Fixed Point Theory, 13(2012), No. 1, 307-317
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

CONVERGENCE OF MODIFIED ISHIKAWA ITERATIVE PROCESSES FOR AN INFINITE FAMILY OF NONEXPANSIVE MAPPINGS

YU LI

Research Institute of Management Science and Engineering
Henan University, Kaifeng 475004, China
E-mail: henuyuli@163.com

Abstract. The purpose of this article is to modify Ishikawa iterative process to have strong convergence for an infinite family nonexpansive mappings. Convergence theorems are established in a real Banach space. The results presented in this paper mainly improve the corresponding results announced in [3], [15] and [31].
Key Words and Phrases: Nonexpansive mapping, fixed point, strong convergence, control sequence, Ishikawa iterative process.
2010 Mathematics Subject Classification: 47H09, 47J25, 47H10.

1. Introduction and preliminaries

Let E be a real Banach space with the normalized duality mapping J from E into 2^{E^*} give by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \quad \|f\| = \|x\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be Gâteaux differentiable if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S(E)$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if, for each $y \in S(E)$, the above limit is attained uniformly for $x \in S(E)$. The norm of E is said to be Fréchet differentiable if, for each $x \in S(E)$, the above limit is attained uniformly for $y \in S(E)$. It is well known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E.

Recall that, if K and D are nonempty subsets of a Banach space E such that K is nonempty closed convex and $D \subset K$, then a mapping $Q : K \to D$ is sunny ([2],[21]) provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in K$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in K$. A sunny nonexpansive retraction is a sunny retraction, which is also a nonexpansive mapping.

Let C be a nonempty closed and convex subset of E and $T : C \to C$ is a nonlinear mapping. In this paper, we use $F(T)$ to denote the fixed point set of the mapping T.

Recall that T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Recall that a mapping $f : C \to C$ is a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

For such a case, f is also said to be an α-contraction. In this paper, we use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f | f : C \to C \text{ a contraction}\}$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([1], [20], [28]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tf(x) + (1 - t)Tx, \quad \forall x \in C, \tag{1.1}$$

where $f \in \Pi_C$. Then Banach's Contraction Principle guarantees that T_t has a unique fixed point x_t in C.

Xu [28] proved that, if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from Π_C onto $F(T)$.

Iterative methods are popular tools to approximate fixed points of nonlinear mappings. Recall that the normal Mann's iteration was introduced by Mann [11] in 1953. Recently, construction of fixed points for nonlinear mappings via the normal Mann's iteration has been extensively investigated by many authors. The normal Mann's iteration generates a sequence $\{x_n\}$ in the following manner:

$$x_0 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \ge 0, \tag{1.2}$$

where x_0 is an initial value and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Next, we recall another popular iteration: Ishikawa iteration. Ishikawa iteration was introduced by Ishikawa [9] in 1974. Ishikawa iteration generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_0 \in C, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \quad \forall n \ge 0, \end{cases} \tag{1.3}$$

where x_0 is an initial value and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$.

Process (1.3) is indeed more general than process (1.2). But research has been concentrated on (1.2) due probably to the reasons that the process (1.2) is simpler than that of (1.3) and that a convergence theorem for the process (1.2) may possibly lead to a convergence theorem for process (1.3) provided the sequence $\{\beta_n\}$ satisfies certain appropriate conditions. However, the introduction of the process (1.3) has its own right. As a matter of fact, process (1.2) may fail to converge while process (1.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space [7]. Both processes (1.2) and (1.3) have only weak convergence, in general (see [8] for an example). For example, Reich [22] shows that if the underlying space E is uniformly convex and has a Frechet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)=\infty$, then the sequence $\{x_n\}$ generated by the process (1.2) converges weakly to a member of $F(T)$. (An extension of this result to the process (1.3) can be found in [27].)

Therefore, many authors try to modify processes (1.2) and (1.3) to have strong convergence for nonexpansive mappings (see [3]-[5], [10], [12]-[19], [31]) and the references therein).

