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A NEW FAMILY OF CHEBYSHEV-HALLEY LIKE METHODS FREE FROM SECOND DERIVATIVE

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Abstract. A new family of Chebyshev-Halley like methods free from second derivative for nonlinear equations is presented in this paper. The family is at least of third order convergence and includes one fourth order method as special case. It uses only two function evaluations and one first derivative evaluation per iteration. A general error analysis is given. Several numerical examples are given to illustrate the performance of the presented methods by comparing with some other methods.
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1. Introduction

One of the most important and challenging problems in mathematics, physics and other sciences is to find the solutions of nonlinear equations. In this paper, we consider some new iterative methods to find a simple root for the nonlinear equation

$$f(x) = 0, \tag{1.1}$$

where $f : D \subseteq R \to R$ is a scalar function defined on an open interval D and sufficiently smooth in a neighborhood of the root, say α.

The typical method for solving (1.1) is Newton's method (NM), which is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.2}$$

and converges quadratically in some neighborhood of α in D.

In [6], a family of third order methods was given which requires evaluations of one function, one first derivative and one second derivative per iteration. It was given as follows,

$$x_{n+1} = x_n - \Big(1 + \frac{1}{2}\frac{L_f(x_n)}{1-\beta L_f(x_n)}\Big)\frac{f(x_n)}{f'(x_n)}, \tag{1.3}$$

where

$$L_f(x) = \frac{f''(x)f(x)}{f'^2(x)}, \quad \beta \in R. \tag{1.4}$$

One can see that Chebyshev's method (CM) ($\beta = 0$), Halley's method (HAM) ($\beta = \frac{1}{2}$) and Super-Halley's method (SHM) ($\beta = 1$) [1, 6, 10] are the special cases of (1.3).

The family in (1.3) requires the evaluation of the second derivative. However, it is usually expensive to compute the second derivative, and the practical applications of such methods are very inconvenient. Several authors introduced and analyzed methods free from second derivatives, see for example, [2]-[5], [7]-[9] and the references therein. Motivated by their work, in this paper, we modify (1.3) to give a new family of third order methods and a new fourth order method. Each iteration requires only one first derivative and two function evaluations.

The improved iterative methods and their convergence analyses are presented in Section 2. With the help of Maple, we also obtain the error equations in the section. In Section 3, we present some special cases. And in Section 4, several numerical examples are compared with other famous iterative methods to illustrate the performance of the presented methods.

2. Improved methods and convergence analyses

In order to derive an approximation to $f''(x)$ in (1.4), we consider approximating $f(x)$ around the point $(x_n, f(x_n))$ by a parabola in the following form:

$$ay^2 + y + bx + c = 0. \tag{2.1}$$

Let us impose the tangency conditions:

$$y(x_n) = f(x_n), \quad y'(x_n) = f'(x_n), \quad y(\varphi_n) = f(\varphi_n), \tag{2.2}$$

where x_n is the nth iteration and

$$\varphi_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2.3}$$

By imposing the tangency conditions (2.2), we have

$$a = -\frac{f(\varphi_n)}{f(\varphi_n)^2 + f(x_n)^2}. \tag{2.4}$$

On the other hand, from (2.1), we have

$$y''(x) = -\frac{2ay'(x)^2}{2ay(x)+1}. \tag{2.5}$$

Using (2.2), (2.4) and (2.5), we can obtain

$$f''(x_n) \approx y''(x_n) = \frac{2f(\varphi_n)f'(x_n)^2}{[f(x_n) - f(\varphi_n)]^2}. \tag{2.6}$$

Substituting (2.6) into (1.4), we have

$$L_f(x_n) \approx \frac{2f(\varphi_n)f(x_n)}{(f(x_n)-f(\varphi_n))^2}. \tag{2.7}$$

Therefore, by (1.3) and (2.7), we can obtain a new family of methods as follows,

$$x_{n+1} = x_n - \Big(1 + \frac{f(\varphi_n)f(x_n)}{[f(x_n)-f(\varphi_n)]^2 - 2\beta f(\varphi_n)f(x_n)}\Big)\frac{f(x_n)}{f'(x_n)}, \tag{2.8}$$

where $\beta \in R, \varphi_n$is defined in (2.3).

The following theorem gives the convergence orders and the error equations of the new iterative methods defined in (2.8).

Theorem 2.1. *Let $\alpha \in D$ be a simple root of a sufficiently differentiable function $f : D \subseteq R \to R$ for an open interval D. If x_0 is sufficiently close to α, then the iterative methods defined in (2.8) are at least of order three for nonzero $\beta \in R$ or of order four for $\beta = 0$, and their error equations are given by*

$$e_{n+1} = Me_n^4 + Ne_n^3 + O(e_n^5), \tag{2.9}$$

where

$$\begin{aligned} M &= 7tC_2^3 + 7C_2C_3 - 4tC_2C_3 - t^2C_2^3 - 8C_2^3, & (2.10)\\ N &= (2-t)C_2^2, & (2.11)\\ t &= 2(1+\beta), & (2.12)\\ C_k &= \frac{f^{(k)}(\alpha)}{k!f'(\alpha)},\ k = 2, 3, \cdots, & (2.13)\\ e_n &= x_n - \alpha. & (2.14) \end{aligned}$$

Proof. For any fixed $n \geq 0$, let

$$r = \frac{f(x_n)f(\varphi_n)}{f(x_n)^2 + f(\varphi_n)^2}, \tag{2.15}$$

$$\phi(r,\beta) = \frac{r}{1-2(1+\beta)r} = \frac{r}{1-tr}, \tag{2.16}$$

where t is defined in (2.12).

