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SOME RESULTS ON ASYMPTOTICALLY HEMI-PSEUDOCONTRACTIVE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. In this paper, a mapping which is asymptotically hemi-pseudocontractive in the intermediate sense is introduced. Hybrid projection methods are considered for the class of mappings. Strong convergence theorems for common fixed points are established in the framework of Hilbert spaces.

Key Words and Phrases: Asymptotically hemi-pseudocontractive mapping, asymptotically quasi-pseudocontractive mapping in the intermediate sense, fixed point, hybrid projection method.
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1. Introduction and preliminaries

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $T : C \to C$ be a mapping. In this paper, we denote the fixed point set of T by $F(T)$.

Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \, n \geq 1.$$

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The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [8] as a generalization of the class of nonexpansive mappings. They proved that, if C is a nonempty closed convex bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C, then T has a fixed point.

T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x-y\|\big) \le 0. \tag{1.1}$$

Observe that if we define

$$\tau_n = \max\left\{0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|)\right\},$$

then $\tau_n \to 0$ as $n\to\infty$. It follows that (1.1) is reduced to

$$\|T^n x - T^n y\| \le \|x-y\| + \tau_n, \quad \forall x,y\in C,\, n\ge 1.$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was considered by Bruck, Kuczumow and Reich [4] and Kirk [12]. It is known that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point.

Recall that T is said to be *strictly pseudocontractive* if there exists a constant $\kappa\in[0,1)$ such that

$$\|Tx - Ty\| \le \|x-y\|^2 + \kappa\|(I-T)x - (I-T)y\|^2, \quad \forall x,y\in C.$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [3].

Recall that T is said to be an *asymptotically strict pseudocontraction* if there exist a constant $\kappa\in[0,1)$ and a sequence $\{k_n\}\subset[1,\infty)$ with $k_n\to 1$ as $n\to\infty$ such that

$$\|T^n x - T^n y\|^2 \le k_n\|x-y\|^2 + \kappa\|(I-T^n)x - (I-T^n)y\|^2, \quad \forall x,y\in C.$$

The class of asymptotically strict pseudocontractions was introduced by Qihou [19].

Recently, Sahu, Xu and Yao [28] introduced a class of new mappings: asymptotically strict pseudocontractive mappings in the intermediate sense. Recall that T is said to be an *asymptotically strict pseudocontraction in the intermediate sense* if

$$\begin{aligned}\limsup_{n\to\infty} \sup_{x,y\in C} \big(&\|T^n x - T^n y\|^2 - k_n\|x-y\|^2 \\ &- \kappa\|(I-T^n)x - (I-T^n)y\|^2\big) \le 0,\end{aligned} \tag{1.2}$$

where $\kappa\in[0,1)$ and $\{k_n\}\subset[1,\infty)$ such that $k_n\to 1$ as $n\to\infty$. Put

$$\xi_n = \max\left\{0, \sup_{x,y\in C} \big(\|T^n x - T^n y\|^2 - k_n\|x-y\|^2 - \kappa\|(I-T^n)x - (I-T^n)y\|^2\big)\right\}.$$

It follows that $\xi_n\to 0$ as $n\to\infty$. Then, (1.2) is reduced to the following:

$$\begin{aligned}\|T^n x - T^n y\|^2 \le{}& k_n\|x-y\|^2 + \kappa\|(I-T^n)x - (I-T^n)y\|^2 \\ &+ \xi_n, \quad \forall x,y\in C.\end{aligned}$$

Recall that T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n \|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

It is clear that (1.3) is equivalent to

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C.$$

The class of asymptotically pseudocontractive mappings was introduced by Schu [29]. In [27], Rhoades gave an example to show that the class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings; see [27] for more details. In 2009, Zhou [34] showed that every uniformly Lipschitz and asymptotically pseudocontractive mapping which is also uniformly asymptotically regular has a fixed point.

Recently, Qin, Cho and Kim [21] introduced the mappings which are asymptotically pseudocontractive mappings in the intermediate sense.