In 2008, Qin, Su and Shang [15] considered the following iterative process.

$$\begin{cases} x_0 \in C, \\ z_n = \gamma_n x_n + (1-\gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{cases}$$

where x_0 in an initial value, $u \in C$ is a fixed element, T is a nonexpansive mapping, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. They proved that the sequence $\{x_n\}$ defined by the above iterative process converges strongly to some fixed point of T in a uniformly smooth Banach space.

The problem of approximating a common fixed point of a family of nonexpansive mappings has been considered by many authors (see [6],[12],[16]-[18],[24]-[26],[30],[33] and the references therein). In 2001, Shimoji and Takahashi [24] considered the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = r_n T_n U_{n,n+1} + (1-r_n)I, \\ U_{n,n-1} = r_{n-1}T_{n-1}U_{n,n} + (1-r_{n-1})I, \\ \cdots \\ U_{n,k} = r_k T_k U_{n,k+1} + (1-r_k)I, \\ u_{n,k-1} = r_{k-1}T_{k-1}U_{n,k} + (1-r_{k-1})I, \\ \cdots \\ U_{n,2} = r_2 T_2 U_{n,3} + (1-r_2)I, \\ W_n = U_{n,1} = r_1 T_1 U_{n,2} + (1-r_1)I, \end{cases} \tag{1.4}$$

where r_1, r_2, $\cdots$ are real numbers such that $0 \leq r_n \leq 1$, $T_1, T_2, \cdots$ be an infinite family of nonexpansive mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n, we have the following lemmas which are important to prove our main results.

Lemma 1.1. ([24]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty}F(T_n)$ is nonempty and $r_1, r_2, \cdots$ be real numbers such that $0 < r_n \leq \gamma < 1$ for any $n \geq 1$. Then, for any $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty}U_{n,k}x$ exists.*

Using Lemma 1.1, one can define the mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C. \tag{1.5}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \cdots$ and $r_1, r_2, \cdots$.

Throughout this paper, we always assume that $0 < r_n \leq \gamma < 1$ for all $n \geq 1$.

Lemma 1.2. ([24]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty and $r_1, r_2, \cdots$ be real numbers such that $0 < r_n \leq \gamma < 1$ for any $n \geq 1$. Then $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Recently, Cho, Kang and Qin [3] considered the following process for an infinite family nonexpansive mappings in a uniformly smooth Banach space

$$\begin{cases} x_0 = x \in C \quad \textit{arbitrarily chosen}, \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where W_n is generated by (1.4) and $f \in \Pi_C$. They proved that the sequence $\{x_n\}$ defined by (1.6) converges to point in $\cap_{i=1}^{\infty} F(T_i)$ under some appropriate assumptions imposed on the control sequences.

Motivated by Cho, Kang and Qin [3], Kim and Xu [10], Qin, Su and Shang [15], Qin, Su and Wu [19], Yao, Chen and Yao [31], we study the problem of modifying Ishikawa iterative process to have strong convergence for an infinite family nonexpansive mappings without any compact assumption in a real Banach space.

In order to prove our main results, we need the following lemmas.

Lemma 1.3. ([28]) *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $f \in \Pi_C$. Then the sequence $\{x_t\}$ defined by*

$$x_t = t f(x_t) + (1-t) T x_t$$

converges strongly to a point in $F(T)$. If we define a mapping $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t \to 0} x_t, \quad \forall f \in \Pi_C.$$

Then $Q(f)$ solves solves the following variational inequality:

$$\langle (I-f)Q(f), J(Q(f)-p) \rangle \leq 0, \quad f \in \Pi_C, p \in F(T).$$

Lemma 1.4. ([23]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n) y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5. *In a Banach space E, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle, \quad \forall x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 1.6. ([29]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a uniformly smooth and strictly convex Banach space. Let* $T_i : C \to C$ *be a nonexpansive mapping for each* $i \in \mathbb{Z}^+$ *and* $f : C \to C$ *an* α*-contraction. Assume that* $\mathcal{F} = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be real number sequences in* $(0,1)$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:* $x_0 \in C$ *and*

$$\begin{cases} z_n = \gamma_n W_n x_n + (1-\gamma_n)x_n, \\ y_n = \beta_n W_n z_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{Υ}$$

where W_n *is generated in* (1.4). *Assume that the following restrictions are satisfied.*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(b) *there exist constants* $b, b' \in (0,1)$ *such that* $0 < b \leq \beta_n \leq b' < 1$, $\forall n \geq 0$;

(c) *there exits a constant* $a \in (0, b]$ *such that* $\gamma_n \leq \frac{b-a}{2-b}$, $\forall n \geq 0$;

(d) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

Proof. The proof is split into four steps.

