HYBRID-EXTRAGRADIENT TYPE METHODS
FOR A GENERALIZED EQUILIBRIUM PROBLEM
AND VARIATIONAL INEQUALITY PROBLEMS
OF NONEXPANSIVE SEMIGROUPS

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Abstract. We study and introduce modified mann iterative algorithms for finding a common ele-
ment of the set of solutions of a generalized equilibrium problem, the set of solutions of variational
inequalities and the set of fixed points for nonexpansive semigroups. Then, we prove strong con-
vergence theorems in a real Hilbert space by using the hybrid-extragradient type methods in the
mathematical programming under some appropriate control conditions.

Key Words and Phrases: Generalized equilibrium problem, variational inequalities, Strong con-
vergence, Nonexpansive, Semigroup, Hilbert space, Extragradient method, Hybrid method.

2010 Mathematics Subject Classification: 46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner
product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $G : C \rightarrow H$ be a nonlinear mapping. In 2008
Takahashi and Takahashi [21] and Peng and Yao [14, 15] considered the following
generalized equilibrium problem: Find $x \in C$ such that

$$F(x, y) + \langle Gx, y - x \rangle \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GEP(F, G)$. In the case of $G = 0$, then the problem (1.1) becomes the following equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.2}$$
The set of solutions of (1.2) is denoted by $EP(F)$. If $F = 0$ for all $x, y \in C$, then the problem (1.1) becomes the following variational inequality problem is to find $x \in C$ such that

$$\langle Gx, y - x \rangle \geq 0 \text{ for all } y \in C.$$  \hspace{1cm} (1.3)

The set of solutions of (1.3) is denoted by $VI(C,G)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others; see for instance [2, 3, 6, 11, 21]. Recently, many authors considered the problem of finding a common element of the set of solutions to the equilibrium problem (1.2) and variational inequality problem (1.3) and of the set of fixed points of nonexpansive mapping in Hilbert spaces; see, for example, [2, 16, 11, 12, 14, 15, 21] and the references therein.

Recall that $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of fixed points of $T$ by $F(T)$, that is $F(T) = \{x \in C : x = Tx\}$. A family $T = \{T(t) : t \geq 0\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s + t)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C, s \mapsto T(s)x$ is continuous.

We denote by $F(T)$ the set of all common fixed points of $T$, that is,

$$F(T) = \bigcap_{t=0}^{\infty} F(T(t)) = \{x \in C : T(t)x = x, \ 0 \leq t < \infty \}.$$  \hspace{1cm} (1.5)

It is know that $F(T)$ is closed and convex.

In 1953, Mann [10] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$$  \hspace{1cm} (1.4)

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [17]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [8]. Attempts to modify the Mann iteration method (1.4) so that strong convergence is guaranteed have recently been made. Generally speaking, the algorithm suggested by Takahashi and Toyoda [22] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of “hybrid” or “outer-approximation” types of methods was originally introduced by Haugazeau in 1968; see [1] for more details.

In 2002, Suzuki [19] was the first one to introduced the following implicit iteration process in Hilbert spaces:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)(x_n), \ n \geq 1,  \hspace{1cm} (1.5)$$

for the nonexpansive semigroup. In 2007, Xu [24] established a Banach space version of the sequence (1.5) of Suzuki [19]. In [4], Chen and He considered the viscosity
approximation process for a nonexpansive semigroup and proved another strong convergence theorem for a nonexpansive semigroup in Banach spaces, which is defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \quad (1.6)$$

where, $f : C \to C$ be a fixed contractive mapping. Korpelevich [9] introduced the following so-called extragradient method also:

$$\begin{align*}
x_0 &= x \in C, \\
y_n &= P_C(x_n - \lambda Ax_n), \\
x_{n+1} &= P_C(x_n - \lambda Ay_n),
\end{align*} \quad (1.7)$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k})$, $C$ is a closed convex subset of $\mathbb{R}^n$ and $A$ is a monotone and $k$-Lipschitz continuous mapping of $C$ into $\mathbb{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.7), converge to the same point $z \in VI(C, A)$.

In 2008, Saejung [18] proved the strong convergence theorems for nonexpansive semigroups without Bochner integrals in Hilbert spaces. The sequence $\{x_n\}$ defined by

$$\begin{align*}
y_n &= \alpha_n x_n + (1 - \alpha_n)T(t_n)x_n, \\
C_{n+1} &= \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} &= P_{C_{n+1}}x_n, \quad \forall n \geq 0,
\end{align*} \quad (1.8)$$

and

$$\begin{align*}
y_n &= \alpha_n x_n + (1 - \alpha_n)T(t_n)x_n, \\
C_n &= \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n}x_n, \quad \forall n \geq 0,
\end{align*} \quad (1.9)$$

where $\{t_n\}$ is a real sequence, $\{\alpha_n\} \subset [0, 1)$ and $\{T(t) : t \geq 0\}$ is a nonexpansive semigroup on $C$.

In the same year, Takahashi and Takahashi [21] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. The sequence $\{x_n\}$ defined by: $u, x_1 \in C$ and

$$\begin{align*}
F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)u_n],
\end{align*} \quad (1.10)$$

for all $n \geq 0$. Where $F$ is a bifunction from $C \times C$ into $\mathbb{R}$, $A : C \to H$ are an inverse-strongly monotone mapping and $S$ is a nonexpansive mapping of $C$ into itself. They proved some strong convergence theorems under suitable conditions.

