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AN ABSTRACT POINT OF VIEW ON ITERATIVE APPROXIMATION OF FIXED POINTS: IMPACT ON THE THEORY OF FIXED POINT EQUATIONS

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Abstract. Let $(X, \to)$ be an L-space, $G : X \times X \to X$ and $f : X \to X$ be two operators. Let $f_G : X \to X$ be defined by $f_G(x) := G(x, f(x))$. If the operator G satisfies the following conditions:

(A_1) $G(x, x) = x, \ \forall x \in X$;

(A_2) $G(x, y) = x \Rightarrow y = x$,

then we call f_G admissible perturbation of f.

We introduce some iterative algorithms in terms of admissible perturbations. We suppose that these algorithms are convergent.

In this paper we study the impact of this hypothesis on the theory of fixed point equations: Gronwall lemmas (when $(X, \to, \leq)$ is an ordered L-space), data dependence, stability and shadowing property (when (X, d) is a metric space). Some open problems are presented.

Key Words and Phrases: fixed point, admissible perturbation, iterative method, Gronwall lemma, comparison lemma, data dependence, stability, Ulam-Hyers stability, limit shadowing property, open problem.

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Introduction

The iterative approximation of fixed points is one of the basic tools in the theory of equations. Various aspects of the theory of iterative algorithms appear in subjects such as:

- Numerical analysis in abstract spaces ([2], [3], [5], [6], [1], [17], ...).
- Difference equations ([22], [21], [31], [4], [23], [34], ...).
- Dynamical systems ([16], [25], [18], [14], [30], ...).
- Operator theory ([2], [3], [5], [6], [39], [45], [42], [46], [1], [8], [13], [26], [29], [39], [43], ...).

The weakly Picard operator theory (see [32], [33] and [39]) offers a solution for the following problem: which properties have the solutions of a fixed point equation for which the Picard iteration converges. In [41] we have studied a similar problem in the case of the Krasnoselskii iteration. In this paper we shall study the same problem corresponding to some general iterative algorithms.

The plan of the paper is the following:

1. The admissible perturbation of an operator
2. Iterative algorithms in terms of admissible perturbations
3. Gronwall lemmas
4. Comparison lemmas
5. Data dependence
6. Stability of an iterative algorithm
7. Open problems

1. The admissible perturbation of an operator

Let X be a nonempty set and $G : X \times X \to X$ be an operator. We suppose that:

(A_1) $G(x, x) = x, \ \forall x \in X$;

(A_2) $x, y \in X, \ G(x, y) = x$ imply $y = x$.

Let $f : X \to X$ be an operator. Then we consider the operator $f_G : X \to X$ defined by

$$f_G(x) := G(x, f(x)).$$

We remark that $F_{f_G} = F_f := \{x \in X \mid f(x) = x\}$. In general (see, for example, Remark 3.1 in [41]) $F_{f_G^n} \neq F_{f^n}, \ n \geq 2$. We call f_G the *admissible perturbation* of f corresponding to G.

Example 1.1. *Let $(V, +, \mathbb{R})$ be a vectorial space, $X \subset V$ a convex subset, $\lambda \in]0, 1[$, $f : X \to X$ and $G : X \times X \to X$ be defined by $G(x, y) := (1-\lambda)x + \lambda y$. Then f_G is an admissible perturbation of f. We shall denote f_G by f_λ and call it the Krasnoselskii perturbation of f.*

Example 1.2. *Let $(V, +, \mathbb{R})$ be a vectorial space, $X \subset V$ a convex subset, $\chi : X \times X \to]0, 1[$, $f : X \to X$ and $G(x, y) := (1 - \chi(x, y))x + \chi(x, y)y$. Then f_G is an admissible perturbation of f.*

Example 1.3. *Let X be a nonempty set endowed with an F-convex structure of Gudder (see [11]), where $F : [0, 1] \times X \times X \to X$ is an operator satisfying some conditions. Let $Y \subset X$ be an F-convex set, $\lambda \in]0, 1[$ and $f : Y \to Y$. We consider the operator $G : Y \times Y \to Y$, defined by $G(x, y) := F(\lambda, x, y)$. Then f_G is an admissible perturbation of f.*

Example 1.4. *Let (X, d) be a metric space endowed with a W-convex structure of Takahashi (see [45]). Here $W : X \times X \times [0, 1] \to X$ is an operator with the following property*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \ \forall x, y, u \in X, \ \lambda \in [0, 1].$$

We additionally suppose that $\lambda \in]0, 1[, \ W(x, y, \lambda) = x$ implies $y = x$.

Now, let $\lambda \in]0, 1[$, $Y \subset X$ a W-convex set, $f : Y \to Y$ and $G(x, y) := W(x, y, \lambda)$. In the above conditions, the operator f_G is an admissible perturbation of the operator f.

