Fixed Point Theory, 13(2012), No. 1, 73-83
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SOME GENERALIZATIONS OF KANNAN'S FIXED POINT THEOREM IN K-METRIC SPACES

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Abstract. We extend some known fixed point results for mappings satisfying Kannan type conditions to the context of K-metric spaces. Firstly, we prove a common fixed point result for non-commuting maps. A generalization of Kannan's fixed point theorem is given in some class of spaces including K-metric spaces.
Key Words and Phrases: Fixed point, Kannan contractive mapping, K-metric spaces.
2010 Mathematics Subject Classification: 47H10, 54E35, 54E50.

1. Introduction

In the last decades the study of fixed points for mappings defined in a K-metric space over a Banach space ordered by means of a cone (also called cone metric spaces) have received great attention ([1]-[9], [11]-[16], [18], [19]). For example, the extension of the Banach contraction principle to the framework of K-metric spaces has been very useful in the theory of differential and integral equations ([13], [14], [19] and the bibliography therein).

Recently, from a paper by L.-G. Huang and X. Zhang [8] many authors have taken up the question of the existence of fixed points for mappings satisfying some contractive conditions in K-metric spaces ([1]-[2], [9]-[11]). Some of them have also studied the topological structure of a K- metric space, although it is a open question if this topology is metrizable ([11], [12], [18]). It is worth pointing out that in the above-mention papers it is usually required the cone to be normal with nonempty interior. Last condition on the cone is very restrictive, for example, the positive cone of the space of functions L_1 and the positive cone of the space of sequences ℓ_p $(1 \leq p < \infty)$ have empty interior.

The purpose of this paper is to generalize some classical fixed point results for contractions in the sense of Kannan [10] in K-metric spaces. We emphasize that in our approach just a normality assumption is imposed on the cone.

It is noteworthy that a metric type space can be defined from a K-metric and thus, as shows by M.A. Khamsi in a very recent paper [11], many results concerning fixed point for contractions in K-metric spaces can be easily derived of the corresponding results in a metric type spaces. However, this is not the case for Kannan mappings as we show in this paper. In fact, we use the structure of a metric type space as a complementary tool to prove our results.

2. Preliminaries

Let $(V, ||\cdot||)$ be a Banach space. A set $K \subset V$ is called a cone if and only if:

(1) K is nonempty and $K \neq \{0_v\}$.
(2) If $\alpha, \beta \in K$ and $a, b \in \mathbb{R}_0^+$, then $a\alpha + b\beta \in K$.
(3) $K \cap (-K) = \{0_v\}$.

For a given cone $K \subset V$, we can define a partial ordering $\leq$ with respect to K by $\alpha \leq \beta$ if and only if $\beta - \alpha \in K$. We shall write $\alpha < \beta$ to indicate that $\alpha \leq \beta$ but $\alpha \neq \beta$. We will refer $(V, ||\cdot||, K)$ as an ordered Banach space.

The following definitions relate the norm $||\cdot||$ with the cone K:

- The cone K is called normal if there exists a number $\lambda \geq 1$ such that for all $\alpha, \beta \in V$, $0_v \leq \alpha \leq \beta \quad$ implies $\quad ||\alpha|| \leq \lambda ||\beta||$. The least positive number satisfying above is called the normal constant of K.
- The cone K is called closed if K is closed with respect to the topology induced by the norm.

Definition 2.1. *Let X be a set and $d_K : X \times X :\to K$ a mapping. We say that d_K is a K-metric, if for all $x, y, z \in X$, one has*

(1) $d_K(x, y) = 0_v \Leftrightarrow x = y$;
(2) $d_K(x, y) = d_K(y, x)$;
(3) $d_K(x, y) \leq d_K(x, z) + d_K(z, y)$.

The pair (X, d_K) is said to be a K-metric space.

Example 2.2. *Let X be a nonempty set and $(V, ||\cdot||, K) = (\mathbb{R}^n, ||\cdot||, K)$ such that $||\alpha|| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$ for all $\alpha \in \mathbb{R}^n$ and*

$$K = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \geq 0, \text{ for all } i \in \{1, 2, \ldots, n\}\}.$$

It is clear that, K is a closed normal cone with normal constant $\lambda = 1$. Now, we define $d_K : X \times X \to K$ such that

$$d_K(x, y) = (d_1(x, y), d_2(x, y), \ldots, d_n(x, y)),$$

where d_i is a metric in X for all $i \in \{1, 2, \ldots, n\}$. We have that d_K is a K-metric.

