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FIXED POINT RESULTS FOR WEAK CONTRACTIVE MAPPINGS IN ORDERED K-METRIC SPACES

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Abstract. In this paper, we derive new coincidence and common fixed point theorems for self-maps satisfying a weak contractive condition in an ordered K-metric space. As application, the obtained results are used to prove an existence theorem of solutions of a nonlinear integral equation.
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1. Introduction and preliminaries

One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction mapping principle [5, 9]. This principle has been generalized in different directions in different spaces by mathematicians over the years.

The concept of weak contractions in Hilbert spaces was defined by Alber and Guerre-Delabriere [3] in 1997 and was extended to metric spaces by Rhoades [41].

Definition 1.1. *A map $T : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if*

$$d(Tx, Ty) \leq d(x, y) - \Phi(d(x, y)),\ \forall\, x, y \in X, \tag{1}$$

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where $\Phi : [0, +\infty) \to [0, +\infty)$ *is continuous and non-decreasing function such that* $\Phi(t) = 0$ *if and only if* $t = 0$.

The result of Rhoades [41] is the following.

Theorem 1.2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a weakly contractive map. Then,* T *admits a unique fixed point.*

Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works, some of which are noted in [6, 10, 11, 18, 44, 50].

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton's approximation method [25, 26, 46, 49] and in optimization theory [14]. Fixed point theory in K-metric and K-normed spaces was developed by A.I. Perov and his consortiums [28, 35, 36], E.M. Mukhamadijev and V.J. Stetsenko [30] and others. The main idea consists to use an ordered Banach space instead of the set of real numbers, as the codomain for a metric. For more details on fixed point theory in K-metric and K-normed spaces, we refer the reader to [49]. Without mentioning these previous works, Huang and Zhang [19] reintroduced such spaces under the name of cone metric spaces but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also proved some fixed point theorems in such spaces in the same work. After that, fixed point results in cone metric spaces have been studied by many other authors. References [1, 15, 16, 20, 21, 24, 27, 38, 40, 45, 47, 48] are some works in this line of research. However, very recently Wei-Shih Du in [17] used the scalarization function and investigated the equivalence of vectorial versions of fixed point theorems in cone metric spaces and scalar versions of fixed point theorems in metric spaces. He showed that many of the fixed point theorems for maps satisfying contractive conditions of a linear type in cone metric spaces can be considered as the corollaries of corresponding theorems in metric spaces. Nevertheless, the fixed point theory in cone metric spaces proceeds to be actual, since it is unknown if it is possible to adapt the method of scalarization to maps satisfying contractive conditions of a nonlinear type in cone metric spaces.

Recently, in [11], B. S. Choudhury and N. Metiya established a unique fixed point result for maps satisfying nonlinear weak contractive contractions in cone metric spaces. Before stating the main theorem in [11], we begin by recalling some definitions and mathematical preliminaries.

Let $\mathbf{B}$ be a Banach space over $\mathbb{R}$ with respect to a given norm $\|\cdot\|_B$. We denote by 0_B the zero vector of $\mathbf{B}$.

Definition 1.3. (Zabrejko [49]) *A non-empty subset* $\mathbf{K}$ *of* $\mathbf{B}$ *is called a cone if and only if:*
(i) $\overline{\mathbf{K}} = \mathbf{K}$, $\mathbf{K} \neq \{0_B\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in \mathbf{K} \Rightarrow ax + by \in \mathbf{K}$,
(iii) $\mathbf{K} \cap (-\mathbf{K}) = \{0_B\}$,
where $\overline{\mathbf{K}}$ *denotes the closure of* $\mathbf{K}$.

Definition 1.4. *Recall that a binary relation* $\leq_A$ *on a non-empty set* A *is said to be an order relation (and* A *equipped with* $\leq_A$ *is called a partially ordered set) if it*

satisfies the following three properties:
(i) *reflexivity:* $x \leq_A x$ *for all* $x \in X$,
(ii) *antisymmetry:* $x \leq_A y$ *and* $y \leq_A x$ *imply* $x = y$,
(iii) *transitivity:* $x \leq_A y$ *and* $y \leq_A z$ *imply* $x \leq_A z$.

Let $\mathbf{K}$ be a cone in $\mathbf{B}$. We denote by $\mathrm{int}(\mathbf{K})$ the interior of $\mathbf{K}$. We suppose that $\mathrm{int}(\mathbf{K})$ is non-empty. We denote by $\leq_B$ the binary relation on $\mathbf{B}$ defined by:

$$x, y \in \mathbf{B}, \quad x \leq_B y \Longleftrightarrow y - x \in \mathbf{K}.$$

The notation $x <_B y$ indicates that $x \leq_B y$ and $x \neq y$ while $x \ll y$ will show $y - x \in \mathrm{int}(\mathbf{K})$.