In this paper, we prove the strong convergence theorems of modified Mann iterative algorithms for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of two variational inequalities and the set of solutions of nonexpansive semigroups in a Hilbert space under some appropriate control conditions by using the new hybrid-extragradient methods in the mathematical programming. The results presented in this paper extend and improve the corresponding ones announced by Saejung [18], Takahashi and Takahashi [21] and many others.
2. Preliminaries

Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$ and let $C$ be a closed convex subset of $H$. Then

$$\| x - y \|^2 = \| x \|^2 - 2 \langle x, y \rangle$$  \hspace{1cm} (2.1)

and

$$\| \lambda x + (1 - \lambda) y \|^2 = \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda (1 - \lambda) \| x - y \|^2$$  \hspace{1cm} (2.2)

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

A space $X$ is said to satisfy Opial’s condition [13], if for each sequence $\{x_n\}$ in $X$ which converges weakly to a point $x \in X$, we have

$$\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \|, \ \forall y \in X, \ y \neq x.$$  

Recall that, for every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\| x - P_C x \| \leq \| x - y \| \quad \text{for all } y \in C.$$  

$P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \| P_C x - P_C y \|^2$$  \hspace{1cm} (2.3)

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0,$$  \hspace{1cm} (2.4)

$$\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2$$  \hspace{1cm} (2.5)

for all $x \in H, y \in C$.

Hilbert space $H$ satisfies the Kadec-Klee property [7, 20], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\| x_n \| \to \| x \|$ together imply $\| x_n - x \| \to 0$.

For solving the equilibrium problem, let us give the following assumptions for the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) $F$ is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \to 0} F(t z + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We need the following lemmas for proving our main results.

Lemma 2.1. ([Blum and Oettli [3]]) Let $C$ be a nonempty closed convex subset of $H$ and let $F$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfies (A1)-(A4). Let $r > 0$ and $z \in H$. Then, there exists $x \in C$ such that

$$F(x, y) + \frac{1}{r} \langle y - x, x - z \rangle \geq 0, \ \forall y \in C.$$  \hspace{1cm} (2.6)

Lemma 2.2. ([Combettes and Hirstoaga [5]]) Let $C$ be a nonempty closed convex subset of $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfies (A1)-(A4). For $r > 0$ and $z \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(z) = \{ x \in C : F(x, y) + \frac{1}{r} \langle y - x, x - z \rangle \geq 0, \ \forall y \in C \}$$
for all \( z \in H \). Then, the following hold:

1. \( T_r \) is single-valued;
2. \( T_r \) is firmly nonexpansive, i.e., for any \( x, y \in H \),
   \[ \|T_r x - T_r y\|^2 \leq (T_r x - T_r y, x - y); \]
3. \( F(T_r) = EP(F) \);
4. \( EP(F) \) is closed and convex.

**Remark 2.3.** Replacing \( z \) with \( z - rGz \in H \) in (2.6), then there exists \( x \in C \), such that
\[ F(x, y) + \langle Gz, y - x \rangle + \frac{1}{r}(y - x, x - z) \geq 0, \forall y \in C. \]

3. Main results

In this section, we prove strong convergence theorems for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of two variational inequalities and the set of fixed points for a nonexpansive semigroup in a real Hilbert space.

3.1. The hybrid method.

**Theorem 3.1.** Let \( C \) be a nonempty bounded closed convex subset of a real Hilbert space \( H \). Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on \( C \), let \( F \) be a bifunction of \( C \times C \) into real numbers \( \mathbb{R} \) satisfying (A1) – (A4) and let \( G, A, B : C \rightarrow H \) be three \( \alpha, \beta, \lambda \)-inverse-strongly monotone mappings, respectively. Suppose that \( \Omega := (\cap_{t>0} F(S(t))) \cap VI(C, A) \cap VI(C, B) \cap EP(F, G) \neq \emptyset \). Let \( \{\alpha_n\} \subset [0, a) \subset [0, 1), \{\beta_n\} \subset [0, b] \subset (0, 2\beta), \{\lambda_n\} \subset [0, l] \subset (0, 2\lambda), \{r_n\} \subset [0, r'] \subset (0, 2a) \) and \( \{t_n\} \subset [0, \infty) \) satisfying \( \lim \inf t_n = 0, \lim \sup t_n > 0, \) and \( \lim \inf (t_{n+1} - t_n) = 0 \). For \( x_0 \in H \), let the sequences \( \{x_n\}, \{u_n\}, \{v_n\}, \{y_n\} \) and \( \{z_n\} \) be generated by \( u_n \in C \) and
\[
\begin{cases}
F(u_n, y) + \langle Gx_n, y - u_n \rangle + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in C, \\
v_n = P_C(u_n - \lambda_n Bu_n), \\
z_n = P_C(v_n - \beta_n Av_n), \\
y_n = \alpha_n u_n + (1 - \alpha_n) S(t_n) z_n, \\
C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \forall n \geq 0.
\end{cases}
\]

Then the sequence \( \{x_n\} \) converges strongly to \( P_{\Omega} x_0 \).