The proof of the following lemma easily follows from the definition of K-metric.

Lemma 2.3. *Let (X, d_K) be a K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone with normal constant λ. Then the function $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||d_K(x, y)||$ satisfies the following properties:*

(1) $D(x, y) = 0 \Leftrightarrow x = y$;
(2) $D(x, y) = D(y, x)$;

(3) $D(x, y) \leq \lambda[D(x, z) + D(z, y)]$ *for all* $x, y, z \in X$.

Recently, M.A. Khamsi [11] introduced the concept of a metric type space: for an arbitrary set X, the pair (X, D) is called a metric type space if $D : X \times X \to [0, \infty)$ is a function satisfying properties [1], [2] and [3] in the above lemma. Defining a topology on this class of spaces, he obtained some metric and topological fixed point theorems (see [12]). The previous lemma is used to carry such fixed point results to the framework of K-metric spaces. This fact suggests that many fixed point results proved in K-metric spaces do not need full structure because they could follow by using the real valued function D. However, this approach does not work in order to prove our results, as we show after Theorem 3.1.

Example 2.4. *Let* $c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : \lim\limits_{i\to\infty} x_i = 0\}$ *and* $(\mathbb{R}^n, ||\cdot||, K)$ *such that* $||\alpha|| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$ *for all* $\alpha \in \mathbb{R}^n$ *and*

$$K = \{\alpha \in \mathbb{R}^n : \alpha_i \geq 0,\ \textit{for all } i \in \{1, 2, \dots, n\}\}.$$

Fix $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in K$ *and define* $d_K : c_0 \times c_0 \to K$ *such that*

$$\begin{aligned} d_K(x, y) &= (\alpha_1||x - y||_\infty, \alpha_2||x - y||_\infty, \dots, \alpha_n||x - y||_\infty) \\ &= ||x - y||_\infty(\alpha_1, \alpha_2, \dots, \alpha_n) = ||x - y||_\infty \sum_{i=1}^{n} \alpha_i e_i, \end{aligned}$$

Thus, (c_0, d_K) *is a* K*-metric and* $D(x, y) = ||x - y||_\infty ||\alpha||$.

Given a K-metric space (X, d_K), the concept of sequence will be the usual. We next define convergence and completeness.

Definition 2.5. *Let* (X, d_K) *be a* K*-metric space over an ordered Banach space* $(V, ||\cdot||, K)$ *such that* K *is a closed normal cone,* $x \in X$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset X$. *We say that* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *in* X, *if* $\lim\limits_{n} D(x_n, x) = 0$. *We write* $x_n \to x$ *to denote that the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is convergent to* x, *that is,* $x_n \to x \Leftrightarrow D(x_n, x) \to 0$.

From this definition, it is easy to deduce the following result.

Lemma 2.6. *Let* (X, d_K) *be a* K*-metric space over an ordered Banach space* $(V, ||\cdot||, K)$ *such that* K *is a normal cone. Then the limit is unique.*

Definition 2.7. *Let* (X, d_K) *be a* K*-metric space over an ordered Banach space* $(V, ||\cdot||, K)$ *such that* K *is a closed normal cone and* $\{x_n\}_{n\in\mathbb{N}}$ *a sequence in* X. *We say that* $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence, if for every positive real number* a, *there exists* $\hat{N} \in \mathbb{N}$ *such that for all* $n, m \geq \hat{N}$, *we have that* $D(x_n, x_m) < a$.

Definition 2.8. *Let* (X, d_K) *be a* K*-metric space over an ordered Banach space* $(V, ||\cdot||, K)$ *such that* K *is a closed normal cone. Then* X *is called a complete* K*-metric space, if every Cauchy sequence is convergent in* X.

Let $(V, ||\cdot||, K)$ an ordered Banach space. Recall that a linear operator $Q : V \to V$ is positive if $Q(K) \subseteq K$.