Step 1. Show that $\{x_n\}$ is bounded.

Fixing $p \in \mathcal{F}$, we see that

$$\begin{aligned} \|z_n - p\| &\leq \gamma_n \|W_n x_n - p\| + (1-\gamma_n)\|x_n - p\| \\ &\leq \gamma_n \|x_n - p\| + (1-\gamma_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)(\beta_n \|W_n z_n - p\| + (1-\beta_n)\|x_n - p\|) \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq [1 - \alpha_n(1-\alpha)]\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned}$$

By simple induction, we obtain that

$$\|x_n - p\| \leq \max\{\frac{\|p - f(p)\|}{1-\alpha}, \|x_0 - p\|\}, \quad \forall n \geq 0.$$

This shows that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Step 2. Show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.1}$$

Put $l_n = \frac{x_{n+1}-(1-\beta_n)x_n}{\beta_n}$. It follows that

$$x_{n+1} = \beta_n l_n + (1-\beta_n)x_n, \quad \forall n \geq 0. \tag{2.2}$$

Now, we compute $\|l_{n+1} - l_n\|$. Note that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1})y_{n+1} - (1-\beta_{n+1})x_{n+1}}{\beta_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + (1-\alpha_n)y_n - (1-\beta_n)x_n}{\beta_n} \\ &= \frac{\alpha_{n+1}\big(f(x_{n+1}) - y_{n+1}\big)}{\beta_{n+1}} - \frac{\alpha_n\big(f(x_n) - y_n\big)}{\beta_n} + W_{n+1}z_{n+1} - W_n z_n. \end{aligned}$$

It follows that

$$\begin{aligned} \|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1}}{\beta_{n+1}} \|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{\beta_n}\|y_n - f(x_n)\| \\ + \|z_{n+1} - z_n\| + \|W_{n+1}z_n - W_n z_n\|. \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\|z_{n+1}-z_n\| \leq \|x_{n+1}-x_n\| + \|W_n x_n - x_n\||\gamma_{n+1}-\gamma_n| + \gamma_{n+1}\|W_{n+1}x_n - W_n x_n\|. \tag{2.4}$$

Since T_i and $U_{n,i}$ are nonexpansive, we see from (1.4) have

$$\begin{aligned} \|W_{n+1}x_n - W_n x_n\| &= \|r_1 T_1 U_{n+1,2}x_n - r_1 T_1 U_{n,2}x_n\| \\ &\leq \gamma_1 \|U_{n+1,2}x_n - U_{n,2}x_n\| \\ &\leq r_1 r_2 \|U_{u+1,3}x_n - U_{n,3}x_n\| \\ &\leq \cdots \\ &\leq r_1 r_2 \cdots r_n \|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\ &\leq M_1 \prod_{i=1}^{n} r_i, \end{aligned} \tag{2.5}$$

where $M_1 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M_1$ for all $n \geq 0$. In a similar way, we can obtain that

$$\|W_{n+1}z_n - W_n z_n\| \leq M_2 \prod_{i=1}^{n} r_i, \tag{2.6}$$

where $M_2 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}z_n - U_{n,n+1}z_n\| \leq M_2$ for all $n \geq 0$. Combining (2.4) with (2.5), we arrive at

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \|W_n x_n - x_n\||\gamma_{n+1} - \gamma_n| + \gamma_{n+1} M_1 \prod_{i=1}^{n} r_i. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.3), we see that

$$\begin{aligned}
&\|l_{n+1} - l_n - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{\beta_n}\|y_n - f(x_n)\| \\
&\quad + \|W_n x_n - x_n\||\gamma_{n+1} - \gamma_n| + \gamma_{n+1} M_1 \prod_{i=1}^{n} r_i + M_2 \prod_{i=1}^{n} r_i \qquad (2.8)\\
&\le \frac{\alpha_{n+1}}{\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{\beta_n}\|y_n - f(x_n)\| + M_3(|\gamma_{n+1} - \gamma_n| + 2\prod_{i=1}^{n} r_i),
\end{aligned}$$