**Proof.** It is obvious that \( C_n \) and \( Q_n \) are closed and convex for all \( n \geq 0 \). Thus that \( C_n \cap Q_n \) is closed and convex for all \( n \geq 0 \). Let \( x^* \in \Omega \) and \( \{T_{r_n}\} \) be a sequence of mappings defined as in Lemma 2.2 then, \( x^* = T_{r_n}(x^* - r_n Gx^*) = P_C(x^* - \beta_n Ax^*) = P_C(x^* - \lambda_n Bx^*) \) and \( u_n = T_{r_n}(x_n - r_n Gx_n) \in C \). Note that \( I - r_n G \) is nonexpansive for all \( n \geq 0 \), for all \( u, v \in C \) and \( \{r_n\} \subset (0, 2a) \), we have
\[
\begin{align*}
\| (I - r_n G) u - (I - r_n G) v \|^2 &= \| (u - v) - r_n (G u - G v) \|^2 \\
&= \| u - v \|^2 - 2 r_n \langle u - v, G u - G v \rangle + r_n^2 \| G u - G v \|^2 \\
&\leq \| u - v \|^2 + r_n (r_n - 2a) \| G u - G v \|^2 \leq \| u - v \|^2.
\end{align*}
\]
By the same method, we obtain that
\[ \| (I - \beta_n A) u - (I - \beta_n A) v \| \leq \| u - v \| \]
and
\[ \| (I - \lambda_n B) u - (I - \lambda_n B) v \| \leq \| u - v \|. \]
We note that
\[ \| u_n - x^* \| = \| T_{r_n}(x_n - r_n G x_n) - T_{r_n}(x^* - r_n G x^*) \| \]
\[ \leq \| (x_n - r_n G x_n) - (x^* - r_n G x^*) \| \]
\[ \leq \| x_n - x^* \| \] (3.3)
and
\[ \| v_n - x^* \| = \| P_C(u_n - \lambda_n B u_n) - P_C(x^* - \lambda_n B x^*) \| \]
\[ \leq \| (u_n - \lambda_n B u_n) - (x^* - \lambda_n B x^*) \| \]
\[ \leq \| u_n - x^* \| \]
\[ \leq \| x_n - x^* \| \] (3.4)
hence
\[ \| z_n - x^* \| = \| P_C(v_n - \beta_n A v_n) - P_C(x^* - \beta_n A x^*) \| \]
\[ \leq \| (v_n - \beta_n A v_n) - (x^* - \beta_n A x^*) \| \]
\[ \leq \| v_n - x^* \| \]
\[ \leq \| x_n - x^* \|. \] (3.5)
It follows by (3.3), we obtain
\[ \| y_n - x^* \| = \| \alpha_n u_n + (1 - \alpha_n) S(t_n) z_n - x^* \| \]
\[ \leq \alpha_n \| u_n - x^* \| + (1 - \alpha_n) \| S(t_n) z_n - x^* \| \]
\[ \leq \alpha_n \| u_n - x^* \| + (1 - \alpha_n) \| z_n - x^* \| \]
\[ \leq \alpha_n \| u_n - x^* \| + (1 - \alpha_n) \| u_n - x^* \| \]
\[ = \| u_n - x^* \| \]
\[ \leq \| x_n - x^* \|. \] (3.6)
Therefore, \( \Omega \subset C_n \) for all \( n \geq 0 \).
By induction, we show that \( \Omega \subset C_n \cap Q_n \) for all \( n \geq 0 \). Form \( x_1 = P_C x_0 \), we have
\[ \langle x_1 - y, x_0 - x_1 \rangle \geq 0 \] for all \( y \in C \),
and hence \( Q_1 = C \). So, we have \( \Omega \subset Q_1 \). Then, \( \Omega \subset C_1 \cap Q_1 \). Suppose that \( \Omega \subset C_k \cap Q_k \) for some \( k \geq 0 \). From \( x_{k+1} = P_{C_k \cap Q_k} x_0 \), we have
\[ \langle x_{k+1} - y, x_0 - x_{k+1} \rangle \geq 0 \] for all \( y \in C_k \cap Q_k \).
Since \( \Omega \subset C_k \cap Q_k \), we have
\[ \langle x_{k+1} - u, x_0 - x_{k+1} \rangle \geq 0 \] for all \( u \in \Omega \),
and hence \( \Omega \subset Q_{k+1} \). Since \( \Omega \subset C_n \) for all \( n \geq 0 \), we have \( \Omega \subset C_{k+1} \cap Q_{k+1} \). So, we have that \( \Omega \subset C_n \cap Q_n \) for all \( n \geq 0 \). Then, \( \{ x_n \} \) is well-defined.
Let $z_0 = P_{\Omega}x_0$. From $x_{n+1} = P_{C_n \cap Q_n}x_0$ and $z_0 \in \Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

(3.7)

for all $n \geq 0$. Therefore, \{x_n\} is bounded. So, \{u_n\}, \{v_n\}, \{y_n\} and \{z_n\} are also bounded.

Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}x_0$, we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, for all $n \geq 0$. It follows that \{x_n\} in nondecreasing and from \{x_n\} bounded. So there exists the limit of $\|x_n - x_0\|$.