Lemma 2.9. *Let* $(V, ||\cdot||, K)$ *be an ordered Banach space such that* K *is a normal cone with normal constant* λ *and* $Q : V \to V$ *be a linear positive operator. If* $||Q^n(\alpha)|| \to 0$ *for* $\alpha \in K - \{0_v\}$, *then* $Q(\alpha) - \alpha \notin K - \{0_v\}$.

Proof. Consider $\alpha \in K - \{0_v\}$ such that $||Q^n(\alpha)|| \to 0$. Suppose that $Q(\alpha) - \alpha \in K$, that is $\alpha \leq Q(\alpha)$. Since Q is a linear and positive operator we have that $\alpha \leq Q^n(\alpha)$ for all $n \in \mathbb{N}$. Hence $||\alpha|| \leq \lambda ||Q^n(\alpha)||$ for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ we have $||\alpha|| = 0_v$, and so $\alpha = 0_v$ which is a contradiction. □

Along the paper we will use the following notation:

- $B^+(V) = \{Q : V \to V/Q$ is a positive bounded linear operator $\}$.
- $||Q||_o = \sup\{||Q(\alpha)|| : ||\alpha|| \leq 1\}$.

3. Main results

We begin this section with a common fixed point result for mappings satisfying a Kannan type condition. This result is actually an extension of a result of [17] in the context of a K-metric space.

Theorem 3.1. *Let (X, d_K) be a complete K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone with normal constant λ. If T and S are two operators mapping X into itself such that for some $p, q \in \mathbb{N}$ we have that*

$$d_K(T^p x, S^q y) \leq Q(d_K(x, T^p x)) + R(d_K(y, S^q y)),$$

for all $x, y \in X$, where $Q, R \in B^+(V)$ and $||Q||_o + ||R||_o < 1$, then T and S have a unique and common fixed point in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}_{n=1}^{\infty}$ in X as follows

$$\begin{array}{ll} x_1 = T^p x_0, & x_2 = S^q x_1, \\ x_3 = T^p x_2, & x_4 = S^q x_3, \\ \vdots & \vdots \\ x_{2k-1} = T^p x_{2(k-1)}, & x_{2k} = S^q x_{2k-1}, \end{array}$$

for all $k \in \mathbb{N}$. From the assumption we have

$$\begin{aligned} d_K(x_1, x_2) = d_K(T^p x_0, S^q x_1) &\leq Q(d_K(x_0, T^p x_0)) + R(d_K(x_1, S^q x_1)) \\ &= Q(d_K(x_0, T^p x_0)) + R(d_K(x_1, x_2)). \end{aligned}$$

Hence, $[I - R](d_K(x_1, x_2)) \leq Q(d_K(x_0, T^p x_0))$ and thus we get that $d_K(x_1, x_2) \leq [[I - R]^{-1} \circ Q](d_K(x_0, T^p x_0))$. We put $W_1 := [I - R]^{-1} \circ Q$. Thus,

$$d_K(x_1, x_2) \leq W_1(d_K(x_0, T^p x_0)),$$

which implies that

$$D(x_1, x_2) \leq \lambda ||W_1||_o D(x_0, T^p x_0).$$

On the other hand,

$$\begin{aligned} d_K(x_2, x_3) = d_K(S^q x_1, T^p x_2) &\leq Q(d_K(x_2, T^p x_2)) + R(d_K(x_1, S^q x_1)) \\ &= Q(d_K(x_2, x_3)) + R(d_K(x_1, x_2)). \end{aligned}$$

Thus we obtain $[I-Q](d_K(x_2,x_3)) \leq R(d_K(x_1,x_2))$ and hence we get that $d_K(x_2,x_3) \leq [[I-Q]^{-1}\circ R](d_K(x_1,x_2))$. If we put $W_2 := [I-Q]^{-1}\circ R$, it follows that