2. Iterative algorithms in terms of admissible perturbations

Recall first two important abstract concepts, see [32].

Definition 2.1. *Let (X,d) be a metric space. An operator $f : X \to X$ is Picard operator (briefly PO) if:*
(i) $F_f = \{x^*\}$;
(ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 2.2. *Let (X,d) be a metric space. An operator $f : X \to X$ is weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.*

Let $(X,\to)$ be an L-space (see [32], [39], [42]), $f : X \to X$ and $G, G_n : X\times X \to X$, $n \in \mathbb{N}$.

Example 2.1 (GK-algorithm)**.** *We consider the iterative algorithm*

$$x_0 \in X,\ x_{n+1} = G(x_n, f(x_n)),\ n \in \mathbb{N}.$$

By definition, this iterative process is convergent iff

$$x_n \to x^*(x_0) \in F_f \text{ as } n \to \infty, \text{ for all } x_0 \in X.$$

We remark that $x_n = f_G^n(x_0)$. So, this algorithm is convergent if and only if f_G is WPO. If f_G is WPO and an admissible perturbation of f, then $f_G^\infty : X \to F_f$ is a set retraction. We call this algorithm, Krasnoselskii algorithm corresponding to G or GK-algorithm.

For the particular case given in Example 1.1, see [2], [3] *and* [5]*. For the impact of this algorithm on the theory of fixed point equations see* [41].

Example 2.2 ($\mathbb{G}M$-algorithm)**.** *We consider the iterative method*

$$x_0 \in X,\ x_{n+1} = f_{G_n}(x_n),\ n \in \mathbb{N}.$$

We suppose that f_{G_n} is an admissible perturbation of f for all $n \in \mathbb{N}$ and that

$$x_n \to x^*(x_0) \in F_f \text{ as } n \to \infty.$$

In the above conditions, we consider the operator, $f_{\mathbb{G}}^\infty : X \to F_f$, defined by $f_{\mathbb{G}}^\infty(x) := x^*(x)$. We observe that $f_{\mathbb{G}}^\infty : X \to F_f$ is a set retraction. We call this algorithm Mann algorithm corresponding to $\mathbb{G} = (G_n)_{n\in\mathbb{N}}$ or $\mathbb{G}M$-algorithm.

In the particular case when X is a convex subset of a Banach space $\mathbb{B}$ and $G_n(x,y) := (1-\lambda_n)x + \lambda_n y$ with $\lambda_n \in]0,1[$ and $n \in \mathbb{N}$, we have the classical Mann iterative method ([2], [5], *...).*

Example 2.3 ($\mathbb{G}H$-algorithm)**.** *Let $u \in X$. We consider the operator $f_{G_{n,u}} : X \to X$ defined by*

$$f_{G_{n,u}}(x) := G_n(u, f(x)),\ n \in \mathbb{N}.$$

Let us consider the iterative process

$$x_0 \in X,\ x_{n+1} = G_n(u, f(x_n)),\ n \in \mathbb{N}.$$

We suppose that f_{G_n} *is an admissible perturbation for all* $n \in \mathbb{N}$ *and that the algorithm is convergent, i.e.,*

$$x_n \to x^*(u, x_0) \in F_f \text{ as } n \to \infty.$$

Now we define the operator $f^{\infty}_{\mathbb{G}H} : X \to F_f$, $f^{\infty}_{\mathbb{G}H}(x) := x^*(x, x)$. *We remark that* $f^{\infty}_{\mathbb{G}H} : X \to F_f$ *is a set retraction. Indeed, by the definition of* $f^{\infty}_{\mathbb{G}H}$, $f^{\infty}_{\mathbb{G}H}(X) \subset F_f$. *Let us prove that* $f^{\infty}_{\mathbb{G}H}(x) = x$, *for all* $x \in F_f$. *For* $u := x$ *and* $x_0 := x$ *we have:*

$$\begin{aligned} &x_1 = G_0(x, f(x)) = G_1(x, x) = x,\\ &x_2 = G_1(x, f(x_1)) = G_2(x, f(x)) = G_2(x, x) = x,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

By induction, we have that $x_n = x$, $\forall n \in \mathbb{N}$. *So,* $f^{\infty}_{\mathbb{G}H}(x) = x$, $\forall x \in F_f$.

We call this algorithm, Halpern algorithm corresponding to $\mathbb{G} = (G_n)_{n\in\mathbb{N}}$ *or* $\mathbb{G}H$-*algorithm. For some particular cases of this algorithm see* [2], [3] *and* [6].

