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NOVEL COMPUTATIONAL DERIVATIVE-FREE METHODS FOR SIMPLE ROOTS

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Abstract. Some novel computational techniques for solving single variable nonlinear equations are given. The schemes are without memory and free from derivative evaluations per full iteration. They are built by applying the weight function approach alongside an approximation for the first derivative of the function in the second step of a two-step cycle for obtaining optimal fourth-order schemes; and also by adapting a nonlinear fraction in the third-step of a three-step cycle to attain seventh-order techniques. The classical efficiency indices of the proposed two- and three-step derivative-free methods are 1.587 and 1.626, respectively up to now. Further research has also been done via the concept of weight functions to provide optimal eighth-order derivative-free techniques which possess 1.682 as their efficiency index. The superiority of the techniques is illustrated by solving numerical examples.
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1. Introduction

Solving nonlinear scalar equations is considered in this research. Iterative methods are the best choice for doing this when the analytical methods fail for finding the exact roots. It is known that various problems arising in diverse disciplines of engineering and sciences can be described by a nonlinear equation. That is why nonlinear equation solving by iterative processes is of great significance [3, 12]. These iterative methods could be generalized to solve systems of nonlinear equations which are also of grave importance for applicable engineering problems [1, 14].

Root solvers illustrate the central roles of convergence and complexity in scientific computing. Currently, there are so many iterative techniques in the literature; for example see [5, 6, 9, 11, 13] and their bibliographies. Why is it necessary to know more than one method for solving equations? Often, the choice of method will depend on the cost of evaluating the function f and perhaps its derivative(s). If $f(x) = e^x - \sin(x)$, it may take a few thousandths of a second to determine $f(x)$, and its derivative is available if needed. If $f(x)$ denotes the freezing temperature of ethylene glycol solution under x atmosphere of pressure, each function evaluation may require considerable time in a well-equipped laboratory, and determining the derivative may be infeasible. Hence, it is desirable to provide novel derivative-involved and derivative-free methods for solving such equations [2, 10].

The first well-known iterative method was given by Newton as

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

which is also known as Newton-Raphson method. Steffensen approximated $f'(x_n)$, in Newton's iteration, by using forward finite difference (of order one) to obtain its derivative-free form as

$$x_{n+1} = x_n - f(x_n)^2/(f(x_n + f(x_n)) - f(x_n)).$$

Note that backward finite difference of order one could also be used to make the Newton's iteration derivative-free with two function evaluations per iteration. Both techniques reach the quadratically convergence using two evaluations per cycle [12].

A very important aspect of an iterative process is the rate of convergence or the speed of convergence of the sequence $\{x_n\}_{n=0}^{\infty}$, which approximates a solution of $f(x) = 0$. This concept, along with the operational cost associated to the technique; allow establishing the index of efficiency of an iterative process. In this way, the (classical) efficiency index of an iterative process is defined by the value $p^{1/n}$, where p is the convergence rate and n is the whole number of evaluations per cycle [12]. Consequently, Newton-Raphson and Steffensen both have the same efficiency index 1.414. Note that these two schemes are without memory iterations according to the classification of iterative processes for solving nonlinear equations which was also given in the fundamental book of Traub [12].

The remained contents of this work are organized as follows. In Section 2, we present some optimal derivative-free quartically; and seventh-order convergent iterative methods for solving nonlinear single valued equations. The proposed two-step techniques agree with the optimality of two-step methods consisting of three evaluations per full cycle. Since, according to the Kung and Traub hypothesis [4], a without memory iterative scheme is known as an optimal root solver where by using n evaluations per full cycle, reaches the convergence order 2^{n-1}. Then, we generalize the obtained optimal fourth-order derivative-free methods by giving some seventh-order derivative-free techniques via applying a nonlinear fraction for estimating the new-appeared first derivative of the function in the third step of a three-step cycle on which its first two steps are any of the optimal two-step fourth-order derivative-free methods (e.g., our two-step optimal without memory derivative-free methods). Note that this way could also been taken into account as a general class in providing three-step seventh-order methods. Section 3 presents some optimal families of three-step eighth-order derivative-free iterative methods which are consistent with the optimality of Kung-Traub hypothesis [4] to reach the index of efficient 1.682 by using only four function evaluations per full cycle. Analyses of convergence are fully given for the contributions. And a comparison between the existing methods of various orders and our contributed methods is given in Section 4.