$$d_K(x_2,x_3) \leq W_2(d_K(x_1,x_2)) \leq [W_2\circ W_1](d_K(x_0,T^px_0)).$$

Thus

$$D(x_2,x_3) \leq \lambda||W_2||_o||W_1||_oD(x_0,T^px_0).$$

By induction, for each $k\geq 1$ we obtain,

$$D(x_{2k-1},x_{2k}) \leq \lambda||W_1||_o^k||W_2||_o^{k-1}D(x_0,T^px_0),$$

and

$$D(x_{2k},x_{2k+1}) \leq \lambda||W_1||_o^k||W_2||_o^kD(x_0,T^px_0).$$

Denote by $r_1 := ||W_1||_o$ and $r_2 := ||W_2||_o$. Note that

$$\begin{aligned} r_1 &\leq ||[I-R]^{-1}||_o||Q||_o = ||I+R+R^2+\cdots+R^n+\cdots||_o||Q||_o \\ &\leq (1+||R||_o+||R^2||_o+\cdots+||R^n||_o+\cdots)||Q||_o \\ &= \frac{||Q||_o}{1-||R||_o} < 1, \end{aligned}$$

because $||Q||_o+||R||_o<1$. Analogously $r_2<1$. We will prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For each $m\geq 1$ we have for any $n=2k$, $k\geq 1$

$$\begin{aligned} D(x_n,x_{n+m}) &\leq \lambda[D(x_n,x_{n+1})+D(x_{n+1},x_{n+2})+\cdots+D(x_{n+m-1},x_{n+m})] \\ &\leq \lambda[(r_1^kr_2^k+r_1^{k+1}r_2^k+r_1^{k+1}r_2^{k+1}+\cdots)\lambda D(x_0,T^px_0)] \\ &\leq \lambda^2D(x_0,T^px_0)[(r_1^kr_2^k+r_1^{k+1}r_2^k)\sum_{i=0}^{\infty}r_1^ir_2^i] \\ &= \frac{\lambda^2D(x_0,T^px_0)(1+r_1)}{1-r_1r_2}r_1^kr_2^k \end{aligned}$$

Similarly, for any $n=2k-1$, $k\geq 1$

$$\begin{aligned} D(x_n,x_{n+m}) &\leq \lambda[(r_1^kr_2^{k-1}+r_1^kr_2^k+r_1^{k+1}r_2^k+\cdots)\lambda D(x_0,T^px_0)] \\ &\leq \lambda^2D(x_0,T^px_0)[(r_1^kr_2^{k-1}+r_1^kr_2^k)\sum_{i=0}^{\infty}r_1^ir_2^i] \\ &= \frac{\lambda^2D(x_0,T^px_0)(1+r_2)}{1-r_1r_2}r_1^kr_2^{k-1}. \end{aligned}$$

Therefore, $D(x_n,x_{n+m})\to 0$ as $n\to\infty$. Since X is complete, there exists $x^*\in X$ such that $x_n\to x^*$. We first show that $T^px^*=S^qx^*=x^*$.

Because d_K is a K-metric, we have, for each $n\geq 1$, that

$$d_K(x^*,T^px^*) \leq d_K(x^*,x_n)+d_K(x_n,T^px^*).$$

If n is even, $x_n=S^qx_{n-1}$ and

$$\begin{aligned} d_K(x^*,T^px^*) &\leq d_K(x^*,x_n)+d_K(S^qx_{n-1},T^px^*) \\ &\leq d_K(x^*,x_n)+Q(d_K(x^*,T^px^*))+R(d_K(x_{n-1},S^qx_{n-1})) \end{aligned}$$

which implies $[I-Q](d_K(x^*,T^px^*)) \leq d_K(x^*,x_n)+R(d_K(x_{n-1},x_n))$. Thus, we have $||[I-Q](d_K(x^*,T^px^*))|| \leq \lambda[D(x^*,x_n)+||R||_o D(x_{n-1},x_n)]$. Taking limit as $n \to \infty$, since $D(x^*,x_n) \to 0$ and $D(x_{n-1},x_n) \to 0$, we obtain $||[I-Q](d_K(x^*,T^px^*))|| = 0$. Hence $d_K(x^*,T^px^*) = 0_v$. Hence $T^px^* = x^*$. Similarly $S^qx^* = x^*$.

We shall show that x^* is the unique common fixed point of T^p and S^q. Suppose that there exists another $y^* \in X$ such that $T^py^* = S^qy^* = y^*$. Then

$$d_K(x^*,y^*) = d_K(T^px^*,S^qy^*) \leq Q(d_K(x^*,T^px^*)) + R(d_K(y^*,S^qy^*)) = 0_v.$$

Therefore $d_K(x^*,y^*) = 0_v$ i.e. $x^* = y^*$.