Recall that T is said to be an *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2\big) \leq 0, \tag{1.4}$$

where $\{k_n\}$ is a sequence in $[1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Put

$$\nu_n = \max\left\{0, \sup_{x,y\in C} \big(\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2\big)\right\}. \tag{1.5}$$

It follows that $\nu_n \to 0$ as $n \to \infty$. Then, (1.4) is reduced to the following:

$$\langle T^n x - T^n y, x - y\rangle \leq k_n\|x - y\|^2 + \nu_n, \quad \forall x, y \in C,\ n \geq 1.$$

It is easy to see that the above inequality is equivalent to

$$\begin{aligned}\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 &+ \|(I - T^n)x - (I - T^n)y\|^2 \\ &+ 2\nu_n,\ \forall x, y \in C,\ n \geq 1.\end{aligned} \tag{1.6}$$

We remark that if $\nu_n = 0$ for each $n \geq 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically pseudocontractive mappings.

In this paper, we introduce and consider the following mapping.

Recall that T is said to be an *asymptotically hemi-pseudocontractive mapping in the intermediate sense* if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{x\in C, y\in F(T)} \big(\langle T^n x - y, x - y\rangle - k_n\|x - y\|^2\big) \leq 0, \tag{1.7}$$

where $\{k_n\}$ is a sequence in $[1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Put

$$\nu_n = \max\left\{0, \sup_{x\in C, y\in F(T)} \big(\langle T^n x - y, x - y\rangle - k_n\|x - y\|^2\big)\right\}. \tag{1.8}$$

It follows that $\nu_n \to 0$ as $n \to \infty$. Then, (1.7) is reduced to the following:

$$\langle T^n x - y, x - y\rangle \leq k_n\|x - y\|^2 + \nu_n, \quad \forall x, y \in C,\ n \geq 1.$$

It is easy to see that the above inequality is equivalent to

$$\begin{aligned}\|T^n x - y\|^2 \le (2k_n - 1)\|x - y\|^2 + \|x - T^n x\|^2 \\ + 2\nu_n, \ \forall x \in C, \, y \in F(T), \, n \ge 1.\end{aligned} \tag{1.9}$$

We remark that if $\nu_n = 0$ for each $n \ge 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically hemi-pseudocontractive mappings.

Recall that the normal Mann iteration was introduced by Mann [13] in 1953. Since then, the constructions of fixed points for nonexpansive mappings via the Mann iteration have been extensively investigated by many authors. The Mann iteration generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \ge 1, \tag{1.10}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$.

In 1991, Schu [29] gave an adaptation of the normal Mann iteration for asymptotically nonexpansive mappings as follows:

$$x_1 \in C \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \ge 1, \tag{1.11}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$. Weak convergence theorems are established under certain restrictions imposed on the control sequence $\{\alpha_n\}$.

It is well known that, in an infinite-dimensional Hilbert space, only weak convergence theorems for the normal Mann iteration were established even for nonexpansive mappings, see [7]. Attempts to modify the normal Mann iteration for nonexpansive mappings by hybrid projection methods have recently been made so that strong convergence theorems are obtained; see, for example, [1,2,5,6,9-11,14-18,20-26,30-34] and the references therein.

In 2008, Kim and Xu [11] considered the class of asymptotically strict pseudocontractions based on the hybrid projection method. To be more precise, they proved the following results.

Theorem KX. *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be an asymptotically κ-strict pseudocontraction for some $0 \le \kappa < 1$. Assume that $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be the sequence generated by the following (CQ) algorithm: $x_0 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_n = \big\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad\quad + \big(\kappa - \alpha_n(1 - \alpha_n)\big)\|x_n - T^n x_n\|^2 + \theta_n\big\}, \\ Q_n = \big\{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\big\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.12}$$

where $\theta_n = \Delta_n^2(1-\alpha_n)(k_n-1) \to 0$ as $n \to \infty$ and $\Delta_n = \sup\big\{\|x_n - z\| : z \in F(T)\big\} < \infty$. Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - \kappa$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

For the class of asymptotically strict pseudocontractions in the intermediate sense, Sahu, Xu and Yao [28] obtained the following results.