2. Main Contributions

The general procedure to build high order methods with better efficiency indices is to consider an existing iterative technique at the beginning of a cycle and then perform a Newton's iteration to boost up the convergence rate. Unfortunately, this procedure increases the number of evaluations per full cycle. In fact, if the initial method (of order p) has n evaluations, then the new method (through this structure) will have an efficiency index $(2p)^{1/(n+2)}$, which is worse for $p > 2$ and $n > 2$. For developing and contributing to the efficiency index, we use approximations of the derivative that reduce the number of evaluations without lowering too much of the convergence rate.

The main idea of this work is to present very simple generalizations of the well-known Steffensen's method with higher orders and efficiency indices. Toward this goal, we combine some methods to each other and then approximate the new-appeared first derivatives along with using weight function approach to obtain as high as possible of efficiency indices with special care to the computational burden.

Let us take into consideration the Steffensen's method in the first step of a two-step cycle. Performing now a Newton's step after that as follows

$$\begin{cases} x_0 \text{ given}, \\ y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \end{cases} \tag{1}$$

(1) is a quartically convergent technique using four evaluations per iteration with $4^{1/4} = 2^{1/2} \approx 1.414$ as its efficiency index which is the same as Newton's and Steffensen's. In order to improve the index of efficiency, we estimate the first derivative of the function in the second step using the similar concept as in Secant method; i.e.

$$\begin{cases} x_0 \text{ given}, \\ y_n = x_n - \frac{f(x_n)^2}{f(w_n)-f(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{\frac{f(y_n)-f(w_n)}{y_n-w_n}}, \end{cases} \tag{2}$$

wherein $w_n = x_n + f(x_n)$. Unfortunately, this way decreases the exactness of the approximation of $f'(y_n)$ and subsequently (2) will be of third-order of convergence with the following error equation:

$$e_{n+1} = \frac{(1+c_1)^2 c_2^2}{c_1^2} e_n^3 + O(e_n^4).$$

Based on the still un-proved hypothesis of Kung and Traub, the convergence rate three is not optimal for a without memory iteration consuming three function evaluations per full iteration. Hence, in order to obtain the fourth-order convergence with three evaluations per cycle; we apply the approach of weight function as follows

$$\begin{cases} x_0 \text{ given}, \\ y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f[y_n,w_n]}[1 + \frac{f(y_n)}{f(x_n)}], \end{cases} \tag{3}$$

where $f[y_n, w_n]$ and $f[x_n, w_n]$ are divided differences. As can be seen, (3) has a quite simple structure while its order and efficiency index are optimal. Theorem 1 illustrates this fact. There are many choices to make (2) optimal through weight function approach. As a matter of fact, in the second step, i.e. $y_n - \frac{f(y_n)}{f[y_n,w_n]}G(t_n)$, the weight function $G(t_n)$ should be chosen as the order arrives at four and the coefficient of e_n^3 has been vanished. This can be done by representing a real one-valued weight function $G(t_n)$ where $t_n = f(y_n)/f(x_n)$. That is to say, by Taylor's series expanding in a linearized form, any forms of $G(t_n)$ which make the convergence order four can be constructed. The Taylor's series expansion of the weight function $G(t_n)$ and the last step is as follows:

$$\begin{aligned}
e_{n+1} = &-(1/c_1)(1+c_1)c_2(-1+G(0))e_n^2 + (1/c_1^2)(-c_1(1+c_1)(2+c_1)c_3(-1+G(0)) \\
&+c_2^2(-2+3G(0)+c_1(2+c_1)(-1+2G(0)-G'(0))-G'(0)))e_n^3 \\
&+(1/(2c_1^3))(-2c_1^2(1+c_1)(3+c_1(3+c_1))c_4(-1+G(0))+2c_1c_2c_3(-7+10G(0)-4G'(0) \\
&+c_1(c_1(7+2c_1)(-1+2G(0))-2c_1(4+c_1)G'(0)+2(9G(0)-5(1+G'(0))))) \\
&+c_2^3(2(4-7G(0)+6G'(0)+c_1(5-11G(0)+13G'(0) \\
&+c_1(3+c_1-8G(0)-3c_1G(0)+(10+3c_1)G'(0))))-((1+c_1)^3)G''(0)))e_n^4 + O(e_n^5).
\end{aligned}$$

Thus, any form of the function $G(t_n)$ in which $G(0) = G'(0) = 1$ and $\mid G''(0) \mid < \infty$ can be considered as the weight function to make (2) optimal. Hereby, to reduce any additional computational load on our root solver, we have presented the simplest form.