Since $x_n = P_{Q_n}x_0$ and $x_{n+1} \in Q_n$, we have $\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0$ and hence

$$\|x_n - x_{n+1}\|^2 = \|x_n - x_0 + x_0 - x_{n+1}\|^2$$

$$= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2$$

$$= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle$$

$$+ \|x_0 - x_{n+1}\|^2$$

$$\leq \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n \rangle + \|x_0 - x_{n+1}\|^2$$

$$\leq \|x_n - x_0\|^2 - 2\|x_0 - x_n\|^2 + \|x_0 - x_{n+1}\|^2$$

$$\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.$$

Since $\lim_{n \to \infty} \|x_0 - x_n\|$ exists, implies that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$  \hspace{1cm} (3.8)

Since $x_{n+1} \in C_n$, we have

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$  \hspace{1cm} (3.9)

By (3.2) and (3.6), we obtain

$$\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 + r_n(2\alpha - 2\alpha') \|Gx_n - Gx^*\|^2,$$

therefore,

$$r(2\alpha - r') \|Gx_n - Gx^*\|^2 \leq r_n(2\alpha - r_n) \|Gx_n - Gx^*\|^2 \leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2$$

$$\leq (\|x_n - x^*\|^2 + \|y_n - x^*\|) \|x_n - y_n\|.$$

It follows from (3.9) and since \{x_n\} and \{y_n\} are bounded that

$$\lim_{n \to \infty} \|Gx_n - Gx^*\| = 0.$$  \hspace{1cm} (3.10)

By the same method, we have

$$\lim_{n \to \infty} \|Av_n - Ax^*\| = 0,$$  \hspace{1cm} (3.11)

and

$$\lim_{n \to \infty} \|Bu_n - Bx^*\| = 0.$$  \hspace{1cm} (3.12)
For \( x^* \in \Omega \), from Lemma 2.2, we have
\[
\|u_n - x^*\|^2 = \|T_n x_n - r_n G x_n - T_n (x^* - r_n G x^*)\|^2 \\
\leq \|T_n (x_n - r_n G x_n) - T_n (x^* - r_n G x^*)\| \\
\leq \|x_n - r_n G x_n - (x^* - r_n G x^*)\| \\
= \frac{1}{2} \left( \|u_n - x^*\|^2 + \|x_n - r_n G x_n - (x^* - r_n G x^*)\|^2 \right) \\
\leq \frac{1}{2} \left( \|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \right) \\
+ 2r_n (G x_n - G x^*, x_n - u_n) - r_n^2 \|G x_n - G x^*\|^2, \\
\]
hence,
\[
\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n (G x_n - G x^*, x_n - u_n). \\
\]
By (3.6), it follows that
\[
\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n (G x_n - G x^*, x_n - u_n), \\
\]
therefore,
\[
\|x_n - u_n\|^2 \leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2r_n (G x_n - G x^*, x_n - u_n) \\
\leq \left( \|x_n - x^*\| + \|y_n - x^*\| \right) \|x_n - u_n\| \\
+ 2r_n \|G x_n - G x^*\| \|x_n - u_n\|. \\
\]
From (3.9) and (3.10), we obtain
\[
\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{3.13} \\
\]
For \( x^* \in \Omega \), from (2.3) and (3.4), we have
\[
\|z_n - x^*\|^2 = \|P_C (v_n - \beta_n A v_n) - P_C (x^* - \beta_n A x^*)\|^2 \\
\leq \|(v_n - \beta_n A v_n) - (x^* - \beta_n A x^*)\|^2 \\
\leq \frac{1}{2} \left( \|z_n - x^*\|^2 + \|v_n - \beta_n A v_n - (x^* - \beta_n A x^*)\|^2 \right) \\
- \|(v_n - \beta_n A v_n - (x^* - \beta_n A x^*) - (z_n - x^*))\|^2 \right) \\
\leq \frac{1}{2} \left( \|z_n - x^*\|^2 + \|v_n - x^*\|^2 - \|(v_n - z_n) - \beta_n (A v_n - A x^*)\|^2 \right) \\
\leq \frac{1}{2} \left( \|z_n - x^*\|^2 + \|x_n - x^*\|^2 - \|v_n - z_n\|^2 \right) \\
+ 2\beta_n (A v_n - A x^*, v_n - z_n) - r_n^2 \|A v_n - A x^*\|^2, \\
\]
hence,
\[
\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|v_n - z_n\|^2 + 2\beta_n (A v_n - A x^*, v_n - z_n). \\
\]
By (3.5), it follows that
\[\|u_n - x^*\|^2 \leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|v_n - z_n\|^2 + 2\beta_n \langle Av_n - Ax^*, v_n - z_n\rangle,\]
therefore,
\[\|v_n - z_n\|^2 \leq \|x_n - x^*\|^2 - \|u_n - x^*\|^2 + 2\beta_n \langle Av_n - Ax^*, v_n - z_n\rangle \leq \left(\|x_n - x^*\| + \|u_n - x^*\|\right)\|x_n - u_n\| + 2\beta_n \|Av_n - Ax^*\||v_n - z_n|.