Example 2.4 ($\mathbb{G}_1\mathbb{G}_2 I$-algorithm)**.** *Let* $(X, \to)$ *be an* L*-space,* $G_{1n}, G_{2n} : X \times X \to X$, $n \in \mathbb{N}$ *and* $f : X \to X$. *We suppose that* G_{1n}, G_{2n} *are admissible perturbations of* f *for all* $n \in \mathbb{N}$, *and that the algorithm*

$$x_0 \in X,\ x_{n+1} = G_{2n}(x_n, f(G_{1n}(x_n, f(x_n)))),\ n \in \mathbb{N}$$

is convergent, i.e.,

$$x_n \to x^*(x_0) \in F_f \text{ as } n \to \infty.$$

In the above conditions, we consider the operator $f^{\infty}_{\mathbb{G}_1\mathbb{G}_2} : X \to F_f$, $f^{\infty}_{\mathbb{G}_1\mathbb{G}_2}(x) := x^*(x)$. *We remark that* $f^{\infty}_{\mathbb{G}_1\mathbb{G}_2} : X \to F_f$ *is a set retraction. We call this algorithm Ishikawa algorithm corresponding to* $\mathbb{G}_1 = (G_{1n})_{n\in\mathbb{N}}$ *and* $\mathbb{G}_2 = (G_{2n})_{n\in\mathbb{N}}$, *or,* $\mathbb{G}_1\mathbb{G}_2 I$-*algorithm.*

For some particular cases of this algorithm see [2], [3] *and* [6] *and the references therein.*

Remark 2.1. *In the above example and throughout the paper in* G_{1n} *and* G_{2n} *1 and* n, *2 and* n *are indices.*

3. Gronwall lemmas

There are many results in the abstract theory of Gronwall lemmas (see [24], [32], [33], [37], [39], [41], [7], [8], [29], [43] and the references therein). For the basic problems of this theory see [37]. In this section we shall present Gronwall lemmas corresponding to an iterative process.

3.1. **The case of GK-algorithm.** Let $(X, \to, \leq)$ be an ordered L-space, $f : X \to X$ be an operator and f_G be an admissible perturbation of f. Then the following results hold.

Lemma 3.1. *We suppose that:*

(i) f_G *is a PO;*
(ii) *the operators* G *and* f *are increasing.*

Then $F_f = \{x^*\}$ *and*

(a) $x \in X,\ x \leq f(x) \Rightarrow x \leq x^*$;

(b) $x \in X,\ x \geq f(x) \Rightarrow x \geq x^*$.

Proof. Since f_G is an admissible perturbation of f, we have that $F_{f_G} = F_f$. From (i) it follows that $F_{f_G} = \{x^*\}$. So, $F_f = \{x^*\}$. The condition (ii) implies that the operator f_G is increasing. From $x \leq f(x)$ it follows that $x \leq f_G(x)$. Now the proofs for (a) and (b) follow from Gronwall lemma for POs (see [32], [33] and [37]). □

Lemma 3.2. *We suppose that:*

(i) f_G *is a* WPO;

(ii) *the operator* G *and* f *are increasing.*

Then:

(a) $x \in X,\ x \leq f(x) \Rightarrow x \leq f_G^\infty(x)$;

(b) $x \in X,\ x \geq f(x) \Rightarrow x \geq f_G^\infty(x)$.

3.2. **The case of $\mathbb{G}M$-algorithm.** Let $(X, \to, \leq)$ be an ordered L-space, $f : X \to X$ be an operator and f_{G_n}, $n \in \mathbb{N}$, be admissible perturbations of f. Then, we have the following result.

Lemma 3.3. *We suppose that:*

(i) *the* $\mathbb{G}M$*-algorithm is convergent;*

(ii) *the operators* f *and* G_n, $n \in \mathbb{N}$ *are increasing.*

Then:

(a) $x \in X,\ x \leq f(x) \Rightarrow x \leq f_{\mathbb{G}}^\infty(x)$;

(b) $x \in X,\ x \geq f(x) \Rightarrow x \geq f_{\mathbb{G}}^\infty(x)$.

Proof. Let us prove, for example, (a). Let $x \in X$ be such that $x \leq f(x)$. From (ii) we have that f_{G_n} is increasing for all $n \in \mathbb{N}$. This implies that

$$x \leq f_{G_n} \circ f_{G_{n-1}} \circ \ldots \circ f_{G_0}(x),\ \forall n \in \mathbb{N}.$$

Since $(X, \to, \leq)$ is an ordered L-space, from (i) it follows that, $x \leq f_{\mathbb{G}}^\infty(x)$. □

3.3. **The case of $\mathbb{G}H$-algorithm.** Let $(X, \to, \leq)$ be an ordered L-space, $f : X \to X$ be an operator and f_{G_n}, $n \in \mathbb{N}$, be admissible perturbations of the operator f. Then, we have the following result.

Lemma 3.4. *We suppose that:*

(i) *the* $\mathbb{G}H$*-algorithm is convergent;*

(ii) *the operators* f, G_n $n \in \mathbb{N}$ *are increasing.*

Then:

(a) $x \in X,\ x \leq f(x) \Rightarrow x \leq f_{\mathbb{G}H}^\infty(x)$;

(b) $x \in X,\ x \geq f(x) \Rightarrow x \geq f_{\mathbb{G}H}^\infty(x)$.