Finally we prove that x^* is the unique common fixed point of T and S. It is clear that $T^px^* = x^*$. Hence $T(T^px^*) = Tx^*$, that is, $T^p(Tx^*) = Tx^*$. Thus $Tx^* = x^*$. In the same way we obtain that $Sx^* = x^*$. Since a common fixed point of T and S is also a common fixed point of T^p and S^p the proof is complete.

□

Remark 3.2. Note that the use of a K-metric cannot be avoided in the proof of the previous theorem (unlike the case of contractions as shows in [11]). Indeed, from the statement of theorem 3.1 ($p, q = 1$) we can just obtain

$$D(Tx,Sy) \leq \lambda(aD(x,Tx) + bD(y,Sy)),$$

where $a + b < 1$. When $\lambda > 1$, it is easy to find an example of a pair of fixed point free mappings satisfying the above condition.

As a consequence of the preceding theorem we obtain the extension of Kannan's fixed point theorem [10] to K-metric spaces.

Corollary 3.3. *Let (X, d_K) be a complete K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone. If T and S are two operators mapping X into itself such that for all $x, y \in X$ we have that*

$$d_K(Tx,Sy) \leq Q(d_K(x,Tx) + d_K(y,Sy)),$$

where $Q \in B^+(V)$ and $||Q||_o < 1/2$, then T and S have a unique and common fixed point in X.

Remark 3.4. It should be note that the above corollary improves in different senses some results from [1] and [15]. Specifically, while in [15] the same result is obtained by considering a vector-valued metric, in [1] the authors state an analogue theorem where a numerical constant is taken instead of a linear operator on the contractive condition satisfied by the maps.

Corollary 3.5. *Let (X, d_K) be a complete K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone. If T is an operator mappings X into itself such that for some $p, q \in \mathbb{N}$ we have that*

$$d_K(T^px,T^qy) \leq Q(d_K(x,T^px)) + R(d_K(y,T^qy)),$$

for all $x, y \in X$, where $Q, R \in B^+(V)$ and $||Q||_o + ||R||_o < 1$, then T has a unique fixed point in X.

In the situation $T = S$ as in Corollary 3.5, we can, in fact, carry our results to a more general setting. Following some ideas from [2] we introduce next a class of spaces including K-metric spaces.

Definition 3.6. *Let X be a set and $d_K : X \times X :\to K$ a mapping. We say that d_K is a N-polygonal K-metric, if for all $x, y \in X$ and for all distinct points $z_1, z_2, \ldots, z_N \in X$, each of them different from x and y, one has*

(1) $d_K(x, y) = 0_v \Leftrightarrow x = y$;
(2) $d_K(x, y) = d_K(y, x)$;
(3) $d_K(x, y) \leq d_K(x, z_1) + d_K(z_1, z_2) + \cdots + d_K(z_{N-1}, z_N) + d_K(z_N, y)$.

The pair (X, d_K) is said to be a N-polygonal K-metric space.

If $N = 1$ the pair (X, d_K) is a K-metric space. The concept of 2-polygonal K-metric space is referred in [2] as cone rectangular metric space. It is clear that a K-metric space is a N-poligonal K-metric space, for all $N \geq 2$.

Example 3.7. *Let $X = \mathbb{N}$, $(V, ||\cdot||) = (\mathbb{R}^2, ||\cdot||)$ with $||\alpha|| = \sqrt{\alpha_1^2 + \alpha_2^2}$ and*

$$K = \{(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \geq 0\}.$$

We define $d_K : X \times X \to K$ as follow:

$$d_K(x, y) = \begin{cases} (0,0) & : \textit{if} \quad x = y, \\ (3a, 3) & : \textit{if} \quad x \in \{1,2\} \wedge y \in \{1,2\} \wedge x \neq y, \\ (a, 1) & : \textit{if} \quad (x \in \{1,2\}^c \vee y \in \{1,2\}^c) \wedge x \neq y. \end{cases}$$

with $a \in (0, \infty)$. Then d_K is a rectangular K-metric but it is not a K-metric because it lacks the triangular property:

$$d_K(1,2) = (3a, 3) > d_K(1,3) + d_K(3,2) = (2a, 2).$$

From the definition of N-polygonal K-metric we can easily deduce the following lemma.