where M_3 is an appropriate positive constant such that

$$M_3 = \max\{\sup_{n\ge 0}\{\|W_n x_n - x_n\|\}, M_1, M_2\}.$$

It follows from the conditions (a), (b), (d) and $0 < r_n \le \gamma < 1$ that

$$\limsup_{n\to\infty} \big(\|l_{n+1} - l_n - \|x_{n+1} - x_n\|\big) \le 0.$$

In view of Lemma 1.4, we see that $\lim_{n\to\infty} \|l_n - x_n\| = 0$, which combines with (2.2) shows that (2.1) holds.

Step 3. Show that

$$\lim_{n\to\infty} \|W x_n - x_n\| = 0. \qquad (2.9)$$

Note that

$$\begin{aligned}
&\|x_n - W_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - W_n z_n\| + \|W_n z_n - W_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\| + (1-\beta_n)\|x_n - W_n z_n\| + \|z_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\| + (1-\beta_n)\|x_n - W_n x_n\| \\
&\quad + (2-\beta_n)\|z_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\| + (1-\beta_n)\|x_n - W_n x_n\| \\
&\quad + (2-\beta_n)\gamma_n\|W_n x_n - x_n\|.
\end{aligned}$$

From the condition (c), we arrive at

$$a\|x_n - W_n x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\|,$$

In view of the conditions (a), (b) and (2.1), we obtain that

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0.$$

From [Remark 3.1, 32], we see that (2.9) holds.

Step 4. Show that $x_n \to x^*$ as $n \to \infty$.

First, we show that

$$\limsup_{n\to\infty}\langle x^* - f(x^*), J(x^* - p)\rangle \le 0,$$

where $x^* = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus we have

$$\|x_t - x_n\| = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|.$$

On the other hand, for any $t \in (0,1)$, we see that

$$\begin{aligned}
\|x_t - x_n\|^2 &= (1-t)\big(\langle Wx_t - Wx_n, J(x_t - x_n)\rangle + \langle Wx_n - x_n, J(x_t - x_n)\rangle\big)\\
&\quad + t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + t\langle x_t - x_n, J(x_t - x_n)\rangle\\
&\le (1-t)\big(\|x_t - x_n\|^2 + \|Wx_n - x_n\|\|x_t - x_n\|\big)\\
&\quad + t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + t\|x_t - x_n\|^2\\
&\le \|x_t - x_n\|^2 + \|Wx_n - x_n\|\|x_t - x_n\| + t\langle f(x_t) - x_t, J(x_t - x_n)\rangle.
\end{aligned}$$

It follows that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{1}{t}\|Wx_n - x_n\|\|x_t - x_n\|, \quad \forall t \in (0,1).$$

In view of (2.9), we see that

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{2.10}$$

Since the fact that J is strong to weak* uniformly continuous on bounded subsets of E, we see that

$$\begin{aligned}
&|\langle f(x^*) - x^*, J(x_n - x^*)\rangle - \langle x_t - f(x_t), J(x_t - x_n)\rangle|\\
&\le |\langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle| + |\langle f(x^*) - x^* + x_t - f(x_t), J(x_n - x_t)\rangle|\\
&\le \|f(x^*) - x^*\|\|J(x_n - x^*) - J(x_n - x_t)\|\\
&\quad + \|f(x^*) - x^* + x_t - f(x_t)\|\|x_n - x_t\| \to 0, \quad \text{as } t \to 0.
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0,\delta)$ the following inequality holds

$$\langle f(x^*) - x^*, J(x_n - x^*)\rangle \le \langle x_t - f(x_t), J(x_t - x_n)\rangle + \epsilon.$$

This implies that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \le \limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle + \epsilon.$$

Since ϵ is arbitrary and (2.10), we see that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \le 0. \tag{2.11}$$