Theorem 1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq R \to R$ in an open interval I. If x_0 is sufficiently close to α, then the method defined by (3) has the optimal convergence order four.*

Proof. We expand any terms of (3) around the simple root α in the nth iterate where $c_j = \frac{f^{(j)}(\alpha)}{j!}$, $j \geq 1$, and $e_n = x_n - \alpha$. Thus, we write

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5),$$

and also

$$\begin{aligned}
x_n + f(x_n) = &-(-1+c_1+e_n(c_2+e_n(c_3+c_4e_n)))(c_1-c_2e_n(-1+c_1 \\
&+e_n(c_2+e_n(c_3+c_4e_n)))e_n + c_3e_n^2 + O(e_n^3).
\end{aligned}$$

Subsequently, we attain

$$\begin{aligned}
x_n - \frac{f(x_n)}{f[x_n, w_n]} &= \alpha + (1+\frac{1}{c_1})c_2e_n^2 \\
&+\frac{(-(2+(2+c_1)c_1)c_2^2+c_1(1+c_1)(2+c_1)c_3)}{c_1^2}e_n^3 + O(e_n^4). \qquad (4)
\end{aligned}$$

Now we should expand $f(y_n)$ around the simple root by using (4). We get that

$$f(y_n) = (1+c_1)c_2e_n^2 + (-\frac{(2+c_1(2+c_1))c_2^2}{c_1} + (1+c_1)(2+c_1)c_3)e_n^3 + O(e_n^4). \qquad (5)$$

Using (5), we have

$$f[y_n, w_n] = c_1 + (1+c_1)c_2e_n + ((2+1/c_1)c_2^2 + (1+c_1)^2c_3)e_n^2$$

$$+((-(2+c_1(2+c_1))c_2^3+c_1(3+4c_1(2+c_1))c_2c_3+c_1^2(1+c_1)^3c_4)e_n^3)/c_1^2+O(e_n^4).$$

Additionally by providing the Taylor's series expanding, in the second step of (3), we get that

$$\frac{f(y_n)}{f[y_n,w_n]}[1+\frac{f(y_n)}{f(x_n)}]=(1+\frac{1}{c_1})c_2e_n^2$$
$$+\frac{(-(2+c_1(2+c_1))c_2^2+c_1(1+c_1)(2+c_1)c_3)}{c_1^2}e_n^3+O(e_n^4). \tag{6}$$

Using (4), (6), the second step of (3) and simplifying, we attain

$$e_{n+1}=x_{n+1}-\alpha=\frac{(1+c_1)^2c_2((3+c_1)c_2^2-c_1c_3)}{c_1^3}e_n^4+O(e_n^5). \tag{7}$$

This shows that (3) is an optimal fourth-order derivative-free method consuming three evaluations per cycle. Hence, the proof is completed. □

Remark 1. The optimal without memory derivative-free method (3) was constructed using the concept of weight function in the second step alongside an approximation for the new-appeared first derivative. Here, we should note that if one chooses any optimal quadratically derivative-free method in the first step of (3), e.g. choosing the backward finite difference instead of forward finite difference to approximate the first derivative in the Newton's iteration, then a new optimal quartically method will be furnished.

Remark 2. It should be remarked that if one approximates $f'(y_n)$ by $\frac{f(y_n)-f(x_n)}{y_n-x_n}$, in the denominator of the second step of (1), then the simplest form of the weight function should be chosen as comes next to provide the optimal fourth-order convergence

$$\begin{cases} x_0 \ given, \\ y_n=x_n-\frac{f(x_n)}{f[x_n,w_n]}, \\ x_{n+1}=y_n-\frac{f(y_n)}{f[y_n,x_n]}[1+\frac{f(y_n)}{f(w_n)}], \end{cases} \tag{8}$$

that is to say, $1+\frac{f(y_n)}{f(w_n)}$ should be chosen as the weight function. More precisely, the weight function $H(\mu_n)$ where $\mu_n=f(y_n)/f(w_n)$ should be expanded to find the possible cases in which the order arrives at four, i.e. any forms of the function $H(\mu_n)$ in which $H(0)=H'(0)=1$ and $\mid H''(0)\mid<\infty$ can be paid heed to provide optimal two-step methods by the given approximation. Thus, (8) is another novel optimal fourth-order derivative-free technique for solving one variable nonlinear equations which has the following error equation

$$e_{n+1}=x_{n+1}-\alpha=\frac{(1+c_1)c_2((3+2c_1)c_2^2-c_1(1+c_1)c_3)}{c_1^3}e_n^4+O(e_n^5).$$

Remark 3. Some particular forms of the weight functions $G(t_n)$ and $H(\mu_n)$ are given in Table 1 below.