Proposition 1.5. ([4, 49]) $(\mathbf{B}, \leq_B)$ *is a partially ordered set.*

Definition 1.6. ([4, 34, 49]) *The cone* $\mathbf{K}$ *is called normal if there is a constant* $L > 0$ *such that the order inequalities* $\xi \leq_B \eta$ $(\xi, \eta \in \mathbf{K})$ *imply the scalar inequality* $\|\xi\|_B \leq L\|\eta\|_B$. *The least positive number* L *satysfying the above inequality is called the normal constant of* $\mathbf{K}$. *Clearly,* $L \geq 1$. *In fact, taking* $\xi = \eta \neq 0_B$ *in the above inequality, we have* $L \geq 1$.

Definition 1.7. *The cone* $\mathbf{K}$ *is said to be regular if every increasing sequence which is bounded from above is convergent, that is, if* $\{x_n\}$ *is a sequence in* $\mathbf{B}$ *such that* $x_1 \leq_B x_2 \leq_B \cdots \leq_B y$ *for some* $y \in \mathbf{B}$, *then there is* $x \in \mathbf{B}$ *such that* $\lim_{n \to +\infty} \|x_n - x\|_B = 0$.

Remark 1.8. *Equivalently, the cone* $\mathbf{K}$ *is said to be regular if every decreasing sequence which is bounded from below is convergent.*

Lemma 1.9. ([40]) *Every regular cone is normal.*

Definition 1.10. (Zabrejko [49]) *Let* $\mathbf{X}$ *be a non-empty set and* $\rho : \mathbf{X} \times \mathbf{X} \to \mathbf{K}$ *satisfies:*
(i) *The equality* $\rho(x, y) = 0_B$ *is equivalent to the equality* $x = y$,
(ii) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in \mathbf{X}$,
(iii) $\rho(x, y) \leq_B \rho(x, z) + \rho(z, y)$ *for all* $x, y, z \in \mathbf{B}$.
Then ρ *is called a* K*-metric (or a cone metric) on* $\mathbf{X}$ *and* $(\mathbf{X}, \rho)$ *is called a* K*-metric space (or a cone metric space).*

In [19], Huang and Zhang presented the notion of convergence of sequences in a K-metric space. However this notion is not new and existed before Huang and Zhang (see for example [12, 49]).

Definition 1.11. ([12, 19, 49]) *Let* $(\mathbf{X}, \rho)$ *be a* K*-metric space,* $\{x_n\}$ *is a sequence in* $\mathbf{X}$ *and* $x \in \mathbf{X}$.
(i) *If for every* $c \in \mathbf{B}$ *with* $0_B \ll c$, *there is* $N \in \mathbb{N}$ *such that* $\rho(x_n, x) \ll c$ *for all* $n \geq N$, *then* $\{x_n\}$ *is said to be convergent to* x. *This limit is denoted by* $\lim_{n \to +\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to +\infty$.
(ii) *If for every* $c \in \mathbf{B}$ *with* $0_B \ll c$, *there is* $N \in \mathbb{N}$ *such that* $\rho(x_n, x_m) \ll c$ *for all* $n, m > N$, *then* $\{x_n\}$ *is called a Cauchy sequence in* $\mathbf{X}$.
(iii) *If every Cauchy sequence in* X *is convergent in* $\mathbf{X}$, *then* $(\mathbf{X}, \rho)$ *is called a complete cone metric space.*

Lemma 1.12. ([19, 49]) *Let $(\mathbf{X}, \rho)$ be a cone metric space with $\mathbf{K}$ a normal cone.*
(i) *A sequence $\{x_n\}$ in $\mathbf{X}$ converges to $x \in \mathbf{X}$ if and only if $\|\rho(x_n, x)\|_B \to 0$ as $n \to +\infty$.*
(ii) *A sequence $\{x_n\}$ in $\mathbf{X}$ is Cauchy if and only if $\|\rho(x_n, x_m)\| \to 0$ as $n, m \to +\infty$.*
(iii) *If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbf{X}$ such that $x_n \to x \in \mathbf{X}$ as $n \to +\infty$ and $y_n \to y \in \mathbf{X}$ as $n \to +\infty$, then $\rho(x_n, y_n) \to \rho(x, y)$ as $n \to +\infty$.*