Substituting (2.12), (2.14), (2.15) and (2.16) into (2.8) , we have

$$e_{n+1} = e_n - \Big(1 + \phi(r,\beta)\Big)\frac{f(x_n)}{f'(x_n)}. \tag{2.17}$$

Using Taylor expansion around α and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)(e_n + C_2e_n^2 + C_3e_n^3 + C_4e_n^4 + \cdots), \tag{2.18}$$

$$f'(x_n) = f'(\alpha)(1 + 2C_2e_n + 3C_3e_n^2 + 4C_4e_n^3 + \cdots). \tag{2.19}$$

Dividing (2.18) by (2.19), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - C_2e_n^2 + 2(C_2^2 - C_3)e_n^3 + (7C_2C_3 - 4C_2^3 - 3C_4)e_n^4 + \cdots. \tag{2.20}$$

Let $\varepsilon_n = \varphi_n - \alpha$, then

$$\begin{aligned}\varepsilon_n &= e_n - \frac{f(x_n)}{f'(x_n)}\\ &= C_2e_n^2 - (2C_2^2 - 2C_3)e_n^3 + (4C_2^3 + 3C_4 - 7C_2C_3)e_n^4 + \cdots .\end{aligned}$$

Expanding $f(\varphi_n)$ at α, we have

$$\begin{aligned}f(\varphi_n) &= f'(\alpha)(e_n + C_2e_n^2 + C_3e_n^3 + C_4e_n^4 + C_5e_n^5 + C_6e_n^6 + \cdots)\\ &= f'(\alpha)\big(C_2e_n^2 - 2(C_2^2 - C_3)e_n^3 + (5C_2^3 - 7C_2C_3 + 3C_4)e_n^4 + \cdots\big).\end{aligned} \tag{2.21}$$

Hence, from (2.18) and (2.21), one has

$$f(x_n)f(\varphi_n) = f'(\alpha)^2\big(C_2 + (2C_3 - C_2^2)e_n + (3C_2^3 + 3C_4 - 4C_2C_3)e_n^2 + \cdots\big)e_n^3, \tag{2.22}$$

$$\begin{aligned}f(x_n)^2 + f(\varphi_n)^2 = f'(\alpha)^2\big(1 &+ 2C_2e_n + (2C_2^2 + 2C_3)e_n^2 + (2C_4 - 4C_2^3\\ &+ 6C_2C_3)e_n^3 + \cdots\big)e_n^2.\end{aligned} \tag{2.23}$$

Thus,

$$\begin{aligned}r &= \frac{f(x_n)f(\varphi_n)}{f(x_n)^2 + f(\varphi_n)^2}\\ &= C_2e_n + (-3C_2^2 + 2C_3)e_n^2 + (7C_2^3 - 10C_2C_3 + 3C_4)e_n^3 + \cdots ,\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\phi(r,\beta) &= \frac{r}{1 - tr}\\ &= C_2e_n + (tC_2^2 - 3C_2^2 + 2C_3)e_n^2 + (t^2C_2^3 + 4tC_2C_3\\ &\quad + 7C_2^3 - 10C_2C_3 - 6tC_2^3 + 3C_4)e_n^3 + \cdots .\end{aligned} \tag{2.25}$$

Therefore, by substituting (2.20) and (2.25) into (2.17), we obtain the error equations for the methods defined by (2.8),

$$e_{n+1} = Me_n^4 + Ne_n^3 + O(e_n^5),$$

where M, N, t are shown in (2.10)–(2.12). The proof is complete.

Remark 2.1. *From Theorem 2.1, when $\beta = 0$, the iterative method defined in (2.8) is fourth-order convergence and its error equation is given by*

$$e_{n+1} = (2C_2^3 - C_2C_3)e_n^4 + O(e_n^5),$$

where C_k, e_n are shown in (2.13), (2.14).

3. Several special cases

When β takes different values, we obtain various iterative methods.

