x_{n_k}\| = \lim_{k\to\infty} \|z_k - x_{n_k}\| \leq \frac{1}{2}$$

and similarly $\|v\|_{\tilde{Y}} \leq \frac{1}{2}$.

We have

$$\begin{aligned}\|y\|_{\tilde{Y}} &= \lim_U \||x_{n_k}| - |x_{n_k}| \wedge |x_{m_k}|\|_1 \geq \lim_U \left(\|x_{n_k}\|_1 - \||x_{n_k}| \wedge |x_{m_k}|\|_1\right) \\ &= \lim_U \|x_{n_k}\|_1 \geq \frac{1}{d} \lim_U \|x_{n_k}\| = \frac{1}{d} > r'.\end{aligned}$$

From Lemma 2.1 (ii) it follows that $|u-v| \geq |y|$. Moreover, using Lemma 2.1 (i), we get

$$\begin{aligned}
\||u-v|-|y|\|_{\tilde{Y}} &\geq \lim_U \|||x_{n_k}|-|x_{m_k}||-|x_{n_k}|+|x_{n_k}|\wedge|x_{m_k}|\|_1 \\
&= \lim_U \||x_{m_k}|-|x_{n_k}|\wedge|x_{m_k}|\|_1 \\
&\geq \lim_U \left(\|x_{m_k}\|_1 - \||x_{n_k}|\wedge|x_{m_k}|\|_1\right) \\
&= \lim_U \|x_{m_k}\|_1 \geq \frac{1}{d}\lim_U \|x_{m_k}\| = \frac{1}{d} > r',
\end{aligned}$$

Next, by Lemma 2.1 (iii) we obtain

$$\begin{aligned}
1 = \lim_{k\to\infty}\|z_k\| &\leq d\lim_{k\to\infty}\|z_k\|_1 = d\lim_U \|z_k\|_1 \\
&\leq d\lim_U (\||x_{n_k}-z_k|\vee|x_{m_k}-z_k|\|_1 + \||x_{n_k}|\wedge|x_{m_k}|\|_1) \\
&= d\lim_U \||x_{n_k}-z_k|\vee|x_{m_k}-z_k|\|_1 < \frac{1}{r'}\||u|\vee|v|\|_{\tilde{Y}},
\end{aligned}$$

and we see that $\||u|\vee|v|\|_{\widetilde{Y}} > r'$. It follows that $\widetilde{Y}$ is not r'-OUNC, so by Theorem 3.5, Y is not r-OUNC. ☐

Corollary 3.11. *Let $(X, \|\cdot\|)$ be a Banach lattice with the property τ-(N). Assume that X is τ-orthogonal and OUNC. Then X has the τ-FPP.*

In [16], Sims proved that every strong weak-orthogonal Banach lattice has the weak fixed point property. Our next theorem generalizes his result.

Theorem 3.12. *Let $(X, \|\cdot\|)$ be a Banach space with the property τ-(N). Then X has the τ-FPP if there exist a strong τ-orthogonal Banach lattice $Y = (X, \|\cdot\|_1)$ and $0 < d < \frac{4}{3}$ such that $\|x\|_1 \leq \|x\| \leq d\|x\|_1$ for every $x \in X$.*

Proof. Assume that X fails to have τ-FPP. There exist a nonempty τ- sequentially compact convex bounded subset $C \subset X$ and nonexpansive mapping $T : C \to C$ which has no fixed point. We can assume that C is minimal invariant for T and $\operatorname{diam} C = 1$. There exists in C an approximate fixed point sequence (x_n) for T. We can assume that (x_n) is τ-convergent to 0 (in particular $0 \in C$). By τ orthogonality of Y,

$$\lim_{n\to\infty}\||x_n|\wedge|x|\|_1 = 0$$

for $x \in X$ and by Lemma 3.6

$$\lim_{n\to\infty}\|x_n - x\| = 1$$

for every $x \in C$. In particular for a fixed $m \in \mathbb{N}$

$$\lim_{n\to\infty}\||x_n|\wedge|x_m|\|_1 = 0 \qquad \text{and} \qquad \lim_{n\to\infty}\|x_n - x_m\| = 1.$$

Let $\{\varepsilon_k\} \subseteq \mathbb{R}^+$ satisfy $\lim_{k\to\infty}\varepsilon_k = 0$. There exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$

$$\||x_n|\wedge|x_1|\|_1 < \varepsilon_1 \qquad \text{and} \qquad |\|x_1 - x_n\| - 1| < \varepsilon_1.$$

Having defined n_k we choose $n_{k+1} > n_k$ so that for all $n \geq n_{k+1}$

$$\||x_n|\wedge|x_{n_k}|\|_1 < \varepsilon_{k+1} \qquad \text{and} \qquad |\|x_{n_k} - x_n\| - 1| < \varepsilon_{k+1}.$$

Since

$$\||x_{n_{k+1}}| \wedge |x_{n_k}|\|_1 < \varepsilon_{k+1} \qquad \text{and} \qquad |\|x_{n_{k+1}} - x_{n_k}\| - 1| < \varepsilon_{k+1}$$

we have

$$\lim_{k\to\infty} \||x_{n_{k+1}}| \wedge |x_{n_k}|\|_1 = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = 1.$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Denote $\widetilde{a}_1 = (x_{n_k})_{\mathcal{U}}$, $\widetilde{a}_2 = (x_{n_{k+1}})_{\mathcal{U}}$ and $\widetilde{x} = (x)_{\mathcal{U}}$. Then the following conditions hold.