Table 1. Some typical forms of $G(t_n)$ and $H(\mu_n)$ where $\lambda\in R$

Weight Functions	Form 1	Form 2	Form 3
$G(t_n)$	$1+t_n+\lambda(t_n^2+t_n^3)$	$1+\frac{t_n}{1+\lambda t_n}$	$(1+\lambda t_n)^{1/\lambda},\lambda\neq 0$
$H(\mu_n)$	$1+\mu_n+\lambda(\mu_n^2+\mu_n^3)$	$1+\frac{\mu_n}{1+\lambda\mu_n}$	$(1+\lambda\mu_n)^{1/\lambda},\lambda\neq 0$

As could be observed, (3) and (8) (or any variant by considering more complicated weight functions as in Table 1) are simple generalizations of the well-known Steffensen's method, by using only three evaluations of the function, in which the convergence rate and the classical index of efficiency are optimal in the sense of Kung and Traub.

Now, for improving the convergence rate and the index of efficiency more; we compute a Newton's step as comes next in the third step of a three-step cycle in which the first two steps are the new obtained optimal derivative-free techniques, i.e. (3) or (8) or any variant of them,

$$\begin{cases} x_0 \textit{ given}, \\ y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[y_n,w_n]}[1 + \frac{f(y_n)}{f(x_n)}], \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \end{cases} \tag{9}$$

Clearly (9) is an eighth-order method with five evaluations per iteration to reach the efficiency index $8^{1/5} \approx 1.516$. This index of efficiency is lower than that of (3) or (8). Accordingly, we should approximate $f'(z_n)$ using an adequate approximation which provides high order of convergence and reduce the total number of evaluations per full step. We use the following known data $f(x_n)$, $f(y_n)$, $f(z_n)$ in approximating $f'(z_n)$. Afterwards, We consider the following rational nonlinear function

$$f(t) \approx \frac{a + (t - x_n)}{b(t - x_n) + c} + f(x_n), \tag{10}$$

whence its derivative is in the following form $f'(t) = c/(b(t-x_n)+c)^2$. At this moment, the three unknown coefficients could be attained by substituting of the known values in (10). That is, by satisfying $f(t)|_{x_n} = f(x_n)$, $f(t)|_{y_n} = f(y_n)$, $f(t)|_{z_n} = f(z_n)$, first we obtain that $a = 0$ and subsequently, we have

$$\begin{cases} b(z_n - x_n) + c = \frac{1}{f[x_n,z_n]}, \\ b(y_n - x_n) + c = \frac{1}{f[x_n,y_n]}, \end{cases} \tag{11}$$

where $f[x_n, y_n]$ and $f[x_n, z_n]$ are forward divided differences. By simplifying and satisfying the derivative form of $f'(t)$, we attain that $f'(z_n) \approx \frac{f[x_n,z_n]f[y_n,z_n]}{f[x_n,y_n]}$. Thus, a novel derivative-free technique will be obtained as follows

$$\begin{cases} x_0 \textit{ given}, \\ y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[y_n,w_n]}[1 + \frac{f(y_n)}{f(x_n)}], \\ x_{n+1} = z_n - \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}. \end{cases} \tag{12}$$

This idea provides a three-step and quite simple improvement for without memory root solvers. As a matter of fact, (12) is of seventh-order convergence and has better efficiency index than the optimal two-step methods. Theorem 2 provides its error equation analytically.

Theorem 2. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq R \to R$ in an open interval I. If x_0 is sufficiently close to α, then the method defined by (12) has the convergence order seven.*

Proof. Use the same definitions and symbolic computations as in the Proof of Theorem 1, we have

$$f(z_n) = \frac{(1+c_1)^2 c_2((3+c_1)c_2^2 - c_1c_3)}{c_1^3} e_n^4 + \frac{1}{c_1^3}((1+c_1)(-(3+c_1)(6+c_1$$
$$(7+3c_1))c_2^4 + c_1(20+c_1(34+c_1(19+3c_1)))c_2^2c_3 - c_1^2(1+c_1)(2+c_1)c_3^2)$$
$$-(2+c_1)(c_1+c_1^2)^2c_2c_4)e_n^5 + \ldots + O(e_n^8). \tag{13}$$

