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KRASNOSEL'SKII TYPE FIXED POINT THEOREM FOR NONLINEAR EXPANSION

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Abstract. A new fixed point theorem on nonlinear expansive operators as defined in this article is firstly pointed out. Subsequently, we establish several fixed point theorems on the sum of $A + B$, where A is a compact operator, B is a nonlinear expansive operator. The results obtained generalize and improve the corresponding results of Avramescu and Xiang in papers [C. Avramescu, C. Vladimirescu, Some remarks on Krasnoselskii's fixed point theorem, Fixed Point Theory, 4 (2003) 3–13, T. Xiang, R. Yuan, A class of expansive-type Krasnosel'skii fixed point theorems, Nonlinear Anal. 71 (2009) 3229–3239]. As applications, the existence theorem of nonnegative solutions for a class of nonlinear integral equation is discussed.

Key Words and Phrases: Fixed point theorem, nonlinear expansive mapping, nonlinear integral equation.
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1. Introduction

In many problems of analysis, one encounters operators which may be split in the form $A + B$, where A is a compact operator and B is contraction in some sense, and itself has neither of these properties. Thus neither the Schauder fixed point theorem nor the Banach contraction principle applies directly in this case, and it becomes desirable to develop fixed point theorems for such situations. An early theorem of this type was given by Krasnosel'skii [1] (or see [2]).

Theorem A. *Let M be a closed convex nonempty subset of Banach space X. Suppose that A and B map M into X such that*

(i) A is continuous and $A(M)$ is contained in a compact set;

(ii) B is a contraction with constant $\alpha < 1$;

(iii) $Ax + By \in M (\forall\ x, y \in M)$. Then the operator equation $Au + Bu = u$ has a solution in M.

Since then there have appeared a large number of papers contributing generalizations or modifications of the Theorem A and their applications. One of the main features of such generalizations is the adopting of generalized forms of the Banach contraction principle or the Schauder fixed point theorem. See [3] and the references therein. Here the operator equation $Au + Bu = u$ may also be considered as a perturbation of $Au = u$. In [1], Krasnosel'skii asserted the existence of a solution of the perturbed equation under the condition of the perturbative item B being a contraction. However, in 2003, Avramescu and Vladimirescu firstly considered that the perturbative item B as an expansion rather than a contraction, in [4] they obtained the following result.

Theorem B. *Let $(X, \|\cdot\|)$ be a Banach space, and let $M = \{x \in X : \|x\| < r\}$. Suppose that A and B map M into X such that*

(i) A is continuous, $A(M)$ is contained in a compact set;

(ii) B is an expansive mapping (i.e. there exists a constant $h > 1$ such that $\|Bx - By\| \geq h\|x - y\|$, for all $x, y \in X$), $B\theta = \theta$ and $I - B$ is surjective;

(iii) $\sup_{x \in M} \|Ax\| \leq r(h - 1)$.

Then the operator equation $Au + Bu = u$ has a solution in M.

Recently, Xiang and Yuan [5] also considered B as an expansion and obtained the following result.

Theorem C. *Let $(X, \|\cdot\|)$ be a Banach space, and let M be a nonempty closed convex subset of X. Suppose that A and B map M into X such that*

(i) A is continuous, $A(M)$ is contained in a compact subset of X;

(ii) B is an expansive mapping;

(iii) $x \in M$ implies $Ax + B(M) \supset M$.

Then the operator equation $Au + Bu = u$ has a solution in M.

The purpose of this paper is to improve and generalize the Theorems B and C. Being directly motivated by [4–5], in this paper, we will define a new class of mappings which is said to be nonlinear expansion, and prove a new fixed point theorem on nonlinear expansions. As main results in present paper we will give several Krasnosel'skii type fixed point theorems for nonlinear expansion, and an application will be also discussed.

The remainder of this paper is organized as follows. In Section 2, we define a new class of nonlinear expansive mappings and prove a fixed point theorem on which. In Section 3, we give the results on Krasnosel'skii type fixed point theorems for nonlinear expansion. In Section 4, the existence of nonnegative solutions of a class of nonlinear integral equation is discussed.