Theorem SXY. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{k_n\}$ such that $F(T)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $0 < \delta \le \alpha_n \le 1-\kappa$ for all n. Let $\{x_n\}$ be the sequence in C generated by the following (CQ) algorithm: $u = x_1 \in C$ and*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \big\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\big\}, \\ Q_n = \big\{z \in C : \langle u - x_n, x_n - z\rangle \ge 0\big\}, \\ x_{n+1} = P_{C_n \cap Q_n} u, \end{cases} \tag{1.13}$$

where

$$\theta_n = \max\Big\{0, \sup_{x,y\in C} \big(\|T^n x - T^n y\|^2 - k_n\|x-y\|^2 - \kappa\|(I-T^n)x - (I-T^n)y\|^2\big)\Big\} + (k_n - 1)\Delta_n$$

and $\Delta_n = \sup\big\{\|x_n - z\| : z \in F(T)\big\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}u$.

For the class of asymptotically pseudocontractive mappings, Zhou [34] obtained the following results.

Theorem Z. *Let C be a closed convex bounded subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a fixed point. Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n \in [a,b]$ for some $a, b \in \big(0, \frac{1}{1+L}\big)$. Let a sequence $\{x_n\}$ be generated in the following manner: $x_0 \in C$ and*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \\ C_n = \big\{z \in C : \alpha_n\big(1-(1+L)\alpha_n\big)\|x_n - T^n x_n\|^2 \\ \qquad \le \langle x_n - z, (I-T^n)y_n\rangle + (k_n - 1)(diam\, C)^2\big\}, \\ Q_n = \big\{z \in C : \langle z - x_n, x_n - z\rangle \ge 0\big\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.14}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

In this paper, motivated by Theorem KX, Theorem SXY and Theorem Z, we consider the class of mappings which are asymptotically hemi-pseudocontractive mappings in the intermediate sense based on hybrid projection methods. Strong convergence theorems for common fixed points are established in the framework of real Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 and Lemma 1.2 are well known results in real Hilbert spaces.

Lemma 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $x \in H$. Then*

$$z = P_C x \quad \Longleftrightarrow \quad \langle x - z, z - y\rangle \ge 0, \quad \forall y \in C.$$

Lemma 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in C.$$

Lemma 1.3. *Let C be a nonempty closed convex bounded subset of H and T be a uniformly L-Lipschitzian and asymptotically hemi-pseudocontractive mapping in the intermediate sense. Then $F(T)$ is a closed convex subset of C.*

Proof. From the continuity of T, we can conclude that $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $p_1, p_2 \in F(T)$. We prove $p = tp_1 + (1-t)p_2 \in F(T)$, where $t \in (0,1)$. Put $y_{(\alpha,n)} = (1-\alpha)p + \alpha T^n p$, where $\alpha \in \big(0, \frac{1}{1+L}\big)$. For all $w \in F(T)$, we see that

$$\begin{aligned}
\|p - T^n p\|^2 &= \frac{1}{\alpha}\langle p - y_{(\alpha,n)}, p - T^n p\rangle \\
&= \frac{1}{\alpha}\big\langle p - y_{(\alpha,n)}, p - T^n p - \big(y_{(\alpha,n)} - T^n y_{(\alpha,n)}\big)\big\rangle \\
&\quad + \frac{1}{\alpha}\big\langle p - w + w - y_{\alpha,n}, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\big\rangle \\
&\leq (1+L)\alpha\|p - T^n p\|^2 + \frac{1}{\alpha}\big\langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\big\rangle \\
&\quad + \frac{1}{\alpha}\langle w - y_{(\alpha,n)}, y_{(\alpha,n)} - w\rangle + \frac{1}{\alpha}\big\langle w - y_{(\alpha,n)}, w - T^n y_{(\alpha,n)}\big\rangle \\
&\leq (1+L)\alpha\|p - T^n p\|^2 + \frac{1}{\alpha}\big\langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\big\rangle \\
&\quad + \frac{(k_n - 1)(diam\, C)^2 + \nu_n}{\alpha},
\end{aligned}$$

where $diam\, C$ denotes the diameter of the set C and

$$\nu_n = \max\Big\{0, \sup_{w \in F(T)} \big(\big\langle w - T^n y_{(\alpha,n)}, w - y_{(\alpha,n)}\big\rangle - k_n\|w - y_{(\alpha,n)}\|^2\big)\Big\}.$$

