***Fixed Point Theory***, 13(2012), No. 1, 147-151
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

# ON DISCUS SPACES

LECH PASICKI

Faculty of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30, 30-059 Kraków, Poland
E-mail: pasicki@agh.edu.pl

**Abstract.** The paper contains some fixed point theorems (Theorem 12, Theorem 13, Theorem 15) and a selection theorem (Theorem 11). Theorem 13 seems to be one of the main results for non-expansive mappings and it is a far extension of Browder-Göhde-Kirk result even for uniformly convex spaces. What is more its proof is simple and natural (cp. the monograph of Dugundji and Granas [1, p. 52]); the shortest way to the proof of this theorem is directly by Definition 7. In addition we give a more thorough investigation of the properties of discus spaces (extension of uniformly convex spaces) which seem to be of importance.
**Key Words and Phrases**: Discus space, multivalued mapping, fixed point, selection, non-expansive mapping.
**2010 Mathematics Subject Classification**: 54H25, 54E35, 54C65, 47H10.

Let us recall two definitions.

**Definition 1** ([2, Def. 1]). *A metric space* $(X,d)$ *is a discus space if there exists a mapping* $\rho : [0,\infty) \times (0,\infty) \to [0,\infty)$ *such that*

$$\rho(\beta, r) < \rho(0, r) = r,\ \beta, r > 0, \tag{1}$$

$$\rho(\cdot, r) \text{ is nonincreasing, } r > 0, \tag{2}$$

$$\rho(\delta, \cdot) \text{ is upper semicontinuous, } \delta \geq 0, \tag{3}$$

$$\text{for each } x, y \in X,\ r, \epsilon > 0 \text{ there exists a } z \in X \text{ such that } B(x,r) \cap B(y,r) \subset B(z, \rho(d(x,y), r) + \epsilon). \tag{4}$$

**Definition 2** ([2, Def. 5]). *Let* $(X,d)$ *be a metric space and* $A$ *a nonempty subset of* $X$. *An* $x \in X$ *is a central point for* $A$ *if*

$$r(A) := inf\{t \in (0,\infty] : \text{ there exists a } z \in X \text{ with } A \subset B(z,t)\} = inf\{t \in (0,\infty] : A \subset B(x,t)\}. \tag{5}$$

*The centre* $c(A)$ *for* $A$ *is the set of all central points for* $A$, *and* $r(A)$ *is the radius of* $A$.

The lemma to follow extends [2, Lemma 6].

**Lemma 3.** *Let $(X,d)$ be a discus space and let $A \subset X$ be nonempty and bounded. Then $c(A)$ consists of at most one point. If in addition $(X,d)$ is complete then $c(A)$ is a singleton.*

*Proof.* Let $\epsilon > 0$ be arbitrary and Let $(r_n)_{n\in N}$ decrease to $r = r(A)$ while $A \subset B(x_n, r_n)$. Suppose $(x_n)_{n\in N}$ is not a Cauchy sequence, i.e. $d(x_n, x_k) \geq \beta > 0$ for infinitely many $k < n$. We have

$$\begin{aligned} A &\subset B(x_n, r_n) \cap B(x_k, r_k) \subset B(x_n, r_k) \cap B(x_k, r_k) \\ &\subset B(z_{n,k}, \rho(d(x_n, x_k), r_k) + \epsilon) \subset B(z_{n,k}, \rho(\beta, r_k) + \epsilon) \end{aligned}$$

(see (4),(2)). Set $2\gamma = r - \rho(\beta, r) = \rho(0, r) - \rho(\beta, r) > 0$ (see (1)). We have $\rho(\beta, r_k) + \epsilon \leq \rho(\beta, r) + \gamma$ for sufficiently large $k$ and small $\epsilon$ (see (3)). Now we obtain $\rho(\beta, r_k) + \epsilon \leq \rho(\beta, r) + \gamma = r - 2\gamma + \gamma = r - \gamma$ and consequently

$$A \subset B(z_{n,k}, \rho(\beta, r_k) + \epsilon) \subset B(z_{n,k}, r - \gamma))$$

which means $r = r(A) \leq r(A) - \gamma$, a contradiction. Therefore $(x_n)_{n\in N}$ is a Cauchy sequence. If $(X,d)$ is complete then $(x_n)_{n\in N}$ converges, say to $x$. Then for any $\beta > 0$ we have $B(x_n, r_n) \subset B(x, r + \beta)$ for all sufficiently large $n$, which means $A \subset B(x, r + \beta)$ for all $\beta > 0$ and consequently $x \in c(A)$. Suppose $x, y \in c(A)$ and $d(x,y) \geq \beta > 0$. Then by (4) for $\gamma$ defined above we obtain