2. On nonlinear expansion

In [6], F.E. Browder proved that the constant α in Banach contraction principle can be replaced by the use of a nondecreasing and right continuous function. Inspiration by this idea we now investigate the nonlinear expansion in present section.

Definition 2.1. *Let (X, d) be a metric space, and let M be a certain subset of X. The mapping $T : M \to X$ is said to be a nonlinear expansion, if there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying (i) ϕ is nondecreasing, (ii) $\phi(t) > t$ for each $t > 0$, (iii) ϕ is right continuous, such that*

$$d(Tx, Ty) \geq \phi(d(x, y)), \text{for all } x, y \in X. \tag{2.1}$$

Remark 2.2. *Note that if one takes $\phi(t) = ht$ with $h > 1$, the nonlinear expansion in (2.1) reduces to an expansion with constant h.*

The main result in this section is the following fixed point theorem on nonlinear expansive mappings.

Theorem 2.3. *Let M be a closed subset of complete metric space (X, d). Assume that mapping $T : M \to X$ is a nonlinear expansion and $T(M) \supset M$, then T has a unique fixed point $u \in M$.*

Proof. Uniqueness is clear from (2.1), so we need only prove existence. Let $\mu = \inf\{d(x, Tx) : x \in M\}$. Now, we prove $\mu = 0$. Suppose that this is not the case. For any $\epsilon > 0$, we can find $x \in M$ such that

$$\mu \leq d(x, Tx) \leq \mu + \epsilon.$$

Since $T(M) \supset M$, there is $y \in M$ such that $x = Ty$, and so $\mu \leq d(y, Ty)$. By virtue of nondecreasing property of ϕ, we have

$$\phi(\mu) \leq \phi(d(y, Ty)) \leq d(Ty, T^2y) = d(x, Tx) < \mu + \epsilon,$$

so $\phi(\mu) \leq \mu$, a contraction. Thus $\mu = 0$.

In fact, it has just been confirmed that there is a sequence $\{x_n\}$ in M satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. We next show that such $\{x_n\}$ is a Cauchy sequence. Suppose that this is not the case. We can find two sequences of integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k \geq k$, such that

$$\alpha = \lim_{p\to\infty,\ k\geq p} \inf d(x_{m_k}, x_{n_k}) > 0.$$

We define the sequence of real numbers $\{\alpha_k\}$ by

$$\alpha_k = d(x_{m_k}, x_{n_k}),\ k = 1, 2, \ldots,$$

then there is a subsequence of α_k which is nonincreasing and converges at α. Without loss of generality, we might suppose that it is $\{\alpha_k\}$. Thus

$$\begin{aligned} \phi(\alpha_k) &= \phi(d(x_{m_k}, x_{n_k})) \\ &\leq d(Tx_{m_k}, Tx_{n_k}) \\ &\leq d(Tx_{m_k}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) \\ &= d(Tx_{m_k}, x_{m_k}) + d(Tx_{n_k}, x_{n_k}) + \alpha_k. \end{aligned} \tag{2.2}$$

Let $k \to \infty$ in (2.2), we obtain

$$\phi(\alpha) \leq \alpha,$$

a contradiction. Therefore $\{x_n\}$ is Cauchy sequence, and

$$\lim_{n\to\infty} x_n = x \in M,$$

for X is complete and M is closed.

Since $M \subset T(M)$, there exist $\{u_n\}$ and u in M such that $x_n = Tu_n$ and $x = Tu$. Finally, we prove u is a fixed point. For any $\epsilon > 0$, there is a nature number N such that $d(x_n, Tx_n) < \frac{\epsilon}{3}$ and $d(x_n, x) < \frac{\epsilon}{3}$, provided that $n > N$. By the nonlinear expansion of T, we have

$$\begin{aligned} d(u, Tu) &\leq d(u, u_n) + d(u_n, x_n) + d(x_n, Tu) \\ &\leq \phi(d(u, u_n)) + \phi(d(u_n, x_n)) + d(x_n, Tu) \\ &\leq d(Tu, Tu_n) + d(Tu_n, Tx_n) + d(x_n, Tu) \\ &= 2d(x, x_n) + d(x_n, Tx_n) < \epsilon, \end{aligned}$$

which implies $Tu = u$.