Proof. (a). Let $x \in X$ be such that $x \leq f(x)$. From (ii) it follows that $x \leq G_0(x, f(x)) \leq G_1(x, G_0(x, f(x))) \leq \ldots \leq G_n(x, x_n(x,x)) = x_{n+1}(x,x) \to f_{\mathbb{G}H}^\infty(x)$. □

3.4. **The case of $\mathbb{G}_1\mathbb{G}_2I$-algorithm.** In a similar way we have (see Example 2.4) the following result.

Lemma 3.5. *Let $(X, \to, \leq)$ be an ordered L-space. We suppose that:*

(i) *the $\mathbb{G}_1\mathbb{G}_2I$-algorithm is convergent;*

(ii) *the operators f, G_{1n}, G_{2n}, $n \in \mathbb{N}$ are increasing.*

Then:

(a) $x \in X,\ x \leq f(x) \Rightarrow x \leq f^{\infty}_{\mathbb{G}_1\mathbb{G}_2}(x)$;

(b) $x \in X,\ x \geq f(x) \Rightarrow x \geq f^{\infty}_{\mathbb{G}_1\mathbb{G}_2}(x)$.

Remark 3.1. *The above results are partial answers for Problem* 1 *and Problem* 2 *in* [37]*:*

Problem 1. *Let $(X, \leq)$ be an ordered set and $f : X \to X$ be an operator. If $F_f = \{x^*\}$, in which conditions we have that:*

(a) $x \in X,\ x \leq f(x) \Rightarrow x \leq x^*$?

(b) $x \in X,\ x \geq f(x) \Rightarrow x \geq x^*$?

Problem 2. *If $F_f \neq \emptyset$, in which conditions there exists a set retraction $r : X \to F_f$ such that:*

(a) $x \in X,\ x \leq f(x) \Rightarrow x \leq r(x)$?

(b) $x \in X,\ x \geq f(x) \Rightarrow x \geq r(x)$?

Remark 3.2. *In some particular cases there are conditions which imply condition* (i) *in the above lemmas (see* [2], [3], [6] *and the references therein). For example see Remark 4.3 in* [41].

4. Comparison lemmas

In this section we present some comparison lemmas corresponding to an iterative algorithm. For the case of Picard iteration see [32], [33], [39], [41] and [24]. We have the following two lemmas.

Lemma 4.1. *Let $(X, \to, \leq)$ be an ordered L-space and $f, g, h : X \to X$ be three operators. We suppose that:*

(i) *f_G, g_G and h_G are $WPOs$;*

(ii) *G and g are increasing;*

(iii) $f \leq g \leq h$.

Then, the following implication holds:

$$x, y, z \in X,\ x \leq y \leq z \Rightarrow f^{\infty}_G(x) \leq g^{\infty}_G(y) \leq h^{\infty}_G(z).$$

Proof. The condition (ii) implies that g_G is increasing. The conditions (ii) and (iii) imply that $f_G \leq g_G \leq h_G$. Now the proof follows from the corresponding lemma for $WPOs$ (see [32] and [33]). □

Lemma 4.2. *Let $(X, \to, \leq)$ be an ordered L-space, $f, g, h : X \to X$ be three operators and f_{G_n}, g_{G_n} and h_{G_n} the corresponding admissible perturbations. We suppose that:*

(i) *the $\mathbb{G}M$-algorithms corresponding to f, g and h are convergent;*

(ii) *the operators f, G_n, $n \in \mathbb{N}$, are increasing;*

(iii) $f \leq g \leq h$.
Then, the following implication holds:

$$x, y, z \in X,\ x \leq y \leq z \Rightarrow f_{\mathbb{G}}^{\infty}(x) \leq g_{\mathbb{G}}^{\infty}(y) \leq h_{\mathbb{G}}^{\infty}(z).$$

Proof. We have

$$\begin{aligned} x \leq y \leq z \Rightarrow & f_{G_n} \circ f_{G_{n-1}} \circ \ldots \circ f_{G_0}(x) \leq g_{G_n} \circ g_{G_{n-1}} \circ \ldots \circ g_{G_0}(x) \\ & \leq g_{G_n} \circ g_{G_{n-1}} \circ \ldots \circ g_{G_0}(y) \leq g_{G_n} \circ g_{G_{n-1}} \circ \ldots \circ g_{G_0}(z) \\ & \leq h_{G_n} \circ h_{G_{n-1}} \circ \ldots \circ h_{G_0}(z). \end{aligned}$$

Since $(X, \rightarrow, \leq)$ is an ordered L-space, from

$$f_{G_n} \circ f_{G_{n-1}} \circ \ldots \circ f_{G_0}(x) \leq g_{G_n} \circ g_{G_{n-1}} \circ \ldots \circ g_{G_0}(y) \leq h_{G_n} \circ h_{G_{n-1}} \circ \ldots \circ h_{G_0}(z)$$

it follows that $f_{\mathbb{G}}^{\infty}(x) \leq g_{\mathbb{G}}^{\infty}(y) \leq h_{\mathbb{G}}^{\infty}(z)$. □

We have similar results for $\mathbb{G}H$-algorithm and for $\mathbb{G}_1\mathbb{G}_2I$-algorithm.