$$A \subset \overline{B}(x, r) \cap \overline{B}(y, r) \subset \overline{B}(z, \rho(\beta, r) + \epsilon) \subset \overline{B}(z, r - \gamma)$$

for a $\gamma > 0$, a contradiction. □

For complete spaces condition (4) is too general

**Lemma 4** ([2, Lemma 4])**.** *If $(X,d)$ is a complete discus space then (4) can be replaced by*

$$\begin{aligned} &\text{for each } x, y \in X \text{ and } r > 0 \text{ there exists a } z \in X \\ &\text{such that } B(x, r) \cap B(y, r) \subset B(z, \rho(d(x,y), r)). \end{aligned} \tag{6}$$

**Definition 5.** *Let $(X,d)$ be a metric space and $\mathcal{A} = \{A_n : n \in N\}$ a family of nonempty subsets of $X$. An $x \in X$ is a central point for $\mathcal{A}$ if*

$$\begin{aligned} r(\mathcal{A}) := inf\{t \in (0,\infty] :\ &\text{there exists } n_0 \text{ such that for each } n > n_0 \\ &\text{there is a } z \in X \text{with } A_n \subset B(z,t)\} = inf\{t \in (0,\infty] : \\ &\text{there exists } n_0 \text{ such that } A_n \subset B(x,t) \text{ for each } n > n_0\}. \end{aligned} \tag{7}$$

*The centre $c(\mathcal{A})$ for $\mathcal{A}$ is the set of all central points for $\mathcal{A}$, and $r(\mathcal{A})$ is the radius of $\mathcal{A}$.*

**Lemma 6.** *Let $(X,d)$ be a discus space and let $\mathcal{A} = \{A_n : n \in N\}$ be a decreasing family of nonempty and bounded subsets of $X$. Then $c(\mathcal{A})$ consists of at most one point. If $\{x_n\} = c(A_n), n \in N$ then $(x_n)_{n\in N}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = x$ means $\{x\} = c(\mathcal{A})$. In particular if $(X,d)$ is a complete discus space then $c(\mathcal{A})$ is a singleton. If $\mathcal{A}$ consists of compact sets then for $A = \bigcap \mathcal{A}$ $c(A) = c(\mathcal{A})$, $r(A) = r(\mathcal{A})$ hold.*

*Proof.* Set $r = r(\mathcal{A})$. We have $A_{n+1} \subset A_n$ and therefore there exist a decreasing sequence $(r_n)_{n\in N}$ convergent to $r$ and a sequence $(x_n)_{n\in N}$ such that $A_n \subset B(x_n, r_n)$ for all $n \in N$ (in particular $\{x_n\} = c(A_n)$ if nonempty). Suppose $(x_n)_{n\in N}$ is not a Cauchy sequence, i.e. $d(x_n, x_k) \geq \beta > 0$ for infinitely many $k < n$. We have

$$A_n \subset A_n \cap A_k \subset B(x_n, r_k) \cap B(x_k, r_k) \subset B(z_{n,k}, \rho(\beta, r_k) + \epsilon)$$

for $k < n$ and consequently $A_n \subset B(z_{n,k}, r - \gamma)$ for a $\gamma > 0$ (see the proof of Lemma 3), a contradiction. Now let $(x_n)_{n\in N}$ converge to $x$. We obtain $A_n \subset B(x, r + \beta)$ for any $\beta > 0$ and sufficiently large $n$. Consequently, $x \in c(\mathcal{A})$. The uniqueness of $x \in c(\mathcal{A})$ can be obtained as in the proof of Lemma 3. We obviously have $r(A) \leq r(\mathcal{A})$, and on the other hand for $x \in c(A)$ we obtain $A_n \subset B(x, r(A) + \beta)$ for any $\beta > 0$ and large $n \in N$, $A_n$ being compact. Thus $r(A) = r(\mathcal{A})$ holds ( $c(A) = c(\mathcal{A})$ is trivial). □

Now we are going to present a lemma which concerns mappings.

Let $2^X$ be the family of all subsets of $X$ and let $F\colon X \to 2^X$ being a multivalued mapping mean that $F(x) \neq \emptyset, x \in X$.