Remark 2.4. *Unlike the formerly several results on expansions, we give an independent proof of Theorem 2.3 which do not rely upon the fixed point theorems on* T^{-1}.

Corollary 2.5. *Let* M *be a closed subset in a metric space* (X, d)*, and let mapping* $T : M \to X$ *be a nonlinear expansive and surjective. Then* T *has a unique fixed point in* M*.*

3. Krasnosel'skii type fixed point theorems for nonlinear expansion

The main result of this section is the following Krasnosel'skii type fixed point theorem under the condition of nonlinear expansion.

Theorem 3.1. *Let* $(X, \|\cdot\|)$ *be a Banach space, and let* M *be a nonempty closed convex subset of* X*. Suppose that* A *and* B *map* M *into* X *such that*

(i) A *is continuous,* $A(M)$ *is contained in a compact subset of* X*;*

(ii) B *is a nonlinear expansion with* ϕ*;*

(iii) $Ax + B(M) \supset M$*, for any* $x \in M$*.*

Then there exists a point $u \in M$ *such that* $Au + Bu = u$*.*

Proof. For any fixed $x \in M$, we define a mapping $T_x : M \to X$ such that

$$T_x(z) = Ax + Bz.$$

Then T_x is a nonlinear expansion, since

$$\|T_x z_2 - T_x z_1\| = \|Bz_2 - Bz_1\| \geq \phi(\|z_2 - z_1\|), for\ all\ z_1, z_2 \in M,$$

and $T_x(M) = Ax + B(M) \supset M$. By using Theorem 2.3, there is a unique fixed point $y \in M$, i.e.

$$y = Ax + By.$$

From the above discussing, a new mapping $S : M \to M$ is defined, such that $S(x) = y$ for any $x \in M$.

Next we will prove that the mapping S is continuous and $S(M)$ is contained in a compact subset of X. For a sequence $\{x_n\}$ and its limit x_0 in M, we set $y_n = S(x_n)$ and $y_0 = S(x_0)$ respectively. Now we show that $\lim_{n\to\infty} y_n = y_0$. Suppose that this is not the case. We can find a subsequence y_{n_k} such that $\eta = \liminf_{p\to\infty,\ k\geq p} \|y_{n_k} - y_0\| > 0$. From the nonlinear expansion of mapping B, we have

$$\begin{aligned} \phi(\|y_{n_k} - y_0\|) &\leq \|By_{n_k} - By_0\| \\ &\leq \|(Ax_{n_k} + By_{n_k}) - (Ax_0 + By_0)\| + \|Ax_{n_k} - Ax_0\| \\ &= \|y_{n_k} - y_0\| + \|Ax_{n_k} - Ax_0\|. \end{aligned} \tag{3.1}$$

Without loss of generality, we might suppose that $\{\|y_{n_k} - y_0\|\}$ is nonincreasing and converges at η. Let $k \to \infty$, by the continuity of A, from (3.1) we obtain

$$\phi(\eta) \leq \eta,$$

a contradiction, which implies $y_n \to y_0$ as $n \to \infty$. The continuity of S has been proved. Furthermore, the inequality

$$\phi(\|y_k - y\|) - \|y_k - y\| \leq \|By_k - By\| - \|y_k - y\| \leq \|Ax_k - Ax\|,$$

implies that for any $\epsilon > 0$, there is a $\delta > 0$ such that $\|y - y_k\| < \epsilon$, provided that $\|Ax - Ax_k\| < \delta$. Now let $Ax_1, \cdots, Ax_k$ be a δ-net in $A(M)$ and $y_i = S(x_i), i = 1, \cdots, k$. Then $y_1, \cdots, y_k$ are ϵ-net in $S(M)$. This proves that $S(M)$ is contained in a compact subset of X. By Schauder fixed point theorem, we know that S has a fixed point $u \in M$ such that $Au + Bu = u$.