Lemma 3.8. *Let (X, d_K) be a N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone with normal constant λ. Then the function $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||d_K(x, y)||$ satisfies the following properties:*

(1) $D(x, y) = 0 \Leftrightarrow x = y$;
(2) $D(x, y) = D(y, x)$;
(3) $D(x, y) \leq \lambda[D(x, z_1) + D(z_1, z_2) + \cdots + D(z_{N-1}, z_N) + D(z_N, y)]$ *for all distinct points $z_1, z_2, \ldots, z_N \in X$, each of them different from x and y.*

Given a N-polygonal K-metric space (X, d_K), the concept of sequence will be the usual. We next define convergence and completeness.

Definition 3.9. *Let (X, d_K) be a N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$. We say that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X, if $\lim_n D(x_n, x) = 0$. We write $x_n \to x$ to denote that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x, that is, $x_n \to x \Leftrightarrow D(x_n, x) \to 0$.*

In general, for a N-polygonal K-metric space with $N \geq 2$, the uniqueness of the limit of a sequence does not hold (see Example 1.6 in [9]). However, the limit is unique for a convergent Cauchy sequence as we show below.

Definition 3.10. *Let (X, d_K) be a N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if for every positive real number a, there exists $\hat{N} \in \mathbb{N}$ such that for all $n, m \geq \hat{N}$, we have that $D(x_n, x_m) < a$.*

Next Lemma is an extension of Lemma 1.10 presented in [9].

Lemma 3.11. *Let (X, d_K) be a N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is an Cauchy sequence, such that satisfies the following conditions:*

(a): $(\exists x, y \in X)(x_n \to x \wedge x_n \to y)$,

(b): $(\exists \hat{N} \in \mathbb{N})(n, m \geq \hat{N} \Rightarrow x_n \neq x_m \wedge x_n \neq x \wedge x_n \neq y)$,

then $x = y$.

Proof. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is an Cauchy sequence satisfying conditions (a) and (b). For all $n \geq \hat{N}$ we have

$$D(x, y) \leq \lambda[D(x, x_n) + D(x_{n+1}, x_{n+2}) + \cdots + D(x_{n+N-1}, y)].$$

Taking limit as $n \to \infty$, we obtain $D(x, y) = 0$ and hence $x = y$. □

Definition 3.12. *Let (X, d_K) be a N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone. Then X is called a complete N-polygonal K-metric space, if every Cauchy sequence is convergent in X.*

Our next fixed point result for Kannan type maps is given in the framework of a complete N-polygonal K-metric space.

Theorem 3.13. *Let (X, d_K) be a complete N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone with normal constant λ. If T is an operator mapping X into itself such that for a positive integer $p \geq 1$ we have that:*

$$d_K(T^p x, T^p y) \leq Q(d_K(x, T^p x)) + R(d_K(y, T^p y)),$$

for all $x, y \in X$, where $Q, R \in B^+(V)$ and $||Q||_o + ||R||_o < 1$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be fixed and, for each $n \geq 1$, define $x_n = T^p x_{n-1} = T^{np} x_0$. By assumptions, we have

$$d_K(x_1, x_2) \leq Q(d_K(x_0, T^p x_0)) + R(d_K(x_1, T^p x_1)).$$

This implies $[I - R](d_K(x_1, x_2)) \leq Q(d_K(x_0, T^p x_0))$. Thus, we have $d_K(x_1, x_2) \leq [[I - R]^{-1} \circ Q](d_K(x_0, T^p x_0))$. Now, denote $W := [I - R]^{-1} \circ Q$. Clearly

$$D(x_1, x_2) \leq \lambda ||W||_o D(x_0, T^p x_0).$$

Since $||Q||_o + ||R||_o < 1$ it follows $||W||_o < 1$, as in the proof of Theorem 3.1.