This implies that

$$\begin{aligned}
&\alpha\big(1 - (1+L)\alpha\big)\|p - T^n p\|^2 \\
&\quad \leq \langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\rangle \\
&\qquad + (k_n - 1)(diam\, C)^2 + \nu_n, \quad \forall w \in F(T).
\end{aligned} \tag{1.15}$$

Taking $w = p_i$, $i = 1, 2$ in (1.15), multiplying t and $(1-t)$ on the both sides of (1.15), respectively and adding up, we see that

$$\alpha\big(1 - (1+L)\alpha\big)\|p - T^n p\|^2 \leq (k_n - 1)(diam\, C)^2 + \nu_n.$$

This shows that $T^n p - p \to 0$ as $n \to \infty$. Note that T is uniformly L-Lipschitzian. It follows that $T^{n+1}p - Tp \to 0$ as $n \to \infty$. This is, $p \in F(T)$. This completes the proof.

2. Main results

Now, we are ready to give our main results.

Theorem 2.1. *Let C be a closed convex bounded subset of a real Hilbert space H. Let $T_m : C \to C$ be a mapping which is uniformly L_m-Lipschitz and asymptotically hemi-pseudocontractive in the intermediate sense for each $m \geq 1$. Assume that $\mathcal{F} = \bigcap_{m=1}^{\infty} F(T_m)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily}, \\ C_{(1,m)} = C, \quad C_1 = \bigcap_{m=1}^{\infty} C_{(1,m)}, \\ x_1 = P_{C_1} x_0, \\ y_{(n,m)} = (1 - \alpha_{(n,m)}) x_n + \alpha_{(n,m)} T_m^n x_n, \quad \forall n \geq 1, \\ C_{(n+1,m)} \\ \quad = \big\{ z \in C_{(n,m)} : \alpha_{(n,m)} \big(1 - (1 + L_m)\alpha_{(n,m)}\big) \big\| x_n - T_m^n x_n \big\|^2 \\ \qquad \leq \big\langle x_n - z, y_{(n,m)} - T_m^n y_{(n,m)} \big\rangle \\ \qquad\quad + (k_{(n,m)} - 1)(diam\, C)^2 + \nu_{(n,m)} \big\}, \\ C_{n+1} = \bigcap_{m=1}^{\infty} C_{(n+1,m)}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{2.1}$$

where

$$\nu_{(n,m)} = \max \left\{ 0, \sup_{p \in \mathcal{F}} \left(\langle p - T_m^n y_{(n,m)}, p - y_{(n,m)} \rangle - k_{(n,m)} \| p - y_{(n,m)} \|^2 \right) \right\}.$$

Assume that the control sequences $\{\alpha_{(n,m)}\}$ are chosen so that $\alpha_{(n,m)} \in [a_m, b_m]$ for some $a_m, b_m \in \big(0, \frac{1}{1+L_m}\big)$. Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$, where P is the metric projection from H onto $\mathcal{F}$.

Proof. We divide the proof into six parts.

Step 1. Show that $P_{\mathcal{F}} x_0$ is well defined for each $x_0 \in H$.

From Lemma 1.3, we see that $F(T_m)$ is closed and convex for each $m \geq 1$. Hence $\mathcal{F} = \bigcap_{m=1}^{\infty} F(T_m)$ is closed and convex. This shows that $P_{\mathcal{F}} x_0$ is well defined for each $x_0 \in H$. This completes the proof of Step 1.

Step 2. Show that C_n is closed and convex for all $n \geq 1$.

It suffices to show that for each $m \geq 1$, $C_{(n,m)}$ is closed and convex. This can be proved by induction on n. In fact, for $n = 1$, $C_{(1,m)} = C$ is closed and convex. Assume that $C_{(n,m)}$ is closed and convex for some $n \geq 1$. From the definition of $C_{(n+1,m)}$, we know that $C_{(n+1,m)}$ is also closed and convex for the same $n \geq 1$ and hence $C_{(n,m)}$ is closed and convex for each $n \geq 1$. This completes the proof of Step 2.

Step 3. Show that $\mathcal{F} \subset C_n$, $\forall n \geq 1$.