Remark 4.1. *In some particular cases, we know conditions which imply condition* (i) *in the above lemmas (see* [2], [3], [6] *and the references therein). For example the following results are given in* [19].

Theorem 4.1. *Let Y be a compact convex subset of a Banach space X and $f : Y \to Y$ be a nonexpansive operator. Let (λ_n) be a sequence in $[0, b]$ for some $b < 1$ such that $\sum_{n=0}^{\infty} \lambda_n = \infty$. Then for any starting point $x_0 \in Y$, the $\mathbb{G}M$-algorithm, where $G_n(x, y) := (1 - \lambda_n)x + \lambda_n y$, converges.*

Corresponding to this Ishikawa's theorem we have

Theorem 4.2. *Let f, g and $h : Y \to Y$ be as in Theorem 4.1. Let X be an ordered Banach space. We suppose that:*

(i) *g is an increasing operator;*
(ii) $f \leq g \leq h$.

Then, the following implication holds:

$$x, y, z \in Y,\ x \leq y \leq z \Rightarrow f_{\mathbb{G}}^{\infty}(x) \leq g_{\mathbb{G}}^{\infty}(y) \leq h_{\mathbb{G}}^{\infty}(z).$$

5. Data dependence

Now we shall study the data dependence of the fixed points in the case of GK-algorithm. For the case of Picard iterations see [2], [32], [39], [36], [42], [1], ...

Let (X, d) be a metric space and $f, g : X \to X$ be two operators. Let f_G and g_G be corresponding admissible perturbations. We have the following results.

Lemma 5.1. *We suppose that:*

(i) *f_G is a ψ-PO with respect to d;*
(ii) *there exists $\eta > 0$ such that*

$$d(f(x), g(x)) \leq \eta,\ \forall x \in X;$$

(*iii*) *there exists* $l_2 > 0$ *such that*

$$d(G(x,y), G(x,z)) \leq l_2 d(y,z),\ \forall x,y,z \in X;$$

(*iv*) $F_g \neq \emptyset$.

If we denote by x_f^* *the unique fixed point of* f, *we have*

$$d(x_f^*, x_g^*) \leq \psi(l_2\eta),\ \forall x_g^* \in F_g.$$

Proof. From (*i*) we have that

$$d(x, x_f^*) \leq \psi(d(x, f_G(x))),\ \forall x \in X.$$

From (*ii*) and (*iii*) it follows

$$d(f_G(x), g_G(x)) = d(G(x, f(x)), G(x, g(x))) \leq l_2 d(f(x), g(x)) \leq l_2\eta.$$

So,

$$d(x_g^*, x_f^*) \leq \psi(d(x_g^*, f_G(x_g^*))) = \psi(d(g_G(x_g^*), f_G(x_g^*))) \leq \psi(l_2\eta).$$

□

In a similar way we obtain the following lemma.

Lemma 5.2. *We suppose that:*

(*i*) f_G *and* g_G *are* ψ*-WPOs;*

(*ii*) *there exists* $\eta > 0$ *such that*

$$d(f(x), g(x)) \leq \eta,\ \forall x \in X;$$

(*iii*) *there exists* $l_2 > 0$ *such that*

$$d(G(x,y), G(x,z)) \leq l_2 d(y,z),\ \forall x,y,z \in X.$$

Then,

$$H_d(F_f, F_g) \leq \psi(l_2\eta).$$

Remark 5.1. *Let* f *be as in Lemma 5.1. Then the equation* $x = f_G(x)$ *is generalized Ulam-Hyers stable (see* [38], [40]*). For the case of classical Krasnoselskii iteration see Remark* 6.2 *in* [41].

6. Stability of an iterative algorithm

There are very many hypostasis of data dependence some of them called stability ([2], [13], [16], [15], [28], [38], [40], [47], [36], [27], [9], [25], [20], [22], [21], [30], [12], [44], [31], ...). Taking account of the notions of stability in Difference equations, Dynamical systems, Differential equations, Operator theory and Numerical analysis, we try to unify these notions by the following definitions.

Let (X,d) be a metric space, $f : X \to X$ be an operator and $G, G_n, G_{1n}, G_{2n} : X \times X \to X$ be such that f_G, f_{G_n}, $f_{G_{1n}}$ and $f_{G_{2n}}$, $n \in \mathbb{N}$, are admissible perturbations of f.