Further, we have

$$\frac{f[x_n,z_n]f[y_n,z_n]}{f[x_n,y_n]} = c_1 + ((1+c_1)c_2(c_2^2-c_1c_3)e_n^3)/c_1^2 + 1/c_1^3((3+2c_1)(1+c_1(3+c_1))c_2^4$$
$$+c_1(4+3c_1)c_2^2c_3 - c_1^2(1+c_1)(2+c_1)c_3^2 - c_1^2(1+c_1)c_2c_4)e_n^4 + \cdots + O(e_n^7).$$

We also obtain by using this new relation, (5) and (13) that

$$\frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]} = \frac{(1+c_1)^2c_2((3+c_1)c_2^2 - c_1c_3)}{c_1^3} e_n^4$$
$$+\frac{1}{c_1^4}((1+c_1)(-(3+c_1)(6+c_1(7+3c_1))c_2^4 + c_1(20+c_1(34+c_1(19+3c_1)))c_2^2c_3$$
$$-c_1^2(1+c_1)(2+c_1)c_3^2) - (2+c_1)(c_1+c_1^2)^2c_2c_4)e_n^5 + \ldots + O(e_n^8). \tag{14}$$

Additionally, applying (14) and (7) results in the follow-up error equation

$$e_{n+1} = x_{n+1} - \alpha = \frac{(1+c_1)^3c_2^2(c_2^2 - c_1c_3)((3+c_1)c_2^2 - c_1c_3)}{c_1^6} e_n^7 + O(e_n^8). \tag{15}$$

This ends the proof and shows that (12) is a seventh-order without memory scheme using four function evaluations per iteration. □

There are three notes that required to be mentioned before keep going. First, if one desires to adapt the (new) nonlinear fraction $f(t) \approx \frac{a+(t-w_n)}{b(t-w_n)+c} + f(w_n)$ in the interpolation conditions $f(t)|_{w_n} = f(w_n)$, $f(t)|_{y_n} = f(y_n)$, $f(t)|_{z_n} = f(z_n)$, then a similar approximation for $f'(z_n)$ can be obtained as follows

$$f'(z_n) \approx \frac{f[w_n,z_n]f[y_n,z_n]}{f[w_n,y_n]}$$

which also provides seventh-order iterative methods by considering optimal two-step three-point without memory iterations at the first steps; i.e., for example the following novel scheme can be obtained

$$\begin{cases} x_0 \ given, \\ y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[y_n,w_n]}[1 + \frac{f(y_n)}{f(x_n)}], \\ x_{n+1} = z_n - \frac{f[w_n,y_n]f(z_n)}{f[w_n,z_n]f[y_n,z_n]}, \end{cases} \tag{16}$$

which has the follow-up error equation

$$e_{n+1} = \frac{(1+c_1)^4 c_2^2 (c_2^2 - c_1 c_3)((3+c_1)c_2^2 - c_1 c_3)}{c_1^6} e_n^7 + O(e_n^8).$$

Second, we have not applied all the available data in approximating $f'(z_n)$ to avoid having a method with massive load of computational effort. On the other hand, the seventh-order of convergence by using four pieces of information per step is not optimal in the sense of Kung-Traub. Therefore, using all four available data or building weight function at the end of the third step can produce optimal eighth-order of convergence. This could be done if one bears more computational burden on the attained derivative-free root solver; due to this, we will provide optimal eighth-order methods in the next section by considering the concept of weight function again on the attained seventh-order techniques. And third, such iterative schemes (or any of the higher order developments) can be written in the final form $x_{n+1} = F(x_n)$, $n = 0, 1, 2, \cdots$, and in fact, the roots of the nonlinear single valued equation $f(x) = 0$ is a *fixed point* of the complicated function F.

Remark 4. Choosing any of the other optimal fourth-order derivative-free methods such as (8) in the first two steps of (12) or (16) ends in a new computational three-step seventh-order method free from derivative. That is to say, we also can construct

$$\begin{cases} x_0 \ given, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[y_n, x_n]}[1 + \frac{f(y_n)}{f(w_n)}], \\ x_{n+1} = z_n - \frac{f[x_n, y_n] f(z_n)}{f[x_n, z_n] f[y_n, z_n]}. \end{cases} \tag{17}$$

This way makes our idea in this contribution to be interesting and competitive. In fact, we have given a class of building without memory seventh-order derivative-free methods.