In view of Lemma 1.5, we see that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\alpha_n)(y_n-x^*)+\alpha_n(f(x_n)-x^*)\|^2\\ &\le (1-\alpha_n)^2\|y_n-x^*\|^2+2\alpha_n\langle f(x_n)-x^*, J(x_{n+1}-x^*)\rangle\\ &\le (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\langle f(x_n)-f(x^*), J(x_{n+1}-x^*)\rangle\\ &\quad +2\alpha_n\langle f(x^*)-x^*, J(x_{n+1}-x^*)\rangle\\ &\le (1-\alpha_n)^2\|x_n-x^*\|^2+\alpha_n\alpha(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\ &\quad +2\alpha_n\langle f(x^*)-x^*, J(x_{n+1}-x^*)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*)-x^*, J(x_{n+1}-x^*)\rangle\\ &\le [1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}]\|x_n-x^*\|^2\\ &\quad +\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}[\frac{1}{1-\alpha}\langle f(x^*)-x^*, J(x_{n+1}-x^*)\rangle+\frac{\alpha_n}{2(1-\alpha)}M_4],\end{aligned}$$

where M_4 is an appropriate constant such that $M_4 \ge \sup_{n\ge1}\{\|x_n-x^*\|^2\}$. Put

$$j_n = \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}$$

and

$$t_n = \frac{1}{1-\alpha}\langle f(x^*)-x^*, J(x_{n+1}-x^*)\rangle+\frac{\alpha_n}{2(1-\alpha)}M_4.$$

It follows that

$$\|x_{n+1}-x^*\|^2 \le (1-j_n)\|x_n-x^*\|+j_nt_n, \quad \forall n\ge 0. \tag{2.12}$$

It follows from the conditions (a), (b) and (2.11) that

$$\lim_{n\to\infty} j_n=0,\ \sum_{n=0}^{\infty} j_n=\infty,\ \limsup_{n\to\infty} t_n\le 0.$$

In view of Lemma 1.6, we obtain that $x_n \to x^*$ as $n\to\infty$. This completes the proof.

As corollaries of Theorem 2.1, we have the following results.

Taking $T_i = I$, the identity mapping, we see that $W_n = I$ for $\forall n \ge 1$. Then the strict convexity of E in Theorem 2.1 is not be needed.

Corollary 2.2. *Let C be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : C \to C$ an α-contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$\begin{cases} z_n = \gamma_nTx_n+(1-\gamma_n)x_n,\\ y_n=\beta_nTz_n+(1-\beta_n)x_n,\\ x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)y_n, \quad \forall n\ge 0.\end{cases}$$

Assume that the following restrictions are satisfied.

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0$;

(b) *there exist constants* $b, b' \in (0,1)$ *such that* $0 < b \le \beta_n \le b' < 1$, $\forall n \ge 0$;

(c) *there exits a constant* $a \in (0, b]$ *such that* $\gamma_n \le \frac{b-a}{2-b}$, $\forall n \ge 0$;

(d) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to some point in* $F(T)$.

Putting $\{\gamma_n\} = 0$ for all $n \ge 0$ in Theorem 2.1, we have the following.

Corollary 2.3. (Cho, Kang and Qin [3]). *Let* C *be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space* E. *Let* $T_i : C \to C$ *be a nonexpansive mapping for each* $i \in \mathbb{Z}^+$ *and* $f : C \to C$ *an* α*-contraction. Assume that* $\mathcal{F} = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:* $x_0 \in C$ *and*

$$\begin{cases} y_n = \beta_n W_n x_n + (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad \forall n \ge 0, \end{cases}$$

where W_n *is generated in* (1.4). *Assume that the following restrictions are satisfied.*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0$;

(b) *there exist constants* $b, b' \in (0,1)$ *such that* $0 < b \le \beta_n \le b' < 1$, $\forall n \ge 0$.

Then the sequence $\{x_n\}$ *converges strongly to some point in* $\mathcal{F}$.

Remark 2.4. Comparing Corollary 2.3 with Theorem 1 of Yao, Chen and Yao [31], we improve their results from a single mapping to an infinite family of mappings.

References

[1] F.E. Browder, *Fixed point theorems for noncompact mappings in Hilbert spaces*, Proc. Natl. Acad. Sci. USA, **53**(1965), 1272-1276.

[2] R.E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math., **47**(1973), 341-355.