On the other hand, we have

$$d_K(x_2, x_3) \le Q(d_K(x_1, T^p x_1)) + R(d_K(x_2, T^p x_2)).$$

Hence, $d_K(x_2, x_3) \le W(d_K(x_1, T^p x_1))$. Therefore we get that

$$d_K(x_2, x_3) \le W^2(d_K(x_0, T^p x_0)).$$

By induction we obtain

$$d_K(x_n, x_{n+1}) \le W^n(d_K(x_0, T^p x_0)),$$

which implies

$$D(x_n, x_{n+1}) \le \lambda ||W||_o^n D(x_0, T^p x_0).$$

Now, we prove that T^p has a fixed point. We divide the proof into two cases.

(Case a): Suppose that $x_n = x_m$ for some $n, m \in \mathbb{N}$ such that $n < m$. Hence $T^{np}x_0 = T^{mp}x_0$. If we define $y_0 := T^{np}x_0$ we have $y_0 = T^{sp}y_0$ where $s = m - n$. Therefore

$$d_K(y_0, T^p y_0) = d_K(T^{sp}y_0, T^{(s+1)p}y_0) \le W^s(d_K(y_0, T^p y_0)).$$

Since $W \in B^+(V)$, because $[I - R]^{-1} = I + R + \cdots + R^n + \cdots$ maps K into K and $||W||_o < 1$, by Lemma 2.9 we must have $d_K(y_0, T^p y_0) = 0_v$. Thus y_0 is a fixed point of T^p.

(Case b): Assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Now, we will prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. We observe that for all $n, m \ge 1$ we have

$$\begin{aligned} d_K(x_n, x_{n+m}) &\le Q(d_K(x_{n-1}, T^p x_{n-1})) + R(d_K(x_{n+m-1}, T^p x_{n+m-1})) \\ &\le Q(W^{n-1}(d_K(x_0, x_1))) + R(W^{n+m-1}(d_K(x_0, x_1))). \end{aligned}$$

Thus

$$D(x_n, x_{n+m}) \le \lambda[||Q||_o ||W||_o^{n-1} D(x_0, x_1) + ||R||_o ||W||_o^{n+m-1} D(x_0, x_1)].$$

Therefore $D(x_n, x_{n+m}) \to 0$ as $n \to \infty$. It implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$.

We shall now show that $T^p x^* = x^*$. Without any loss of generality, we can assume $T^{np}x_0 \neq x^*$ and $T^{np}x_0 \neq T^p x^*$ for all $n \ge 1$. Denote $\alpha := d_K(x^*, T^p x^*)$. Since $x_n \neq x_m$ for all $n, m \ge 1$ such that $n \neq m$ we have

$$\begin{aligned} \alpha &\le d_K(x^*, x_n) + d_K(x_n, x_{n+1}) + .. + d_K(x_{n+N-1}, T^p x^*) \\ &\le d_K(x^*, x_n) + W^n(d_K(x_0, x_1)) + .. + Q(d_K(x_{n+N-2}, x_{n+N-1})) + R(\alpha) \\ &\le d_K(x^*, x_n) + W^n(d_K(x_0, x_1)) + .. + Q(W^{n+N-2}(d_K(x_0, x_1))) + R(\alpha). \end{aligned}$$

Thus, $[I - R](\alpha) \le d_K(x^*, x_n) + W^n(d_K(x_0, x_1)) + \cdots + Q(W^{n+N-2}(d_K(x_0, x_1)))$. Therefore

$$||[I - R](\alpha)|| \le \lambda[D(x^*, x_n) + ||W||_o^n D(x_0, x_1) + \cdots + ||Q||_o ||W||_o^{n+N-2} D(x_0, x_1)].$$

Since $D(x^*, x_n) \to 0$ and $||W||_o^n \to 0$ as $n \to \infty$ we have $||[I - R](\alpha)|| = 0$. Hence $d_K(x^*, T^p x^*) = 0_v$ following $T^p x^* = x^*$.