The following is equivalent to [2, Def. 7] as for $F\colon Y \to 2^Y$ we have $F^n(Y) \subset (F^{n_0})(Y)$ for all $n > n_0$

**Definition 7.** *Let $(X, d)$ be a metric space, $\emptyset \neq Y \subset X$ and $F\colon Y \to 2^Y$ a mapping. An $x \in X$ is a central point for $F$ if*

$$\begin{aligned} r(F) := inf\{t \in (0, \infty] : F^n(Y) \subset B(z, t) \text{ for a } z \in X \text{ and} \\ a\ n \in N\} = inf\{t \in (0, \infty] : F^n(Y) \subset B(x, t) \text{ for a } n \in N\}. \end{aligned} \tag{8}$$

*The centre $c(F)$ for $F$ is the set of all central points for $F$, and $r(F)$ is the radius of $F$.*

From Lemma 6 we obtain the following extension of [2, Lemma 8]

**Lemma 8.** *Let $(X, d)$ be a discus space. If $\emptyset \neq Y \subset X$ is bounded and $F\colon Y \to 2^Y$ is a mapping then $c(F)$ consists of at most one point. If $c(F^n(Y)) = \{x_n\}, n \in N$ then $(x_n)_{n\in N}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = x$ means $c(F) = \{x\}$. In particular if $(X, d)$ is a complete discus space then $c(F)$ is a singleton.*

*Proof.* We apply Lemma 6 to $A_n = F^n(Y)$. □

Now we present an analog of Lemma 6 for the Hausdorff distance $D$.

**Lemma 9.** *Let $(X, d)$ be a discus space and let $\mathcal{A} = \{A_n : n \in N\}$ be a family of nonempty and bounded subsets of $X$ such that $\lim_{m,n\to\infty} D(A_m, A_n) = 0$. Then $c(\mathcal{A})$ consists of at most one point. If $\{x_n\} = c(A_n), n \in N$ then $(x_n)_{n\in N}$ is a Cauchy sequence, and $\lim_{n\to\infty} x_n = x$ means $\{x\} = c(\mathcal{A})$. In particular if $(X, d)$ is a complete discus space then $c(\mathcal{A})$ is a singleton. If $A = \lim_{n\to\infty} A_n$ in $(2^X, D)$ then $c(A) = c(\mathcal{A})$, $r(A) = r(\mathcal{A})$ hold.*

*Proof.* Let us consider $C_n = \bigcup_{k=n}^{\infty} A_k$. Clearly $\mathcal{C} = \{C_n : n \in N\}$ is a decreasing family of nonempty and bounded subsets of $X$. In view of Lemma 6 $c(\mathcal{C})$ consists of at most one point. On the other hand we have $c(\mathcal{C}) = c(\mathcal{A})$ and $r(\mathcal{C}) = r(\mathcal{A})$

(see Definition 5). Clearly one can use $\{x_n\} = c(A_n)$ in place of $\{z_n\} = c(C_n)$ for the respectively defined $(r_n)_{n\in N}$. If $\{x\} = c(A)$ and $r = r(A)$ then we have $A_n \subset B(x, r+\beta)$, $n > n_0$ and for $x_n, r_n$ as above we get $A \subset B(x_n, r_n+\beta)$, $n > n_0$ which imply $c(A) = c(\mathcal{A})$ and $r(A) = r(\mathcal{A})$. □

In what follows if $\{x\} = c(F(z))$ then we adopt $(c \circ F)(z) = x$ .

**Lemma 10.** *Let $(Z,\rho)$ be a metric space and $(X,d)$ a discus space. If $F\colon (Z,\rho) \ni z \to F(z) \in (2^X, D)$ is a continuous mapping, $F(z), z \in Z$ are bounded and $c(F(z)) \neq \emptyset$, $z \in Z$ (e.g. if $(X,d)$ is complete) then $c \circ F\colon Z \to X$ is continuous.*

*Proof.* By continuity of $F$ from $\lim_{n\to\infty} z_n = z$ follows $\lim_{n\to\infty} F(z_n) = F(z)$ in $(2^X, D)$ and then by Lemma 9 we have $\lim_{n\to\infty}(c \circ F)(z_n) = (c \circ F)(z)$ which means the continuity of $c \circ F$. □

As a corollary from the previous lemma we obtain

**Theorem 11.** *Let $(Z,\rho)$ be a metric space and $(X,d)$ a discus space.*
*If $F\colon (Z,\rho) \ni z \to F(z) \in (2^X, D)$ is a continuous mapping, $F(z), z \in Z$ are bounded and $\emptyset \neq c(F(z)) \subset F(z)$, $z \in Z$ then $c \circ F$ is a continuous selection for $F$.*