Some different versions of Theorem 3.1 will be discussed subsequently. We introduce a useful lemma in advance.

Lemma 3.2. *Let mapping T be a nonlinear expansion on a certain subset M in a normed linear space X. Then $I - T$ is injective and $(I - T)^{-1}$ is continuous, where I denotes the identity operator.*

Proof. For any $x, y \in M$ and $x \neq y$, we have

$$\begin{aligned} \|(I-T)x - (I-T)y\| &\geq \|Tx - Ty\| - \|x - y\| \\ &\geq \phi(\|Tx - Ty\|) - \|x - y\| > 0. \end{aligned} \tag{3.2}$$

Hence $(I - T)^{-1}$ is existent. For any $\xi, \eta \in (I - T)(M)$, by (3.2), we have

$$\phi(\|(I-T)^{-1}\xi - (I-T)^{-1}\eta\|) - \|(I-T)^{-1}\xi - (I-T)^{-1}\eta\| \leq \|\xi - \eta\|, \tag{3.3}$$

which implies $(I - T)^{-1}$ is continuous on $(I - T)(M)$.

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a Banach space, and let M be a nonempty closed convex subset of X. Suppose that*

(i) $A : M \to X$ is continuous, $A(M)$ is contained in a compact subset of X;

(ii) $B : X \to X$ is a nonlinear expansion with ϕ;

(iii) $A(M) \subset (I - B)(X)$ and $(I - B)^{-1}A(M) \subset M$.

Then there exists a point $u \in M$ such that $Au + Bu = u$.

Proof. For any fixed $x \in M$, by (iii), we have $Ax \in (I - B)(X)$, so the mapping $(I - B)^{-1}A$ is well defined on M. By Lemma 3.2, $(I - B)^{-1}$ is continuous, thus $(I - B)^{-1}A(M)$ is contained in a compact subset of X. By Schauder fixed point theorem, there is a fixed point $u \in M$ such that $Au + Bu = u$.

The following theorem may be considered as a local version of Theorem 3.1. Without loss of generality, we can admit $B\theta = \theta$, or else we can use B_1 instead of B for $B_1x = Bx - B\theta$.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space, and let $M_r = \{x \in X : \|x\| \leq r\}$. Suppose that*

(i) $A : M_r \to X$ is continuous, $A(M_r)$ is contained in a compact subset of X;

(ii) $B : X \to X$ is a nonlinear expansion with $\phi, B\theta = \theta$, and $I - B$ is surjective;

(iii) $\sigma = \inf\limits_{t \in (0,r]} \frac{\phi(t) - t}{t} > 0, \ \sup\limits_{x \in M_r} \|Ax\| \leq \sigma r$.

Then there exists a point $u \in M_r$ such that $Au + Bu = u$.

Proof. By Theorem 3.3, it is only need to prove that $(I - B)^{-1}A(M_r) \subset M_r$. For any $x \in M_r$, we have

$$\sigma \leq \frac{\phi(\|(I-B)^{-1}Ax\|) - \|(I-B)^{-1}Ax\|}{\|(I-B)^{-1}Ax\|}. \tag{3.4}$$

By $B\theta = \theta$, we obtain

$$(I - B)^{-1}\theta = \theta.$$

It follows from (3.3) and (3.4) that

$$\sigma\|(I-B)^{-1}Ax\| \le \phi(\|(I-B)^{-1}Ax\|) - \|(I-B)^{-1}Ax\| \le \|Ax\| \le \sigma r,$$

which means $\|(I-B)^{-1}Ax\| \le r$, i.e. $(I-B)^{-1}A(M_r) \subset M_r$.

Remark 3.5. *Note that Theorems B and C in Section 1 respectively follow as a special case of Theorem 3.4 and Theorem 3.1 if we choose $\phi(t) = ht$ with $h > 1$.*

4. Applications

In this section, we will apply Theorem 3.1 to the following nonlinear integral equation

$$u(t) = (1+t^2)e^{u(t)} - \int_a^b K(t,s,u(s))ds,\ t \in [a,b], \tag{4.1}$$

where $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X = C[a,b]$. Then X is a Banach space with the sup norm $\|u\| = \sup_{0\le t\le 1} |u(t)|$. Denote $M = \{x \in X : \|x\| \le r, x(t) \ge 0, t \in [a,b]\}$, where $r > 0$.