It suffices to show that, for each $m \geq 1$, $\mathcal{F} \subset C_{(n,m)}$. For $n = 1$, $\mathcal{F} \subset C = C_{(1,m)}$. Assume that $\mathcal{F} = C_{(n,m)}$ for some $n \geq 1$. Next, we prove that $\mathcal{F} \subset C_{(n+1,m)}$ for the same $n \geq 1$. For any $p \in \mathcal{F}$, we see that

$$
\begin{aligned}
&\big\|x_n - T_m^n x_n\big\|^2 \\
&= \frac{1}{\alpha_{(n,m)}}\big\langle x_n - y_{(n,m)}, x_n - T_m^n x_n\big\rangle \\
&= \frac{1}{\alpha_{(n,m)}}\big\langle x_n - y_{(n,m)}, x_n - T_m^n x_n - \big(y_{(n,m)} - T_m^n y_{(n,m)}\big)\big\rangle \\
&\quad + \frac{1}{\alpha_{(n,m)}}\big\langle x_n - y_{(n,m)}, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle \\
&= \frac{1}{\alpha_{(n,m)}}\big\langle x_n - y_{(n,m)}, x_n - T_m^n x_n - \big(y_{(n,m)} - T_m^n y_{(n,m)}\big)\big\rangle \\
&\quad + \frac{1}{\alpha_{(n,m)}}\big\langle x_n - p + p - y_{(n,m)}, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle \\
&\leq \frac{1+L_m}{\alpha_{(n,m)}}\|x_n - y_{(n,m)}\|^2 + \frac{1}{\alpha_{(n,m)}}\big\langle x_n - p, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle \\
&\quad + \frac{1}{\alpha_{(n,m)}}\big\langle p - y_{(n,m)}, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle \\
&\leq (1+L_m)\alpha_{(n,m)}\big\|x_n - T_m^n x_n\big\|^2 + \frac{1}{\alpha_{(n,m)}}\big\langle x_n - p, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle \\
&\quad + \frac{1}{\alpha_{(n,m)}}\big\langle p - y_{(n,m)}, y_{(n,m)} - p\big\rangle + \frac{1}{\alpha_{(n,m)}}\big\langle p - y_{(n,m)}, p - T_m^n y_{(n,m)}\big\rangle \\
&\leq (1+L_m)\alpha_{(n,m)}\big\|x_n - T_m^n x_n\big\|^2 + \frac{1}{\alpha_{(n,m)}}\big\langle x_n - p, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle \\
&\quad + \frac{(k_{(n,m)}-1)\|p - y_{(n,m)}\|^2 + \nu_{(n,m)}}{\alpha_{(n,m)}} \\
&\leq (1+L_m)\alpha_{(n,m)}\big\|x_n - T_m^n x_n\big\|^2 + \frac{1}{\alpha_{(n,m)}}\big\langle x_n - p, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle \\
&\quad + \frac{(k_{(n,m)}-1)(diam\, C)^2 + \nu_{(n,m)}}{\alpha_{(n,m)}},
\end{aligned}
$$

where

$$
\nu_{(n,m)} = \max\Big\{0, \sup_{p\in\mathcal{F}}\big(\big\langle p - T_m^n y_{(n,m)}, p - y_{(n,m)}\big\rangle - k_{(n,m)}\|p - y_{(n,m)}\|^2\big)\Big\}.
$$

This implies that

$$
\begin{aligned}
&\alpha_{(n,m)}\big(1 - (1+L_m)\alpha_{(n,m)}\big)\big\|x_n - T_m^n x_n\big\|^2 \\
&\quad \leq \big\langle x_n - p, y_{(n,m)} - T_m^n y_{(n,m)}\big\rangle + (k_{(n,m)}-1)(diam\, C)^2 + \nu_{(n,m)}.
\end{aligned}
$$

This shows that $p \in C_{(n+1,m)}$. This implies that $\mathcal{F} \subset C_n$ for all $n \geq 1$. This completes the proof of Step 3.

Step 4. Show that $\{x_n\}$ is a Cauchy sequence in C.