Another consequence of Lemma 10 is the following

**Theorem 12.** *Let $X$ be a nonempty convex set in a discus normed space $(Y, \|\cdot\|)$. If $F\colon X \ni x \to F(x) \in (2^Y, D)$ is a continuous mapping, $\emptyset \neq c(F(x)) \subset F(x)$, $x \in X$ and $\overline{\{(c \circ F)(x) : x \in X\}} \subset X$ is compact then $F$ has a fixed point.*

*Proof.* In view of Theorem 11 $c \circ F$ is a continuous selection for $F$. Consequently $c \circ F\colon X \to X$ is a compact map and by Schauder theorem it has a fixed point. □

In view of Lemma 8 the following theorem is an extension of [2, Th. 11] as for any complete discus space $(X,d)$ and its bounded nonempty subset $Y$ the set $c(f_{|Y})$ is a singleton. On the other hand our result extends the well known theorem of Browder-Göhde-Kirk for Hilbert spaces [1, Th. (1.3), p. 52] and for uniformly convex spaces ([1, (C.1) (b), p. 76]). In addition we do not demand the space to be complete.

**Theorem 13** (cp. [2, Th. 11]). *Let $(X,d)$ be a metric space and let $f\colon X \to X$ be a mapping. Assume that $\emptyset \neq Y \subset X$ is such that $f_{|Y}\colon Y \to Y$ and $c(f_{|Y}) = \{x\}$ (a singleton). If the following*

$$d(f(x), f(y)) \leq d(x,y) \text{ for all } y \in Y \tag{9}$$

*holds then $x$ is a fixed point for $f$.*

*Proof.* We have $f(Y) \subset Y$ and $Y$ is bounded (otherwise $c(f_{|Y})$ would not be a singleton). If $f^{n-1}(Y) \subset B(x,t)$ then $f^n(Y) \subset f(Y \cap B(x,t))$ holds. For $d(x,y) < t$ we obtain $d(f(x), f(y)) \leq d(x,y) < t$ (see (9)), which means $f(y) \in B(f(x), t)$ and consequently $f^n(Y) \subset f(Y \cap B(x,t)) \subset B(f(x), t)$, which implies $f(x) \in c(f_{|Y})$ (see Definition 7). Now it is clear that $f(x) = x$ as both belong to $c(f_{|Y})$ which is a singleton. □

The previous theorem is in fact a method of proving fixed point theorems (also for the case of non-expansive mappings where condition (9) is satisfied for $Y = X$ and all $x \in X$). It is sufficient to investigate the properties of $(X, d)$ and of $Y$ under which $c(f_{|Y})$ is a singleton.

One can note that Theorem 13 is a particular case of the following general statement (details are the problem).

**Observation 14.** *If $x$ is the only point satisfying condition (W) and $f(x)$ satisfies (W) then we have $f(x) = x$.*

**Theorem 15.** *Let $(X, d)$ be a metric space and let $F\colon X \to 2^X$ be a mapping with $c(F(x)) \subset F(x)$, $x \in X$. Assume that $\emptyset \neq Y \subset X$ is such that for $f = c \circ F$ we have $f_{|Y}\colon Y \to Y$ (e.g. if $F_{|Y}\colon Y \to 2^Y$ ) and $c(f_{|Y}) = \{x\}$ (a singleton). If condition (9) is satisfied then $x$ is a fixed point for $F$.*

*Proof.* In view of Theorem 13 the element $x$ is a fixed point for $f$. We have $x = f(x) \in c(F(x)) \subset F(x)$. □

**Remark 16.** *Clearly for any set $A$ in a discus space we have $c(A) = c(\overline{A})$, $r(A) = r(\overline{A})$. For any symmetric bounded set $A$ in a normed discus space we have $c(A) \subset A$. The same holds for any bounded complete and convex set $A$ in a normed discus space whenever the sections of balls and hyperplanes are symmetric (e.g. in unitary space).*

**Problem 17.** *Let $A$ be a bounded complete convex set in a discus normed space. Prove that $c(A) \subset A$ .*

## References

[1] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, 2003.
[2] L. Pasicki, *A basic fixed point theorem*, Bull. Pol. Ac. Math., **54**(2006), 85-88.

*Received: November 24, 2010; Accepted: December 15, 2010.*