Definition 6.1$_\mathbf{a}$. *The operator* f *has stable Picard iterates at* $x_0 \in X$ *if for every* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that*

$$x \in X,\ d(x, x_0) < \delta(\varepsilon) \Rightarrow d(f^n(x), f^n(x_0)) < \varepsilon,\ \forall n \in \mathbb{N}.$$

The operator f has stable Picard iterates on $Y \subset X$ if it has stable Picard iterates at all $x_0 \in Y$.

The operator f has attractive iterates at $x_0 \in X$ if there exists $\delta > 0$ such that

$$x \in X,\ d(x, x_0) < \delta \Rightarrow d(f^n(x), f^n(x_0)) \to 0 \text{ as } n \to \infty.$$

Definition 6.1$_\mathbf{b}$. *The operator f has stable GK-sequence at x_0 if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that:*

$$x_0 \in X,\ x_{n+1} = G(x_n, f(x_n)),\ n \in \mathbb{N}$$
$$y_0 \in X,\ y_{n+1} = G(y_n, f(y_n)),\ n \in \mathbb{N}$$

and

$$d(x_0, y_0) < \delta(\varepsilon),\ \text{imply that } d(x_n, y_n) < \varepsilon,\ \text{for all } n \in \mathbb{N}.$$

The operator f has stable GK-sequences on $Y \subset X$ if it has stable GK-sequence at all $x_0 \in Y$.

The operator f has attractive GK-sequence at x_0 if there exists $\delta > 0$ such that:

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Definition 6.1$_\mathbf{c}$. *The operator f has stable $\mathbb{G}M$-sequence at x_0 if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that:*

$$x_0 \in X,\ x_{n+1} = f_{G_n}(x_n),\ n \in \mathbb{N}$$
$$y_0 \in X,\ y_{n+1} = f_{G_n}(y_n),\ n \in \mathbb{N}$$

and

$$d(x_0, y_0) < \delta(\varepsilon),\ \text{imply that } d(x_n, y_n) < \varepsilon,\ \text{for all } n \in \mathbb{N}.$$

The operator f has stable $\mathbb{G}M$-sequences on $Y \subset X$ if it has stable $\mathbb{G}M$-sequence at all $x_0 \in Y$.

The operator f has attractive $\mathbb{G}M$-sequence at x_0 if there exists $\delta > 0$ such that:

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Definition 6.1$_\mathbf{d}$. *The operator f has stable $\mathbb{G}H$-sequence at x_0 if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that:*

$$x_0 \in X,\ x_{n+1} = G_n(x_0, f(x_n)),\ n \in \mathbb{N}$$
$$y_0 \in X,\ y_{n+1} = G_n(y_0, f(y_n)),\ n \in \mathbb{N}$$

and

$$d(x_0, y_0) < \delta(\varepsilon),\ \text{imply that } d(x_n, y_n) < \varepsilon,\ \text{for all } n \in \mathbb{N}.$$

The operator f has stable $\mathbb{G}H$-sequences on $Y \subset X$ if it has stable $\mathbb{G}H$-sequence at all $x_0 \in Y$.

The operator f has attractive $\mathbb{G}H$-sequence at x_0 if there exists $\delta > 0$ such that

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Definition 6.1$_{\mathbf{e}}$. *The operator f has stable $\mathbb{G}_1\mathbb{G}_2I$-sequence at x_0 if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that:*

$$x_0 \in X,\ x_{n+1} = G_{2n}(x_n, f(G_{1n}(x_n, f(x_n)))),\ n \in \mathbb{N}$$
$$y_0 \in X,\ y_{n+1} = G_{2n}(y_n, f(G_{1n}(y_n, f(y_n)))),\ n \in \mathbb{N}$$

and

$$d(x_0, y_0) < \delta(\varepsilon),\ \textit{imply that } d(x_n, y_n) < \varepsilon,\ \textit{for all } n \in \mathbb{N}.$$

The operator f has stable $\mathbb{G}_1\mathbb{G}_2I$-sequences on $Y \subset X$ if it has stable $\mathbb{G}_1\mathbb{G}_2I$-sequence at all $x_0 \in Y$.

The operator f has attractive $\mathbb{G}_1\mathbb{G}_2I$-sequence at x_0 if there exists $\delta > 0$ such that

$$d(x_0, y_0) < \delta \Rightarrow d(x_n, y_n) \to 0 \ \textit{as } n \to \infty.$$

Definition 6.1. *The operator f has asymptotically stable sequence at x_0, generated by an algorithm, if this sequence is stable and attractive.*

Definition 6.2$_{\mathbf{a}}$. *The operator f has the limit shadowing property with respect to Picard algorithm (i.e. Picard iteration) if*

$$y_n \in X,\ n \in \mathbb{N},\ d(y_{n+1}, f(y_n)) \to 0 \ \textit{as } n \to \infty$$

imply that there exists $x_0 \in X$ such that $d(y_n, f^n(x_0)) \to 0$ as $n \to \infty$.