Note that $x_n = P_{C_n} x_0$. In view of $x_{n+1} \in C_{n+1} \subset C_n$, we see that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

This together with the boundedness of $\{x_n\}$ implies that $\lim_{n\to\infty} \|x_0 - x_n\|$ exists. For $i > n \geq 1$, we have that $x_i = P_{C_i} x_0 \in C_i \subset C_n$. It follows from Lemma 1.2 that

$$\|x_i - x_n\|^2 \leq \|x_i - x_0\|^2 - \|x_n - x_0\|^2.$$

Letting $i, n \to \infty$, we get from the existence of $\lim_{n\to\infty} \|x_0 - x_n\|$ that $x_i - x_n \to 0$ as $i, n \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence. This completes the proof of Step 4.

Step 5. Show that $x_n \to q \in \mathcal{F}$ as $n \to \infty$.

Since H is a Hilbert space and C is closed convex, we may assume that

$$x_n \to q \in C \quad \text{as } n \to \infty.$$

In view of $x_{n+1} = P_{C_{n+1}} x_0$, we see from (2.1) that

$$\left\|x_n - T_m^n x_n\right\| \to 0, \quad \forall m \geq 1 \text{ as } n \to \infty.$$

Since T_m is Lipschitz continuous, we find that $q \in F(T_m)$ for each $m \geq 1$, which implies that $q \in \mathcal{F}$. This completes the proof of Step 5.

Step 6. Show that $q = P_{\mathcal{F}} x_0$.

In view of Lemma 1.1, we obtain from $x_n = P_{C_n} x_0$ that

$$\langle x_0 - x_n, x_n - z\rangle \geq 0, \quad \forall z \in \mathcal{F} \subset C_n.$$

Hence, we have

$$\langle x_0 - q, q - z\rangle \geq 0, \quad \forall z \in \mathcal{F}.$$

From Lemma 1.1, we see that $q = P_{\mathcal{F}} x_0$, which completes the proof of Step 6. This proof is completed.

For a single asymptotically hemi-pseudocontractive mapping in the intermediate sense, we can obtain from Theorem 2.1 the following.

Corollary 2.2. *Let C be a closed convex bounded subset of a real Hilbert space H. Let $T : C \to C$ be a mapping which is uniformly L-Lipschitz and asymptotically hemi-pseudocontractive in the intermediate sense with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in H \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1, \\ C_{n+1} = \big\{z \in C_n : \alpha_n\big(1-(1+L)\alpha_n\big)\|p - T^n p\|^2 \\ \qquad\qquad \leq \langle x_n - z, y_n - T^n y_n\rangle + (k_n - 1)(diam\, C)^2 + \nu_n\big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where

$$\nu_n = \max\left\{0, \sup_{p\in F(T)} \big(\langle p - T^n y_n, p - y_n\rangle - k_n\|p - y_n\|^2\big)\right\}, \quad \forall p \in F(T).$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n \in [a, b]$ for some $a, b \in \big(0, \frac{1}{1+L}\big)$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where P is the metric projection from H onto $F(T)$.

Remark 2.3. For mappings, Corollary 2.2 extends Theorem SXY from asymptotically strict pseudocontractions in the intermediate sense to asymptotically hemi-pseudocontractive mappings in the intermediate sense. For iterative algorithms, the set Q_n is removed.

For a single asymptotically hemi-pseudocontractive mapping, we have the following.

Corollary 2.4. *Let C be a closed convex bounded subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschitz and asymptotically hemi-pseudocontractive mapping with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in H \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1, \\ C_{n+1} = \big\{z \in C_n : \alpha_n\big(1 - (1+L)\alpha_n\big)\|p - T^n p\|^2 \\ \qquad\qquad \leq \langle x_n - z, y_n - T^n y_n\rangle + (k_n - 1)(diam\, C)^2\big\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 0. \end{cases}$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n \in [a, b]$ for some $a, b \in (0, \frac{1}{1+L})$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where P is the metric projection from H onto $F(T)$.

Remark 2.5. For mappings, Corollary 2.4 extends Theorem Z from asymptotically pseudocontractive mappings to asymptotically hemi-pseudocontractive mappings. For iterative algorithms, the set Q_n is removed.

Remark 2.6. We do not know whether the restriction that the subset C is bounded can be remove. It is of interest to extend the results presented in this paper to a Banach space.

References

[1] G. Lopez Acedo, H.K. Xu, *Iterative methods for strict pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **67**(2007), 2258-2271.