3. Construction of Some Optimal Eighth-Order Methods

This section focuses on making the attained seventh-order methods optimal in the sense of Kung-Traub hypothesis [4]. Thus, the order should go up only one unit without more evaluation of the function. This end is done by employing the approach of weight function again. The weight function $J(x_n, w_n, y_n, z_n)$ should be constructed such that the coefficient of e_n^7 in the error equations of the presented seventh-order schemes in the previous section vanishes. Note that $J(x_n, w_n, y_n, z_n)$ is not a several variable function, in fact, we have used this notation to show that it is a function of f which only relies on the points x_n, w_n, y_n, z_n. For this reason, we consider the follow up computationally iterative root solver

$$\begin{cases} x_0 \ given, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[y_n, w_n]}[1 + \frac{f(y_n)}{f(x_n)}], \\ x_{n+1} = z_n - \frac{f[x_n, y_n] f(z_n)}{f[x_n, z_n] f[y_n, z_n]} \times J(x_n, w_n, y_n, z_n), \end{cases} \tag{18}$$

where $w_n = x_n + f(x_n)$, $\zeta, \varphi \in R$ and

$$J(x_n, w_n, y_n, z_n) = 1 + \frac{f(z_n)}{f(w_n)} + (-2 - f[x_n, w_n](3 + f[x_n, w_n]))(\frac{f(y_n)}{f(w_n)})^3 + \zeta(\frac{f(z_n)}{f(y_n)})^2 + \varphi(\frac{f(y_n)}{f(x_n)})^4. \quad (19)$$

Theorem 3. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq R \to R$ in an open interval I. If x_0 is sufficiently close to α, then the family of two parameters that defined by (18) has the optimal convergence order eight.*

Proof. Use the same definitions and symbolic computations as in the Proof of Theorem 2, we have (13)-(15) again. Therefore, Taylor's series expanding for the weight function in the third step of (18) is required. Hence, we attain

$$J(x_n, w_n, y_n, z_n) = 1 + ((1 + c_1)c_2(c_2^2 - c_1c_3)e_n^3)/c_1^3 + (1/c_1^4)(-c_1^2(1 + c_1)(2 + c_1)c_2c_4 + c_1^2(1 + c_1)c_3^2(-2 + c_1(-1 + \zeta) + \zeta) + c_1c_2^2c_3(9 + 13c_1 + 5c_1^2 - 2(1 + c_1)^2(3 + c_1)\zeta) + c_2^4 \times(-3+9\zeta+\varphi+c_1(4(-1+6\zeta+\varphi)+c_1(-2+(22+c_1(8+c_1))\zeta+(6+c_1(4+c_1))\varphi))))e_n^4 + (1/(c_1^5))(c_1^2c_2^2c_4(14 + c_1(26 + c_1(18 + 5c_1)) - 2(1 + c_1)^2(2 + c_1)(3 + c_1)\zeta) + c_1^3(1 + c_1) \times c_3c_4(-7 + 4\zeta + c_1(-7 + 2c_1(-1 + \zeta) + 6\zeta)) + c_1^2c_2(-c_1(1 + c_1)(3 + c_1(3 + c_1))c_5 + c_3^2 \times(21 - 24\zeta + c_1(40 + 7c_1(4 + c_1) - 66\zeta - 2c_1(33 + 2c_1(7 + c_1))\zeta))) + (c_2^5)(-11 - 28c_1 - 18c_1^2 - 2c_1^3 - 2(1+c_1)(3+c_1)^2(4+c_1(5+2c_1))\zeta - 4(1+c_1)^3(3+c_1(3+c_1))\varphi) + c_1c_2^3c_3 \times(-22+96\zeta+8\varphi+c_1(-34+284\zeta+36\varphi+c_1(-27+324\zeta+64\varphi+2c_1(-5+2(44+c_1 \times(11 + c_1))\zeta + 2(14 + c_1(6 + c_1))\varphi)))))e_n^5 + \cdots + O(e_n^8). \quad (20)$$

Using (20) and (15) in the last step of (18), results in the follow-up error equation

$$e_{n+1} = (-1/c_1^7)(1 + c_1)^3c_2((3 + c_1)c_2^2 - c_1c_3)(-c_1^2(1 + c_1)c_2c_4 + c_1^2(1 + c_1)c_3^2\zeta - 2c_1(1 + c_1)c_2^2c_3(-2 + (3 + c_1)\zeta) + c_2^4(-3+9\zeta+\varphi+c_1(-5+15\zeta+3\varphi+c_1(-1+(7+c_1)\zeta+(3+c_1)\varphi))))e_n^8+O(e_n^9). \quad (21)$$