From (3.11) and (3.13), we obtain
\[\lim_{n \to \infty} \|v_n - z_n\| = 0. \quad \text{(3.14)}\]

By the same way, using (3.12) and (3.14) we get
\[\lim_{n \to \infty} \|v_n - u_n\| = 0. \quad \text{(3.15)}\]

Since \(y_n - x_n = \alpha_n u_n + (1 - \alpha_n) S(t_n)z_n - x_n = \alpha_n (u_n - x_n) + (1 - \alpha_n) (S(t_n)z_n - x_n)\), it follows that
\[\|x_n - S(t_n)z_n\| = \frac{\alpha_n}{1 - \alpha_n} \|u_n - x_n\| + \frac{1}{1 - \alpha_n} \|v_n - y_n\| \to 0 \text{ as } n \to \infty. \quad \text{(3.16)}\]

Since \(S(t_n)\) is a nonexpansive mapping, we have
\[\|x_n - S(t_n)z_n\| \leq \|x_n - S(t_n)z_n\| + \|S(t_n)z_n - S(t_n)x_n\| \leq \|x_n - S(t_n)z_n\| + \|z_n - x_n\| \leq \|x_n - S(t_n)z_n\| + \|z_n - v_n\| + \|v_n - u_n\| + \|u_n - x_n\|.\]

From (3.13), (3.14), (3.15) and (3.16), we obtain
\[\lim_{n \to \infty} \|x_n - S(t_n)x_n\| = 0. \quad \text{(3.17)}\]

Since \(\{x_n\}\) is bounded, we choose subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) and assume that \(x_{n_i} \to x'\). Let us show that \(x' \in \Omega\). First, we show that \(x' \in \cap_{t=0}^{\infty} F(S(t))\). Suppose that \(x' \notin \cap_{t=0}^{\infty} F(S(t))\), that is \(x' \notin S(t)x'\). From Opial’s condition and (3.17), we have
\[\liminf_{i \to \infty} \|x_{n_i} - x'\| < \liminf_{i \to \infty} \|x_{n_i} - S(t)x'\| \leq \liminf_{i \to \infty} \left(\|x_{n_i} - S(t)x_{n_i}\| + \|S(t)x_{n_i} - S(t)x'\|\right) \leq \liminf_{i \to \infty} \|x_{n_i} - x'\|.
\]

This is a contradiction. Thus, we obtain \(x' \in \cap_{t=0}^{\infty} F(S(t))\).

Next, let us show \(x' \in \text{GEP}(F, G)\). Since \(u_n = T_{r_n}(x_n - r_n Gx_n)\) and
\[F(u_n, y) + (Gx_n, y - u_n) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C.\]

From (A2), we also have
\[\langle Gx_n, y - u_n \rangle + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq F(y, u_n), \quad \forall y \in C,\]
and hence
\[\langle Gx_n, y - u_n \rangle + (y - u_n, \underbrace{u_{n_i} - x_{n_i}}_{r_{n_i}}) \geq F(y, u_{n_i}), \quad \forall y \in C. \quad \text{(3.18)}\]
From (3.13), we get \( u_{n_i} \to x' \). For \( t \) with 0 < \( t \leq 1 \) and \( y \in C \), put \( y_t = ty + (1 - t)x' \).
Since \( y \in C \) and \( x' \in C \), we have \( y_t \in C \). So, from (3.18), we have and hence
\[
\langle Gy_t, y_t - u_{n_i} \rangle \geq \langle Gy_t, y_t - u_{n_i} \rangle - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\rho_{n_i}} \rangle + F(y_t, u_{n_i}) = \langle Gy_t - Gu_{n_i}, y_t - u_{n_i} \rangle + \langle Gu_{n_i} - Gx_{n_i}, y_t - u_{n_i} \rangle - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\rho_{n_i}} \rangle + F(y_t, u_{n_i}).
\]
From \( \|u_{n_i} - x_{n_i}\| \to 0 \), we obtain \( \|Gu_{n_i} - Gx_{n_i}\| \to 0 \). By the \( \alpha \)-inverse-strongly monotonicity of \( G \), we know that \( \langle Gy_t - Gu_{n_i}, y_t - u_{n_i} \rangle \geq 0 \). Since \( \frac{u_{n_i} - x_{n_i}}{\rho_{n_i}} \to 0 \), it follows by (A4) that
\[
F(y_t, x') \leq \lim_{i \to \infty} F(y_t, u_{n_i}) \leq \lim_{i \to \infty} \langle Gy_t, y_t - u_{n_i} \rangle = \langle Gy_t, y_t - x' \rangle.
\]
So, from (A1) and (A4), we have
\[
0 = F(y_t, y) \leq tF(y_t, y) + (1 - t)F(y_t, x') \leq tF(y_t, y) + (1 - t)\langle Gy_t, y - x' \rangle, \quad \text{and hence}
\]
\[
F(y_t, y) + (1 - t)\langle Gy_t, y - x' \rangle \geq 0.
\]
Letting \( t \to 0 \), we have for each \( y \in C \), \( F(x', y) + \langle Gx', y - x' \rangle \geq 0 \). This implies that \( x' \in \text{GEP}(F,G) \).