[2] R.P. Agarwal, Y.J. Cho, X. Qin, *Generalized projection algorithms for nonlinear operators*, Numer. Funct. Anal. Optim., **28**(2007), 1197-1215.

[3] F.E. Browder, W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20**(1967), 197-228.

[4] R.E. Bruck, T. Kuczumow, S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math., **65**(1993), 169-179.
[5] W. Cholamjiak, S. Suantai, *Convergence theorems from monotone hybrid methods for an infinitely countable family of Lipschitz asymptotically quasi-nonexpansive mappings*, Nonlinear Anal. Hybrid Syst., **4**(2010), 524-530.
[6] I. Inchan, S. Plubtieng, *Strong convergence theorems of hybrid methods for two asymptotically nonexpansive mappings in Hilbert spaces*, Nonlinear Anal., **2**(2008), 1125-1135.
[7] A. Genel, J. Lindenstrass, *An example concerning fixed points*, Israel J. Math., **22**(1975), 81-86.
[8] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972), 171-174.
[9] Y. Kimura, W. Takahashi, *On a hybrid method for a family of relatively nonexpansive mappings in a Banach space*, J. Math. Anal. Appl., **357**(2009), 356-363.
[10] T.H. Kim, H.K. Xu, *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal., **64**(2006), 1140-1152.
[11] T.H. Kim, H.K. Xu, *Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions*, Nonlinear Anal., **68**(2008), 2828-2836.
[12] W.A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math., **17**(1974), 339-346.
[13] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4**(1953), 506-510.
[14] G. Marino, H.K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl., **329**(2007), 336-346.
[15] C. Martinez-Yanes, H.K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal., **64**(2006), 2400-2411.
[16] S.Y. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory, **134**(2005), 257-266.
[17] K. Nakajo, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279**(2003), 372-379.
[18] S. Plubtieng, K. Ungchittrakool, *Strong convergence of modified Ishikawa iteration for two asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal., **67**(2007), 2306-2315.
[19] L. Qihou, *Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings*, Nonlinear Anal., **26**(1996), 1835-1842.
[20] X. Qin, S.Y. Cho, H. Zhou, *Strong convergence theorems of fixed point for quasi-pseudo-contractions by hybrid projection algorithms*, Fixed Point Theory, **11**(2010), 347-354.
[21] X. Qin, S.Y. Cho, J.K. Kim, *Convergence results on asymptotically pseudocontractive mappings in the intermediate sense*, Fixed Point Theory Appl., **2010**(2010), Article ID 186874.
[22] X. Qin, Y.J. Cho, S.M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces*, J. Comput. Appl. Math., **225**(2009), 20-30.
[23] X. Qin, Y.J. Cho, S.M. Kang, M. Shang, *A hybrid iterative scheme for asymptotically k-strict pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **70**(2009), 1902-1911.
[24] X. Qin, Y.J. Cho, S.M. Kang, H. Zhou, *Convergence theorems of common fixed points for a family of Lipschitz quasi-pseudocontractions*, Nonlinear Anal., **71**(2009), 685-690.
[25] X. Qin, Y. Su, *Strong convergence theorems for relatively nonexpansive mappings in a Banach space*, Nonlinear Anal., **67**(2007), 1958-1965.
[26] X. Qin, Y. Su, M. Shang, *Strong convergence theorems for asymptotically nonexpansive mappings by hybrid methods*, Kyungpook Math. J., **48**(2008), 133-142.
[27] B.E. Rhoades, *Comments on two fixed point iteration methods*, J. Math. Anal. Appl., **53**(1976), 741-750.
[28] D.R. Sahu, H.K. Xu, J.C. Yao, *Asymptotically strict pseudocontractive mappings in the intermediate sense*, Nonlinear Anal., **70**(2009), 3502-3511.
[29] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **158**(1991), 407–413.
[30] W. Takahashi, Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **341**(2008), 276-286.

[31] C.M. Yanes, H.K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal., **64**(2006), 2400-2411.
[32] H. Zegeye, N. Shahzad, *Strong convergence theorems for a finite family of nonexpansive mappings and semigroups via the hybrid method*, Nonlinear Anal., **72**(2010), 325-329.
[33] H. Zhou, *Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl., **343**(2008), 546-556.
[34] H. Zhou, *Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **70**(2009), 3140-3145.

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