[3] Y.J. Cho, S.M. Kang, X. Qin, *Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces*, Comput. Math. Appl., **56**(2008), 2058-2064.

[4] Y.J. Cho, S.M. Kang, X. Qin, *Convergence theorems of fixed points for a finite family of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2008**(2008), Article ID 856145.

[5] Y.J. Cho, X. Qin, *Convergence of a general iterative method for nonexpansive mappings in Hilbert spaces*, J. Comput. Appl. Math., **228**(2009), 458-465.

[6] L.C. Ceng, J.C. Yao, *Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, Appl. Math. Comput., **198**(2008), 729-741.

[7] C.E. Chidume, S.A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Am. Math. Soc., **129**(2001), 2359-2363.

[8] A. Genel, J. Lindenstrass, *An example concerning fixed points*, Israel J. Math., **22**(1975), 81-86.

[9] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44**(1974), 147-150.

[10] T.H. Kim, H.K. Xu, *Strong convergence of modified Mann iterations,* Nonlinear Anal., **61**(2005), 51-60.

[11] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4**(1953), 506-510.

[12] G. Marino, V. Colao, X. Qin, S.M. Kang, *Strong convergence of the modified Mann iterative method for strict pseudo-contractions*, Comput. Math. Appl., **57**(2009), 455-465.

[13] S. Plubtieng, R. Wangkeeree, *Strong convergence of modified Mann iterations for a countable family of nonexpansive mappings*, Nonlinear Anal., **70**(2009), 3110-3118.
[14] X. Qin, Y. Su, *Approximation of a zero point of accretive operator in Banach spaces*, J. Math. Anal. Appl., **329**(2007), 415-424.
[15] X. Qin, Y. Su, M. Shang, *Strong convergence of the composite Halpern iteration*, J. Math. Anal. Appl., **339**(2008), 996-1002.
[16] X. Qin, Y.J. Cho, J.I. Kang, S.M. Kang, *Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces*, J. Comput. Appl. Math., **230**(2009), 121-127.
[17] X. Qin, Y.J. Cho, S.M. Kang, H. Zhou, *Convergence theorems of common fixed points for a family of Lipschitz quasi-pseudocontractions,* Nonlinear Anal., **71**(2009), 685-690.
[18] X. Qin, Y.J. Cho, S.M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces,* J. Comput. Appl. Math., **225**(2009), 20-30.
[19] X. Qin, M. Shang, Y. Su, *Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems*, Math. Comput. Modelling, **48**(2008), 1033-1046.
[20] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75**(1980), 287-292.
[21] S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl., **44**(1973), 57-70.
[22] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**(1979), 274-276.
[23] T. Suzuki, *Strong convergence of krasnoselskii and mann's type sequences for one-parameter nonexpansive semigroups without bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227-239.
[24] K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math., **5**(2001), 387-404.
[25] M. Shang, Y. Su, X. Qin, *Strong convergence theorems for a finite family of nonexpansive mappings*, Fixed Point Theory Appl., **2007**(2007), Art. ID 76971.
[26] W. Takahashi, *Viscosity approximation methods for countable families of nonexpansive mappings in Banach spaces*, Nonlinear Anal., **70**(2009), 719-734.
[27] K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process,* J. Math. Anal. Appl., **178**(1993), 301-308.
[28] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.
[29] H.K. Xu, *An iterative approach to quadratic optimization,* J. Optim. Theory Appl., **116**(2003), 659-678.
[30] Y. Yao, J.C. Yao, H. Zhou, *Approximation methods for common fixed points of infinite countable family of nonexpansive mappings*, Comput. Math. Appl., **53**(2007), 1380-1389.
[31] Y. Yao, R. Chen, J.C. Yao, *Strong convergence and certain control conditions for modified Mann iteration*, Nonlinear Anal., **68**(2008), 1687-1693.
[32] Y. Yao, Y.C. Liou, J.C. Yao, *Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings*, Fixed Point Theory Appl., **2007**(2007), Article ID 64363.
[33] H. Zhou, L. Wei, Y.J. Cho, *Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings in reflexive Banach spaces*, Appl. Math. Comput., **173**(2006), 196-212.

Received: July 9, 2009; Accepted: January 28, 2011.