i) For every $x \in C$,

$$\|\widetilde{a}_1 - \widetilde{x}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_{n_k} - x\| = \lim_{k\to\infty} \|x_{n_k} - x\| = 1.$$

Similarly

$$\|\widetilde{a}_2 - \widetilde{x}\|_{\mathcal{U}} = 1.$$

In particular, $\|\widetilde{a}_i\|_{\mathcal{U}} = 1$ for $i = 1, 2$.

ii) $\|\widetilde{a}_1 - \widetilde{a}_2\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_{n_k} - x_{n_{k+1}}\| = \lim_{k\to\infty} \|x_{n_k} - x_{n_{k+1}}\| = 1$

iii) $|\widetilde{a}_i| \wedge |\widetilde{x}| = 0$ and $|\widetilde{a}_1| \wedge |\widetilde{a}_2| = 0$ for every $x \in C$ and $i = 1, 2$.

Let

$$\widetilde{W} := \left\{\widetilde{w} \in \widetilde{C} : \|\widetilde{w} - \widetilde{a}_i\|_{\mathcal{U}} \leq \frac{1}{2} \text{ for } i = 1, 2 \text{ and there exists } \widetilde{v} \in JC \text{ with } \|\widetilde{w} - \widetilde{v}\| \leq \frac{1}{2}\right\},$$

where $J : X \to (X)_{\mathcal{U}}$ is given by the formula $Jx = (x)_{\mathcal{U}} = (x, x, \ldots)$. $\widetilde{W}$ is nonempty $\left(\text{since } \frac{\widetilde{a}_1 + \widetilde{a}_2}{2} \in \widetilde{W}\right)$, convex, $\widetilde{T}$-invariant. Due to the P. K. Lin Lemma [13], for every $\varepsilon > 0$ there exists $\widetilde{v} \in \widetilde{W}$ such that $\|\widetilde{v}\|_{\mathcal{U}} > 1 - \varepsilon$. Let $x \in C$ be such that $\|\widetilde{v} - \widetilde{x}\|_{\mathcal{U}} \leq \frac{1}{2}$. Using Lemma 2.2 and inequality between norms, we get

$$\begin{aligned}
1 - \varepsilon < &\|\widetilde{v}\|_{\mathcal{U}} \leq d\|\widetilde{v}\|_{1,\mathcal{U}} \\
\leq &\frac{1}{2} d(\|\widetilde{v} - \widetilde{x}\|_{1,\mathcal{U}} + \|\widetilde{v} - \widetilde{a}_1\|_{1,\mathcal{U}} + \|\widetilde{v} - \widetilde{a}_2\|_{1,\mathcal{U}}) \\
\leq &\frac{1}{2} d(\|\widetilde{v} - \widetilde{x}\|_{\mathcal{U}} + \|\widetilde{v} - \widetilde{a}_1\|_{\mathcal{U}} + \|\widetilde{v} - \widetilde{a}_2\|_{\mathcal{U}}) \\
\leq &\frac{3}{4} d.
\end{aligned}$$

By arbitrariness of $\varepsilon > 0$ we have $d \geq \frac{4}{3}$ which contradicts assumption that $d < \frac{4}{3}$. □

Corollary 3.13. *Let $(X, \|\cdot\|)$ be a strong τ-orthogonal Banach lattice with the property τ-(N). Then X has the τ-FPP.*

References

[1] A.G. Aksoy, M.A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York, 1990.

[2] J.M. Ayerbe Toledano, T. Domínguez Benavides, G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser Verlag, Basel, 1997.

[3] A. Betiuk-Pilarska, S. Prus, *Banach lattices which are order uniformly noncreasy*, J. Math. Anal. Appl., **342**(2008), 1271-1279.

[4] A. Betiuk-Pilarska, S. Prus, *Order uniformly noncreasy Banach lattices and direct sums*, Proceedings of the Eight International Conference on Fixed Piont Theory and its Applications, July 16-22,2007, 139-150.

[5] J.M. Borwein, B. Sims, *Non-expansive mappings on Banach lattices and related topics*, Houston J. Math., **10**(1984), 339-356.
[6] A. Brunel, L. Sucheston, *On B-convex Banach spaces*, Math. System Theory, **7**(1974), 294-299.
[7] T. Domínguez Benavides, *Weak uniform normal structure in direct sum spaces*, Studia Math., **103**(1992), 283-290.
[8] T. Domínguez Benavides, J. García Falset, M.A. Japón Pineda, *The τ-fixed point property for nonexpansive mappings*, Abstr. Appl. Anal., **3**(1998), 343-362.
[9] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Wiley-Interscience, New York, 2001.
[10] W.A. Kirk, B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Acad. Publ., Dordrecht, 2001.
[11] W. Kurc, *A dual property to uniform monotonicity in Banach lattices*, Collect. Math., **44**(1993), 155-165.
[12] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, New York, 1979.
[13] P.K. Lin, *Unconditional bases and fixed points of nonexpansive mappings*, Pacific J. Math., **116**(1985), 69-76.
[14] B. Maurey, *Points fixes des contractions de certains faiblement compacts de L^1*, Seminaire d'Analyse Fonctionelle 1980-81, Exp. No. VIII, École Polytech., Palaiseau, 1981.
[15] S. Prus, *Banach spaces which are uniformly noncreasy*, Nonlinear Anal., **30**(1997), 2317-2324.
[16] B. Sims, *Orthogonality and fixed points of nonexpansive maps*, Proc. Centre Math. Anal. Austral. Nat. Univ., Austral. Nat. Univ., Canberra, **20**(1988).

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