Definition 6.2$_{\mathbf{b}}$. *The operator f has the limit shadowing property with respect to GK-algorithm if*

$$y_n \in X,\ n \in \mathbb{N},\ d(y_{n+1}, G(y_n, f(y_n))) \to 0 \ \textit{as } n \to \infty$$

imply that there exists $x_0 \in X$ such that

$$d(y_n, f_G^n(x_0)) \to 0 \ \textit{as } n \to \infty.$$

Definition 6.2$_{\mathbf{c}}$. *The operator f has the limit shadowing property with respect to $\mathbb{G}M$-algorithm if*

$$y_n \in X,\ n \in \mathbb{N},\ d(y_{n+1}, f_{G_n}(y_n)) \to 0 \ \textit{as } n \to \infty$$

imply that there exists $x_0 \in X$ such that

$$d(y_n, f_{G_n} \circ f_{G_{n-1}} \circ \ldots \circ f_{G_0}(x_0)) \to 0 \ \textit{as } n \to \infty.$$

Definition 6.2$_{\mathbf{d}}$. *The operator f has the limit shadowing property with respect to $\mathbb{G}H$-algorithm if*

$$y_n \in X,\ n \in \mathbb{N},\ d(y_{n+1}, G_n(y_0, f(y_n))) \to 0 \ \textit{as } n \to \infty$$

imply that there exists $x_0 \in X$ such that

$$d(y_n, G_n(x_0, f(\cdot)) \circ \ldots \circ G_0(x_0, f(x_0))) \to 0 \ \textit{as } n \to \infty.$$

Definition 6.2$_{\mathbf{e}}$. *The operator f has the limit shadowing property with respect to $\mathbb{G}_1\mathbb{G}_2I$-algorithm if*

$$y_n \in X,\ n \in \mathbb{N},\ d(y_{n+1}, f_{G_{2n}G_{1n}}(y_n)) \to 0 \ \textit{as } n \to \infty$$

imply that there exists $x_0 \in X$ *such that*

$$d(y_n, f_{G_{2n}} \circ f_{G_{1n}G_{1n}} \circ f_{G_{2n-1}G_{1n-1}} \circ \ldots \circ f_{G_{20}G_{10}}(x_0)) \to 0 \text{ as } n \to \infty.$$

Here $f_{G_{2k}G_{1k}}(x) := G_{2k}(x, f(G_{1k}(x, f(x))))$.

Definition 6.3. *An iterative algorithm (Picard algorithm, GK-algorithm, $\mathbb{G}M$-algorithm, $\mathbb{G}H$-algorithm, $\mathbb{G}_1\mathbb{G}_2I$-algorithm, ...) is stable with respect to an operator f if it is convergent with respect to f and the operator f has the limit shadowing property with respect to this algorithm.*

Remark 6.1. *For a better understanding of the above definition, please look to the following definitions, remarks and examples.*

Definition 6.4 ([9], [47]). *Let (X, d) be a metric space and $f : X \to X$ be an operator. By definition, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ of iterates of $x_0 \in X$ is stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f^n(x), f^n(x_0)) < \varepsilon$ for every $n = 1, 2, \ldots$, whenever $x \in X$ and $d(x, x_0) < \delta$.*

Definition 6.5 ([15], [28], [27]). *Let (X, d) be a complete metric space and T be a self mapping of X. Let $x_{n+1} = f(T, x_n)$ be some iteration procedure in X. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $q \in F(T)$. Let $\{y_n\} \subset X$, and define, $\varepsilon_n = d(y_{n+1}, f(T, y_n))$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$, then the iteration procedure, $x_{n+1} = f(T, x_n)$, is said to be T-stable.*

Remark 6.2. *For the stability of some classical algorithms see* [2], [13], [16], [28], [47], [27], [9], [10], [25], [20], [14], [44], ...

Remark 6.3. *For the shadowing property with respect to Picard iteration see* [30], [36], [35], [20], [12], ...

Remark 6.4. *For the stability of an invariant subset of f (fixed point, periodic orbit, orbit, attractor, ...) with respect to Picard iteration see* [18], [13], [16], [10], [44], ...