Next, let us show that \( x' \in \text{VI}(C,B) \). Let
\[
Uy = \begin{cases} By + N_CY, & y \in C, \\ \emptyset, & y \notin C. \end{cases}
\]
Then \( U \) is maximal monotone. Let \( (y,w) \in G(U) \). Since \( w - By \in N_CY \) and \( v_n \in C \), we have \( \langle y - v_n, w - By \rangle \geq 0 \). On the other hand, from \( v_i = P_C(u_0 - \lambda_nBu_n) \), we have \( \langle y - v_n, v_i - (u_n - \lambda_nBu_n) \rangle \geq 0 \), that is, \( \langle y - v_n, \frac{v_i - u_n}{\lambda_n} + Bu_n \rangle \geq 0 \).

Therefore, we have
\[
\langle y - v_n, w \rangle \geq \langle y - v_n, By \rangle \geq \langle y - v_n, By \rangle - \langle y - v_n, \frac{v_i - u_n}{\lambda_n} + Bu_n \rangle = \langle y - v_n, By - \frac{v_i - u_n}{\lambda_n} - Bu_n \rangle = \langle y - v_n, By - Bu_{n_i} \rangle + \langle y - v_n, Bu_{n_i} - Bu_n \rangle - \langle y - v_n, \frac{v_i - u_n}{\lambda_n} \rangle \geq \langle y - v_n, Bu_{n_i} \rangle - \langle y - v_n, \frac{v_i - u_n}{\lambda_n} + Bu_n \rangle \geq \|y - v_n\|\|Bu_{n_i} - Bu_n\| - \|y - v_n\|\|\frac{v_i - u_n}{\lambda_n}\|.
\]
(3.19)

Notice that \( \|v_i - u_n\| \to 0 \) as \( i \to \infty \) and \( B \) is Lipschitz continuous, hence from (3.19), we obtain \( \langle y - x', w \rangle \geq 0 \) as \( i \to \infty \). Since \( U \) is maximal monotone, we
have \( x' \in U^{-1}0 \), and hence \( x' \in VI(C, B) \). In the same manner as the proof of \( x' \in VI(C, B) \), we obtain \( x' \in VI(C, A) \). Therefore \( x' \in \Omega \).

Finally, we will show that \( x_n \to P_{\Omega}x_0 \). Since \( x' \in \Omega \), we have

\[
\|P_{\Omega}x_0 - x_0\| \leq \|x' - x_0\| \leq \liminf_{i \to \infty} \|x_n_i - x_0\| \leq \limsup_{i \to \infty} \|x_n_i - x_0\| \leq \|P_{\Omega}x_0 - x_0\|.
\]

Thus, we obtain that \( \lim_{i \to \infty} \|x_n_i - x_0\| = \|x' - x_0\| = \|P_{\Omega}x_0 - x_0\| \). Using the Kadec-Klee property of \( H \), we obtain that \( \lim_{i \to \infty} x_n_i = x' = P_{\Omega}x_0 \). Hence the whole sequence must converge to \( x' = P_{\Omega}x_0 \). This completes the proof. \( \square \)

**Corollary 3.2.** [18, Theorem 2.2] Let \( C \) be a nonempty bounded closed convex subset of a real Hilbert space \( H \). Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on \( C \) and let \( F \) be a bifunction of \( C \times C \) into real numbers \( \mathbb{R} \) satisfying (A1) – (A4). Suppose that \( \Omega := (\cap_{t=0}^{\infty} F(S(t))) \cap EP(F) \neq \emptyset \). Let \( \{\alpha_n\} \subset [0, a) \subset [0, 1), \{r_n\} \subset [r, r'] \subset (0, 2\alpha) \) and \( \{t_n\} \subset [0, \infty) \) satisfying \( \liminf_n t_n = 0, \limsup_n t_n > 0 \), and \( \lim_n(t_{n+1} - t_n) = 0 \).

For \( x_0 \in H \), let the sequences \( \{x_n\} \), \( \{u_n\} \) and \( \{y_n\} \) are generated by \( u_n \in C \) and

\[
\begin{aligned}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) &\geq 0, \forall y \in C, \\
y_n &= \alpha_n u_n + (1 - \alpha_n)S(t_n)u_n, \\
C_n &= \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n}x_0, \forall n \geq 0.
\end{aligned}
\tag{3.20}
\]

Then the sequence \( \{x_n\} \) converges strongly to \( P_{\Omega}x_0 \).

**Proof.** If \( G, A, B \equiv 0 \), in Theorem 3.1, we obtain the desired result. \( \square \)

**Corollary 3.3.** Let \( C \) be a nonempty bounded closed convex subset of a real Hilbert space \( H \). Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on \( C \). Suppose that \( \Omega := \cap_{t=0}^{\infty} F(S(t)) \neq \emptyset \). Let \( \{\alpha_n\} \subset [0, a) \subset [0, 1) \) and \( \{t_n\} \subset [0, \infty) \) satisfying \( \liminf_n t_n = 0, \limsup_n t_n > 0 \), and \( \lim_n(t_{n+1} - t_n) = 0 \). For \( x_0 \in H \), let the sequences \( \{x_n\} \) and \( \{y_n\} \) are generated by

\[
\begin{aligned}
y_n &= \alpha_n x_n + (1 - \alpha_n)S(t_n)x_n, \\
C_n &= \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n}x_0, \forall n \geq 0.
\end{aligned}
\tag{3.21}
\]

Then the sequence \( \{x_n\} \) converges strongly to \( P_{\Omega}x_0 \).