Theorem 4.1. *Suppose that*

$$\frac{1+t^2}{b-a} \le K(t,s,x(s)) \le \frac{(1+t^2)e^r - r}{b-a},\ (t,s,x) \in [a,b]\times[a,b]\times M. \tag{4.2}$$

Then the integral equation (4.1) has at least one nonnegative solution $u \in C[a,b]$.

Proof. We can easily convert (4.1) into the following operator equation

$$x(t) = (Ax)(t) + (Bx)(t),\ t \in [a,b],$$

where the operators A and B are defined by

$$(Ax)(t) = -\int_a^b K(t,s,x(s))ds,\ \forall\ t \in [a,b],\ x \in M; \tag{4.3}$$

$$(Bx)(t) = (1+t^2)e^{x(t)},\ \forall\ t \in [a,b],\ x \in M. \tag{4.4}$$

We next verify that the mappings A and B satisfy the conditions (i)-(iii) of Theorem 3.1. Firstly, using the same method as in [7–8], the mapping $A : M \to X$ is continuous, and $A(M)$ is contained in a compact subset of X. Secondly, the mapping $B : M \to X$ is nonlinear expansive. In fact, for any $x, y \in C[a,b]$, we have

$$\begin{aligned} & |(Bx)(t) - (Bx)(t)| = (1+t^2)|e^{x(t)} - e^{y(t)}| \\ \ge\ & |e^{x(t)} - e^{y(t)}| = e^{\min\{x(t),y(t)\}}(e^{|x(t)-y(t)|} - 1) \\ \ge\ & e^{|x(t)-y(t)|} - 1 \ge |x(t)-y(t)| + \tfrac{1}{2}[x(t)-y(t)]^2,\ \forall\ t \in [a,b]. \end{aligned}$$

Setting $\phi(t) = t + \frac{1}{2}t^2$, we obtain

$$\|Bx - By\| \ge \phi(\|x-y\|),$$

i.e. B is a nonlinear expansion. Finally, for any $x, z \in M$, we may define

$$y(t) = \ln\Big(\frac{z(t) + \int_a^b K(t,s,x(s))ds}{1+t^2}\Big),\ t \in [a,b]. \tag{4.5}$$

Consequently for $t \in [a, b]$, from (4.2), we get

$$y(t) = \ln\Big(\frac{z(t) + \int_a^b K(t,s,x(s))ds}{1+t^2}\Big) \geq \ln\Big(\frac{\int_a^b K(t,s,x(s))ds}{1+t^2}\Big) \geq 0,$$

and

$$\begin{aligned} y(t) &= \ln\Big(\frac{z(t)+\int_a^b K(t,s,x(s))ds}{1+t^2}\Big) \\ &\leq \ln\Big(\frac{r+\int_a^b K(t,s,x(s))ds}{1+t^2}\Big) \\ &\leq \ln(e^r) = r. \end{aligned}$$

Thus $y \in M$. By (4.5), (4.3) and (4.4), we have

$$z(t) = -\int_0^1 K(t,s,x(s))ds + (1+t^2)e^{y(t)} = (Ax)(t) + (Bx)(t),\ t \in [a, b],$$

which implies $Ax + B(M) \supset M$. Thus all conditions of Theorem 3.1 are satisfied and the assertion follows.

Remark 4.2. Krasnosel'skii fixed point theorem is interesting in view of the fact that it has a wide range of applications to nonlinear integral equations of mixed type for proving the existence of solutions. However, as far as the authors know, the existence of nonnegative solutions for a class of nonlinear integral equation as (4.1) has nearly not been involved by former results. In a certain sense, we can interpret a class of equations such as (4.1) as follows: if a compact operator has the fixed point property, then this property can be inherited for some perturbation, provided that this perturbation is nonlinear expansive.

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