We observe that x^* is the unique fixed point of T^p. Indeed, suppose that there exists $y^* \in X$ such that $y^* \neq x^*$ and $T^p y^* = y^*$. Thus

$$d_K(x^*, y^*) = d_K(T^p x^*, T^p y^*) \le Q(d_K(x^*, T^p x^*)) + R(d_K(y^*, T^p y^*)) = 0_v.$$

Therefore $d_K(x^*, y^*) = 0_v$ which is a contradiction. So, the fixed point of T^p is unique. Now we prove that $Tx^* = x^*$. It is clear that $T^p x^* = x^*$, in consequence $T(T^p x^*) = Tx^*$. Thus $T^p(Tx^*) = Tx^*$. Therefore $Tx^* = x^*$, because x^* is the unique fixed point of T^p. We also have that x^* is the unique fixed point of T. Suppose that there exists y^* such that $Ty^* = y^*$. It follows that $T^p y^* = y^*$ and thus $x^* = y^*$. □

Corollary 3.14. *Let (X, d_K) be a complete N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone and $T : X \to X$ be a mapping. If for all $x, y \in X$ and $p \in \mathbb{N}$ we have that*

$$d_K(T^p x, T^p y) \leq Q(d_K(x, T^p x) + d_K(y, T^p y)),$$

where $Q \in B^+(V)$ and $||Q||_o < 1/2$, then T has a unique fixed point in X.

Since a rectangular metric space is a 2-polygonal K-metric space, we obtain the following consequence.

Corollary 3.15. ([3], [9]) *Let (X, d_K) be a complete rectangular K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a closed normal cone and $T : X \to X$ be a mapping. If for all $x, y \in X$ we have that*

$$d_K(Tx, Ty) \leq a(d_K(x, Tx) + d_K(y, Ty)),$$

where $a \in [0, 1/2)$, then T has a unique fixed point in X.

References

[1] M. Abbas and B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett., **22**(2009), 511-515.

[2] A. Azam, M. Arshad and I. Beg, *Banach contraction principle on cone rectangular metric spaces*, Appl. Anal. Discrete Math., **3**(2009), No.2, 236-241.

[3] A. Azam and M. Arshad, *Kannan fixed point theorem on generalized metric spaces*, J. Nonlinear Sci. Anal., **1**(2008), No.1, 45-48.

[4] K.J. Chung, *Nonlinear constractions in abstract spaces*, Kodai Math. J., **4**(1981), 288-292.

[5] J. Eisenfeld and V. Lakshmikantham, *Comparison principle and nonlinear contractions in abstract spaces*, J. Math. Anal. Appl., **49**(1975), 504-511.

[6] J. Eisenfeld and V. Lakshmikantham, *Fixed point theorems through abstract cones*, J. Math. Anal. Appl., **52**(1975), 25-35.

[7] A.D. Filip and A. Petruşel, *Fixed point theorems on spaces endowed with vector-valued metrics*, Fixed Point Theory Appl., **2010**(2010), doi:10.1155/2010/281381, 15 pages.

[8] L.-G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468-1476.

[9] M. Jleli and B. Samet, *The Kannan's fixed point theorem in a cone rectangular metric space*, J. Nonlinear Sci. Appl., **2**(2009), No.3, 161-167.

[10] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60**(1968), 71-76.

[11] M.A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory Appl., **2010**(2010), Article ID 315398, 7 pages.

[12] M.A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., **73**(2010), 3123-3129.

[13] E. de Pascale and L. de Pascale, *Fixed points for some non-obviously contractive operators*, Proc. Amer. Math. Soc., **130**(2002), No.11, 3249-3254.

[14] E. de Pascale and P.P. Zabrejko, *Fixed point theorems for operators in spaces of continuous functions*, Fixed Point Theory, **5**(2004), No. 1, 117-129.

[15] I.A. Rus, *Approximation of common fixed point in a generalized metric space*, Anal. Numér. Théor. Approx., **8**(1979), 236-241.

[16] B. Rzepecki, *On fixed point theorems of Maia type*, Publ. Inst. Math. Beograd, **28**(1980), 179-186.

[17] P. Srivastava and V. K. Gupta, *A note on common fixed points*, Yokohama Math. J., **19**(1971), 91-95.

[18] D. Turkoglu and M. Abuloha, *Cone metric spaces and fixed points theorems in diametrically contractive mappings*, Acta Math. Sinica. (Engl. Ser.) **26**(2010), No.3, 489-496.

[19] P.P. Zabrejko, *K-metric and K-normed linear spaces: survey*, Collect. Math., **48**(1997), 825-859.

Received: September 16, 2010; Accepted: November 10, 2010.