**Proof.** If \( F(x, y) \equiv 0 \) for all \( x, y \in C \) and \( G, A, B \equiv 0 \), by Theorem 3.1 we obtain the desired result. \( \square \)

**Corollary 3.4.** Let \( C \) be a nonempty bounded closed convex subset of a real Hilbert space \( H \). Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on \( C \) and let \( G, A, B : C \to H \) be three \( \alpha, \beta, \lambda \)-inverse-strongly monotone mappings, respectively. Suppose that \( \Omega := \cap_{t=0}^{\infty} F(S(t)) \cap VI(C, A) \cap VI(C, B) \cap VI(C, G) \neq \emptyset \). Let \( \{\alpha_n\} \subset [0, a) \subset [0, 1) \), \( \{\beta_n\} \subset [b, b'] \subset (0, 2\beta) \), \( \{\lambda_n\} \subset [\ell, \ell'] \subset (0, 2\lambda) \), \( \{r_n\} \subset [r, r'] \subset (0, 2a) \) and \( \{t_n\} \subset [0, \infty) \) satisfying \( \liminf_n t_n = 0, \limsup_n t_n > 0 \), and \( \lim_n(t_{n+1} - t_n) = 0 \). For
$x_0 \in H$, let the sequences \( \{x_n\} \), \( \{u_n\} \), \( \{y_n\} \) and \( \{z_n\} \) are generated by $u_n \in C$ and

\[
\begin{align*}
\{ u_n = P_C(x_n - r_n Gx_n), \\
v_n = P_C(u_n - \lambda_n Bu_n), \\
z_n = P_C(v_n - \beta_n Av_n), \\
y_n = \alpha_n u_n + (1 - \alpha_n) S(t_n) z_n, \\
C_n = \{ z \in C \mid \|y_n - z\| \leq \|x_n - z\| \}, \\
Q_n = \{ z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \forall n \geq 0.
\end{align*}
\]

Then the sequence \( \{x_n\} \) converges strongly to $P_{T_1} x_0$.

**Proof.** If $F \equiv 0$, then $u_n = P_C(x_n - r_n Gx_n)$ for all $n \geq 0$, by Theorem 3.1, we obtain the desired result. \( \Box \)

3.2. The shrinking projection method.

**Theorem 3.5.** Let $C$ be a nonempty bounded closed convex subset of a real Hilbert space $H$. Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on $C$, let $F$ be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1) – (A4) and let $G, A : C \to H$ be three $\alpha, \beta, \lambda$-inverse-strongly monotone mappings, respectively. Suppose that $\Omega := (\cap_{t=0}^{\infty} S(t)) \cap VI(C, A) \cap VI(C, B) \cap GEP(F, G) \neq 0$. Let \( \{\alpha_n\} \subset [0, a) \subset (0, 1) \), \( \{\beta_n\} \subset (0, \beta) \), \( \{\lambda_n\} \subset (l, \beta) \subset (0, 2\lambda) \), \( \{r_n\} \subset [r, r') \subset (0, 2a) \) and \( \{t_n\} \subset [0, \infty) \) satisfying $\lim_{n \to \infty} t_n = 0$, $\limsup_{n \to \infty} t_n > 0$, and $\lim_{n \to \infty} (t_{n+1} - t_n) = 0$. For $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$, let the sequences \( \{x_n\} \), \( \{u_n\} \), \( \{v_n\} \), \( \{y_n\} \) and \( \{z_n\} \) are generated by $u_n \in C$ and

\[
\begin{align*}
\{ F(u_n, y) + \langle Gx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\
u_n = P_C(u_n - \lambda_n Bu_n), \\
v_n = P_C(v_n - \beta_n Av_n), \\
y_n = \alpha_n u_n + (1 - \alpha_n) S(t_n) z_n, \\
C_{n+1} = \{ z \in C_n \mid \|y_n - z\| \leq \|x_n - z\| \}, \\
x_{n+1} = P_{C_{n+1}} x_0, \forall n \geq 0.
\end{align*}
\]

Then the sequence \( \{x_n\} \) converges strongly to $P_{T_1} x_0$.

**Proof.** Since for any $x^* \in \Omega$ and let \( \{T_{r_n}\} \) be a sequence of mappings defined as in Lemma 2.2, then we have $x^* = T_{r_n}(x^* - r_n Gx^*)$ and $u_n = T_{r_n}(x_n - r_n Gx_n) \in C$ for all $n \geq 0$. We already have (3.3), (3.4), (3.5) and (3.6). Thus, we get $x^* \in C_{n+1}$. This implies that $\Omega \subset C_n$ for all $n \geq 0$. By using the same argument in the proof of [23, Theorem 3.3 pp. 281–282], we obtain that \( \{x_n\} \) bounded and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. As in the proofs of Theorem 3.1, we already have (3.17). Since \( \{x_n\} \) is bounded, we can choose subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) and assume that $x_{n_i} \to x'$. In the same time, as in the proof of Theorem 3.1, we also have $x' \in \Omega$.