This ends the proof and shows that (18) is an optimal eighth-order bi-parametric family. □

We have obtained a novel three-step derivative-free optimal eighth-order bi-parametric family of without memory iterations for solving nonlinear equations by using the approach of weight function on the seventh-order method (12). The application of weight functions to make the proposed seventh-order methods optimal can be considered for any of the contributed seventh-order methods but with different weight functions. For example choosing (17) and the appropriate weight function as comes next can provide a new family of optimal eighth-order derivative-free methods

$$\begin{cases} x_0 \ given, \\ y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[y_n,x_n]}[1 + \frac{f(y_n)}{f(w_n)}], \\ x_{n+1} = z_n - \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}[1 + \frac{f(z_n)}{f(w_n)} + (-2 - f[x_n, w_n])(\frac{f(y_n)}{f(w_n)})^3 + \rho(\frac{f(z_n)}{f(y_n)})^2], \end{cases} \quad (22)$$

where $w_n = x_n + f(x_n)$, $\rho \in R$ and its error equation is as follows

$$e_{n+1} = -(1/(c_1^7))(1+c_1)c_2(-(3+2c_1)c_2^2 + c_1(1+c_1)c_3)$$
$$((1+c_1)^2 c_2(3c_2^3 - 4c_1c_2c_3 + c_1^2c_4) - ((3+2c_1)c_2^2 - c_1(1+c_1)c_3)^2\rho)e_n^8 + O(e_n^9). \quad (23)$$

Similar procedures by using the approach of weight functions can be done for making any of the seventh-order methods of Section 2 optimal in the sense of Kung-Traub conjecture. Now, we mention the efficiency of the various proposed derivative-free methods in Remark 5 below.

Remark 5. The index of efficiency for (3) or (8) is $4^{1/3} \approx 1.587$ and for (12), (16) and (17) is $7^{1/4} \approx 1.626$ which are better than lots of two- and three-point derivative-free and/or derivative-involved schemes. The index of efficiency for (18) and (22) is $8^{1/4} \approx 1.682$ which is optimal according to the Kung-Traub conjecture.

4. Numerical Reports and Conclusion

The main objective of this section is to provide a robust comparison between the presented schemes and the already known methods in the literature. For numerical reports here, we have used the second-order Newton's method (NM), the quadratically scheme of Steffensen (SM), the third-order method given by Soleymani and Sharifi in [8] as (SS), our proposed optimal fourth-order technique (3), the sixth-order method given by Rafiullah and Haleem in [7] as (RH), our presented novel derivative-free seventh-order method (12), the optimal eighth-order three-step derivative-free family of Kung and Traub [4] with $\beta = 1$ as (KT8-1), and our optimal three-step eight-order family (18) with $\zeta = \varphi = 0$. We have found that among the proposed fourth-order methods "(3) and (8)", have similar outputs, thus we only bring the numerical results of (3). As well as due to similarity of the presented seventh-order methods "(12), (16) and (17)", we just provide the results of (12). The same is true for (18) and (22), hence we just reflect the results of (18). The considered nonlinear test functions, their roots and the initial guesses in the neighborhood of the simple roots are furnished in Table 2.

We have used Div. when the iteration diverges for the considered starting point. The results are summarized in Tables 3 and 4 after two and three full iterations respectively. As they show, novel schemes are comparable with all of the methods. All numerical instances were performed by Matlab 7.6 using 500 digits floating point arithmetic (VPA:=500). We have computed the root of each test function for the initial guess x_0 while the iterative schemes were stopped when $|f(x_n)| \leq 10^{-500}$. As can be seen, the obtained results in Tables 3 and 4 are in harmony with the analytical procedure given in Section 2.

Although we have taken into account of guesses enough close to the sough zeros, sometimes if the initial approximation is not sufficiently close to the root, some methods work but the other fail. This raises the problem of the attraction basin (the set of all starting values that cause the method to converge to that particular zero) of the root for such schemes. At such situations, one of the best ways is to combine the high-order derivative-free method with one of the slow-order methods, such as bisection method, in which the convergence is guaranteed. However, in the absence of any intuition about where the zero might lie, a "guess and check" method might narrow the possibilities to a reasonably small interval by appealing to the intermediate value theorem, or as told above, combining the iteration with slow-order iterations in which the convergence is guaranteed can be taken into consideration.