Remark 6.5. *If $x_{n+1} = f(T, x_n)$ is stable as in Definition 6.5, and $f(T, x) = x$, $\forall x \in F_T$, then $F_T = \{x^*\}$. Indeed, let $q, p \in F_T$. Let $x_n \to q$ as $n \to \infty$ and $y_n = p$, $\forall n \in \mathbb{N}$. Since $d(y_{n+1}, y_n) = 0$, it follows that $y_n \to q$ as $n \to \infty$, i.e., $q = p$.*

Remark 6.6. *In the bibliography of the iterative approximation of fixed points, we find the following synonymous terms: method, process, procedure, algorithm.*

Example 6.1. *Let $X = \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$. In this case:*

(a) *$F_f = \emptyset$;*
(b) *f has stable iterates on $\mathbb{R}$;*
(c) *f does not have the shadowing property with respect to the Picard algorithm.*

Example 6.2. *$X = [0, 1] \cup [2, 3]$, $f(x) = \frac{1}{2}x$ for $x \in [0, 1]$ and $f(x) = \frac{1}{2}(x - 2) + 2$ for $x \in [2, 3]$. In this case:*

(a) *$F_f = \{0, 2\}$;*
(b) *f has stable iterates on X;*
(c) *the Picard algorithm is stable with respect to f.*

Example 6.3. *Let* $X = \mathbb{R}$, $f(x) = -2x$, $x \in \mathbb{R}$ *and* $G(x, y) = \frac{1}{2}x + \frac{1}{2}y$. *Then:*

(a) $F_f = \{0\}$;
(b) f *does not have stable iterate on* $\mathbb{R}$;
(c) f *does not have the limit shadowing property with respect to the Picard algorithm;*
(d) f *has stable* GK*-sequence on* $\mathbb{R}$;
(e) *the* GK*-algorithm is stable with respect to* f.

7. Open problems

The above considerations give rise to the following questions:

Problem 7.1. *To apply the abstract results in §3, §4 and §5 to the differential and integral equations for which the Picard algorithm is not convergent.*

For the case of the Picard algorithm see, for example, [2], [32], [39], [31], [7], [8], [26], [29], [43].

Problem 7.2. *To study the data dependence (see* §5*) in the case of:*

(a) $\mathbb{G}M$*-algorithm;*
(b) $\mathbb{G}H$*-algorithm;*
(c) $\mathbb{G}_1\mathbb{G}_2 I$*-algorithm.*

Problem 7.3. *To study the convergence of the following algorithms:*

(a) $\mathbb{G}K$*-algorithm;*
(b) $\mathbb{G}M$*-algorithm;*
(c) $\mathbb{G}H$*-algorithm;*
(d) $\mathbb{G}_1\mathbb{G}_2 I$*-algorithm.*

Problem 7.4. *To study the limit shadowing property of an operator with respect to:*

(a) $\mathbb{G}K$*-algorithm;*
(b) $\mathbb{G}M$*-algorithm;*
(c) $\mathbb{G}H$*-algorithm;*
(d) $\mathbb{G}_1\mathbb{G}_2 I$*-algorithm.*

Problem 7.5. *Given a convergent fixed point algorithm, the problem is to establish the corresponding Gronwall lemma and comparison lemma.*

For example, let us consider Baillon's ergodic theorem (see [46], page 63) let $(H, +, \mathbb{R}, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $X \subset H$ be a nonempty bounded, closed and convex subset of H and $f : X \to X$ be a nonexpansive operator. Let $s_n : X \to X$ be defined by, $s_{f,n}(x) := \frac{1}{n}\sum_{k=0}^{n-1} f^k(x)$. Then:

(a) $F_f \neq \emptyset$.
(b) $\forall\, x \in X$, $s_{f,n}(x) \rightharpoonup x^*(x) \in F_f$ as $n \to \infty$.
(c) the operator $s_f^\infty : X \to F_f$, $s_f^\infty(x) := x^*(x)$, is a nonexpansive retraction.

Let $(H, +, \mathbb{R}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $(H, \leq)$ be a partial ordered set and $X \subset H$ be a nonempty bounded, closed and convex subset. We suppose that $(H, +, \mathbb{R}, \rightharpoonup, \leq)$ is an ordered linear L-space. Then, corresponding to the above convergent algorithm we have the following results.

Theorem 7.1. *Let $f : X \to X$ be an operator. We suppose that:*

(i) *f is nonexpansive;*
(ii) *f is increasing.*

Then:

(a) $F_f \subset F_{s_{f,n}}$, $\forall\ n \in \mathbb{N}^*$;
(b) $(LF)_f \subset (LF)_{s_{f,n}}$ *and* $(UF)_f \subset (UF)_{s_{f,n}}$, $\forall\ n \in \mathbb{N}^*$;
(c) $x \leq f(x) \Rightarrow x \leq s_f^\infty(x)$;
(d) $x \geq f(x) \Rightarrow x \geq s_f^\infty(x)$.

Theorem 7.2. *Let $f, g, h : X \to X$ be such that:*

(i) *f, g and h are nonexpansive;*
(ii) *g is increasing;*
(iii) $f \leq g \leq h$.

Then $x, y, z \in X,\ x \leq y \leq z \Rightarrow s_f^\infty(x) \leq s_g^\infty(y) \leq s_h^\infty(z)$.

Problem 7.6. *In section 1 we gave some examples of operators G which satisfy the conditions (A_1) and (A_2). The problem is to give other such examples.*

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