Finally, we have to show that $x_n \to P_{T_1} x_0$. Since $x' \in \Omega$, we have

\[
\|P_{T_1} x_0 - x_0\| \leq \|x' - x_0\| \leq \liminf_{i \to \infty} \|x_{n_i} - x_0\| \leq \limsup_{i \to \infty} \|x_{n_i} - x_0\| \leq \|P_{T_2} x_0 - x_0\|.
\]

Thus, we obtain that $\lim_{i \to \infty} \|x_{n_i} - x_0\| = \|x' - x_0\| = \|P_{T_2} x_0 - x_0\|$. Using the Kadec-Klee property of $H$, we obtain that $\lim_{i \to \infty} x_{n_i} = x' = P_{T_2} x_0$. Hence the whole sequence must converge to $x' = P_{T_2} x_0$. This completes the proof. \( \Box \)
Corollary 3.6. Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on C and let \( F \) be a bifunction of \( C \times C \) into real numbers \( \mathbb{R} \) satisfying \((A1)\) – \((A4)\). Suppose that \( \Omega := \bigcap_{t=0}^{\infty} F(S(t)) \cap EP(F) \neq \emptyset \). Let \( \{\alpha_n\} \subset [0, a) \subset [0, 1), \ {\{r_n\}} \subset [r, r'] \subset (0, 2a) \) and \( \{t_n\} \subset [0, \infty) \) satisfying \( \liminf_n t_n = 0, \limsup_n t_n > 0, \) and \( \lim_n (t_{n+1} - t_n) = 0. \) For \( x_0 \in H, C_1 = C, x_1 = P_{C_1} x_0, \) let the sequences \( \{x_n\}, \{u_n\} \) and \( \{y_n\} \) are generated by \( u_n \in C \) and

\[
F(u_n, y) + \frac{1}{t_n} (y - u_n, u_n - x_n) \geq 0, \ \forall y \in C, \\
y_n = \alpha_n u_n + (1 - \alpha_n) S(t_n) u_n, \\
C_{n+1} = \{z \in C_n | \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \geq 0.
\]

Then the sequence \( \{x_n\} \) converges strongly to \( P_\Omega x_0. \)

Proof. If \( G, A, B \equiv 0, \) by Theorem 3.1 we obtain the desired result. \( \square \)

Corollary 3.7. \([18, \text{Theorem 2.1}]\) Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on C. Suppose that \( \Omega := \bigcap_{t=0}^{\infty} F(S(t)) \neq \emptyset \). Let \( \{\alpha_n\} \subset [0, a) \subset [0, 1) \) and \( \{t_n\} \subset [0, \infty) \) satisfying \( \liminf_n t_n = 0, \limsup_n t_n > 0, \) and \( \lim_n (t_{n+1} - t_n) = 0. \) For \( x_0 \in H, C_1 = C, x_1 = P_{C_1} x_0, \) let the sequences \( \{x_n\}, \{u_n\}, \{y_n\} \) are generated by

\[
y_n = \alpha_n x_n + (1 - \alpha_n) S(t_n) x_n, \\
C_{n+1} = \{z \in C_n | \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \geq 0.
\]

Then the sequence \( \{x_n\} \) converges strongly to \( P_\Omega x_0. \)

Proof. If \( F(x, y) \equiv 0 \) for all \( x, y \in C \) and \( G, A, B \equiv 0, \) by Theorem 3.1 we obtain the desired result. \( \square \)

Corollary 3.8. Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup on C and let \( G, A, B : C \rightarrow H \) be three \( \alpha, \beta, \lambda \)-inverse-strongly monotone mappings, respectively. Suppose that \( \Omega := \bigcap_{t=0}^{\infty} F(S(t)) \cap VI(C, A) \cap VI(C, B) \cap VI(C, G) \neq \emptyset. \) Let \( \{\alpha_n\} \subset [0, a) \subset [0, 1), \ {\{\beta_n\}} \subset [b, b'] \subset (0, 2\beta), \ {\lambda_n} \subset [l, l'] \subset (0, 2\lambda), \ {\{r_n\}} \subset [r, r'] \subset (0, 2a) \) and \( \{t_n\} \subset [0, \infty) \) satisfying \( \liminf_n t_n = 0, \limsup_n t_n > 0, \) and \( \lim_n (t_{n+1} - t_n) = 0. \) For \( x_0 \in H, C_1 = C, x_1 = P_{C_1} x_0, \) let the sequences \( \{x_n\}, \{u_n\}, \{v_n\}, \{y_n\} \) are generated by

\[
u_n = P_C (x_n - r_n G x_n), \\
v_n = P_C (u_n - \lambda_n B u_n), \\
z_n = P_C (v_n - \beta_n A v_n), \\
y_n = \alpha_n u_n + (1 - \alpha_n) S(t_n) z_n, \\
C_{n+1} = \{z \in C_n | \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \geq 0.
\]

Then the sequence \( \{x_n\} \) converges strongly to \( P_\Omega x_0. \)

Proof. If \( F \equiv 0, \) then \( u_n = P_C (x_n - r_n G z_n) \) for all \( n \geq 0, \) by Theorem 3.1, we obtain the desired result. \( \square \)
References


Received: June 15, 2010; Accepted: February 24, 2011.