Table 2. The test functions considered in this study

Test Functions	Roots	Initial Guesses
$f_1(x) = 3x + \sin(x) - e^x$	$\alpha_1 \approx 0.360421702960324$	$x_0 = 0.9$
$f_2(x) = \sin(x) - 0.5$	$\alpha_2 \approx 0.523598775598299$	$x_0 = 0.3$
$f_3(x) = x^2 - e^x - 3x + 2$	$\alpha_3 \approx 0.257530285439861$	$x_0 = 1.5$
$f_4(x) = x^3 + 4x^2 - 10$	$\alpha_4 \approx 1.365230013414097$	$x_0 = 0.7$
$f_5(x) = xe^{-x} - 0.1$	$\alpha_5 \approx 0.111832559158963$	$x_0 = 0.2$
$f_6(x) = x^3 - 10$	$\alpha_6 \approx 2.15443490031884$	$x_0 = 1.5$
$f_7(x) = 10xe^{-x^2} - 1$	$\alpha_7 \approx 1.679630610428450$	$x_0 = 1.4$
$f_8(x) = \cos(x) - x$	$\alpha_8 \approx 0.739085133215161$	$x_0 = 0.3$

Table 3. Results of comparisons for different methods after two full iterations

Test Functions	NM	SM	SS	(3)	RH	(12)	KT8-1	(18)
$\|f_1(x_2)\|$	0.6e-1	0.6e-1	0.6e-4	0.1e-4	0.2e-6	0.6e-14	Div.	0.3e-11
$\|f_2(x_2)\|$	0.2e-4	0.9e-4	0.4e-10	0.7e-14	0.9e-44	0.1e-46	0.2e-57	0.3e-57
$\|f_3(x_2)\|$	0.9e-2	0.1	0.3e-4	0.8e-5	0.6e-17	0.1e-19	0.2e-24	0.1e-22
$\|f_4(x_2)\|$	1	3.1	0.3e-4	0.1e-1	0.3	0.3e-13	0.8e-9	0.3e-13
$\|f_5(x_2)\|$	0.7e-4	0.4e-3	0.6e-12	0.1e-10	0.2e-33	0.5e-43	0.6e-49	0.6e-50
$\|f_6(x_2)\|$	0.5	11	0.1e-5	0.1e-2	0.7e-5	0.8e-19	0.4e-9	0.7e-19
$\|f_7(x_2)\|$	0.4e-2	0.1	0.6e-10	0.6e-5	0.3e-21	0.1e-23	0.5e-9	0.6e-17
$\|f_8(x_2)\|$	0.1e-2	0.1e-3	0.4e-8	0.2e-15	0.1e-24	0.1e-47	0.1e-58	0.1e-60

Table 4. Results of comparisons for different methods after three full iterations

Test Functions	NM	SM	SS	(3)	RH	(12)	KT8-1	(18)
$\|f_1(x_3)\|$	0.5e-3	0.3e-2	0.3e-14	0.2e-20	0.5e-43	0.3e-101	Div.	0.2e-94
$\|f_2(x_3)\|$	0.1e-9	0.5e-8	0.8e-32	0.3e-56	0.2e-265	0.4e-328	0.9e-462	0.1e-459
$\|f_3(x_3)\|$	0.2e-5	0.2e-2	0.1e-15	0.3e-23	0.4e-109	0.2e-145	0.2e-204	0.3e-189
$\|f_4(x_3)\|$	0.3e-1	2.1	0.4e-17	0.4e-8	0.1e-9	0.1e-98	0.3e-77	0.8e-111
$\|f_5(x_3)\|$	0.6e-8	0.3e-6	0.1e-36	0.2e-41	0.8e-201	0.5e-301	0.7e-391	0.2e-398
$\|f_6(x_3)\|$	0.1e-1	10	0.1e-20	0.3e-12	0.2e-37	0.1e-137	0.4e-79	0.5e-156
$\|f_7(x_3)\|$	0.7e-5	0.1e-1	0.1e-32	0.4e-22	0.2e-130	0.2e-169	0.1e-75	0.1e-140
$\|f_8(x_3)\|$	0.3e-6	0.1e-8	0.8e-27	0.1e-64	0.3e-153	0.3e-339	0.1e-476	0.2e-492

In this work, we have presented some novel schemes of fourth-, seventh- and eighth-order convergence rate. The fourth-order techniques possess 1.587 and the seventh-order derivative-free methods have 1.626 as the efficiency index. Further research by the concept of weight function has also been done in order to provide optimal three-step families of derivative-free iterations with 1.682 as the index of efficiency. Per full cycle, the proposed techniques are free from derivative calculation which is so important in Chemical, Physical and Engineering problems.

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