1 In fact, when $\beta = \pm\infty$, we obtain Newton's method (1.2).

2 When $\beta = \dfrac{1}{2}$, we obtain a modified Halley's method (MHAM)

$$x_{n+1} = x_n - \Big(1 + \frac{f(\varphi_n)f(x_n)}{f(x_n)^2 + f(\varphi_n)^2 - 3f(\varphi_n)f(x_n)}\Big)\frac{f(x_n)}{f'(x_n)}. \tag{3.1}$$

3 When $\beta = 1$, we obtain a modified Super-Halley's method (MSHM)

$$x_{n+1} = x_n - \Big(1 + \frac{f(\varphi_n)f(x_n)}{f(x_n)^2 + f(\varphi_n)^2 - 4f(\varphi_n)f(x_n)}\Big)\frac{f(x_n)}{f'(x_n)}. \tag{3.2}$$

4 When $\beta = -1$, we obtain a new third order method (NEWM1)

$$x_{n+1} = x_n - \Big(1 + \frac{f(\varphi_n)f(x_n)}{f(x_n)^2 + f(\varphi_n)^2}\Big)\frac{f(x_n)}{f'(x_n)}. \tag{3.3}$$

5 When $\beta = -\dfrac{1}{2}$, we obtain another new third order iterative method (NEWM2)

$$x_{n+1} = x_n - \Big(1 + \frac{f(\varphi_n)f(x_n)}{f(x_n)^2 + f(\varphi_n)^2 - f(\varphi_n)f(x_n)}\Big)\frac{f(x_n)}{f'(x_n)}. \tag{3.4}$$

6 When $\beta = 0$, we obtain a modified Chebyshev's method (MCM)

$$x_{n+1} = x_n - \Big(1 + \frac{f(\varphi_n)f(x_n)}{(f(x_n) - f(\varphi_n))^2}\Big)\frac{f(x_n)}{f'(x_n)}. \tag{3.5}$$

Remark 3.1. *From Theorem 2, we can obtain the order of convergence and the error equations for (3.1)–(3.5).*

4. Numerical testing

In this section, we present some numerical tests to illustrate the efficiency of our iterative methods defined in (2.8).

The following methods are compared: Newton's method (NM), Chebyshev's method (CM), Halley's method (HM), Super-Halley's method (SHM), and our new methods defined in (3.1) (MHAM), (3.2) (MSHM), (3.3) (NEWM1), (3.4) (NEWM2) and (3.5) (MCM).

All computations are done by using Maple 14.0 with 128 digits floating point arithmetics (Digits:=128). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i), $|f(x_{n+1})| \leq \epsilon$; (ii), $|x_{n+1} - x_n| \leq \epsilon$. When the stopping criteria are satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this paper, we use the fixed $\epsilon = 10^{-32}$.

Displayed in Table 1 are the numbers of iterations required. Please note that some of $f_i(x)$ $(i = 1 \cdots 8)$ with the iterative methods (NM), (HAM), (SHM) and (CM) were also studied in [3] and [4].

Table 2 shows the values of x^* and $f(x^*)$ after required iterations displayed in Table 1 and all the values of x^* and $f(x^*)$ have 28th decimal places.

We use the following test functions:

$$f_1(x) = x^3 + 4x^2 - 10, \qquad f_2(x) = e^{x^2+7x-30} - 1,$$

$$f_3(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, \qquad f_4(x) = \sin(x) - \frac{x}{2},$$

$$f_5(x) = \sin^2(x) - x^2 + 1, \qquad f_6(x) = x^2 - e^x - 3x + 2,$$

$$f_7(x) = \cos(x) - x,$$

$$f_8(x) = x^2 \sin^2(x) + e^{x^2 \cos(x) \sin(x)} - 28.$$

TABLE 1. Iteration numbers of various iterative methods

$f(x)$	x_0	NM	HAM	SHM	CM	MHAM	MSHM	NEWM1	NEWM2	MCM
f_1	0.8	8	5	5	6	4	5	5	4	4
f_2	3.5	14	9	Div.	9	9	Div.	9	9	7
f_3	-1	8	5	5	5	4	4	4	4	3
f_4	1.5	7	5	5	6	4	4	5	4	4
f_5	2.3	8	5	5	6	4	5	5	5	4
f_6	0.0	6	4	4	4	3	3	3	3	3
f_7	1.7	7	5	5	5	4	3	4	4	3
f_8	3.5	8	5	5	5	4	5	4	4	4

Remark 4.1. *In Table 1, Div. means divergence.*

From the results presented in Table 1, we can see that the iterative methods of this paper converge faster than the other well known iterative methods, such as (NM), (HAM), (SHM) and (CM).

TABLE 2. The values of x^* and $f(x^*)$.

$f(x)$	x^*	$f(x^*)$
f_1	+1.36523001341409684576080682 90	−2.75126163742208476518395582 32 $e(-29)$
f_2	+3.00000000000000000000000000 00	0
f_3	−1.20764782713091892700941675 84	−7.95218378936168116011019276 80 $e(-29)$
f_4	+1.89549426703398094714403573 81	−5.24035857205462425928496364 77 $e(-30)$
f_5	+1.40449164821534122603508681 78	−7.77491325723175284085602793 95 $e(-30)$
f_6	+0.25753028543986076045536730 49	+4.69228258674996033006687922 55 $e(-30)$
f_7	+0.73908513321516064165531208 77	−2.11872566000397233950386759 74 $e(-31)$
f_8	+3.43747174342176627123219630 69	+1.24995190146208474112948047 80 $e(-27)$

5. CONCLUSION

In this paper, we present a new family of Chebyshev-Halley like methods free from second derivative and a new fourth order method. Per iteration of these methods requires only one first derivative and two function evaluations. The error equations are also given in this paper. Some numerical examples are given to illustrate the performance of the presented methods.

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