Lemma 1.13. *Let $(\mathbf{X}, \rho)$ be a K-metric space with cone $\mathbf{K}$. Then*
(i) $\mathrm{int}(\mathbf{K}) + \mathbf{K} \subseteq \mathrm{int}(\mathbf{K})$ *and* $\lambda\,\mathrm{int}(\mathbf{K}) \subseteq \mathrm{int}(\mathbf{K})$, $\lambda > 0$ [45].
(ii) *$a \leq_B b$ and $b \ll c$ imply $a \ll c$* [22].
(iii) *$0_B \leq_B x_n \leq_B y_n$ for each $n \in \mathbb{N}$, $\lim\limits_{n\to+\infty} x_n = x \in \mathbf{B}$ and $\lim\limits_{n\to+\infty} y_n = y \in \mathbf{B}$ imply $0_B \leq_B x \leq_B y$* [22].
(iv) *$\mathbf{K}$ is normal if and only if $x_n \leq_B y_n \leq_B z_n$ and $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} z_n = x$ imply $\lim\limits_{n\to+\infty} y_n = x$* [14].

Now, the main result in [11] is the following.

Theorem 1.14. (Choudhury-Metiya [11]) *Let $(\mathbf{X}, \rho)$ be a complete K-metric space with regular cone $\mathbf{K}$ such that $\rho(x, y) \in \mathrm{int}(\mathbf{K})$, for $x, y \in \mathbf{X}$ with $x \neq y$. Let $T : \mathbf{X} \to \mathbf{X}$ be a mapping satisfying the inequality (1), where the function $\Phi : \mathrm{int}(\mathbf{K}) \cup \{0_B\} \to \mathrm{int}(\mathbf{K}) \cup \{0_B\}$ is continuous and monotone increasing w.r.t. $\leq_B$ with*
(i) *$\Phi(t) = 0_B \Leftrightarrow t = 0_B$,*
(ii) *$\Phi(t) \ll t$ for all $t \in \mathrm{int}(\mathbf{K})$,*
(iii) *either $\Phi(t) \leq_B \rho(x, y)$ or $\rho(x, y) \leq_B \Phi(t)$, for $t \in \mathrm{int}(\mathbf{K}) \cup \{0_B\}$ and $x, y \in \mathbf{X}$.*
Then, T has a unique fixed point in $\mathbf{X}$.

The existence of a fixed point in partially ordered metric spaces has been recently considered in [2, 7, 8, 13, 29, 31, 33, 37, 39, 42, 43] and others. It is of interest to determine the existence of a fixed point in such a setting. The first result in this direction was given by Ran and Reurings in [39], where they extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. The main result obtained in [39] was further extended and refined by many authors.

In this paper, we establish new coincidence and common fixed point theorems in ordered K-metric spaces for self-maps satisfying a weak contraction. Presented theorems extend the recent result of B. S. Choudhury and N. Metiya [11]. An application of our obtained results to prove an existence theorem for an integral equation is given.

2. Main results

Definition 2.1. ([23]) *Let $(\mathbf{X}, \rho)$ be a K-metric space and $f, g : \mathbf{X} \to \mathbf{X}$. If $w = fx = gx$, for some $x \in \mathbf{X}$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self-maps f and g are said to be compatible if $\lim\limits_{n\to+\infty} \rho(fgx_n, gfx_n) = 0_B$, whenever $\{x_n\}$ is a sequence in $\mathbf{X}$ such that $\lim\limits_{n\to+\infty} fx_n = \lim\limits_{n\to+\infty} gx_n = t$ for some t in X.*

Definition 2.2. ([13]) *Let $(\mathbf{X}, \preceq)$ be a partially ordered set and $T, S : \mathbf{X} \to \mathbf{X}$ are maps of $\mathbf{X}$ into itself. One says T is S-non-decreasing w.r.t. $\preceq$ if for $x, y \in \mathbf{X}$,*

$$Sx \preceq Sy \text{ implies } Tx \preceq Ty.$$

Our first result is the following.

Theorem 2.3. *Let $(\mathbf{X}, \preceq)$ be a partially ordered set and suppose that there exists a K-metric ρ in $\mathbf{X}$ such that the K-metric space $(\mathbf{X}, \rho)$ is complete and $\mathbf{K}$ is a regular cone such that $\rho(x, y) \in \operatorname{int}(\mathbf{K})$ for all $x, y \in \mathbf{X}$, $x \neq y$. Let $T, S : \mathbf{X} \to \mathbf{X}$ be such that*
(a) *$T\mathbf{X} \subseteq S\mathbf{X}$,*
(b) *T and S are compatible,*
(c) *T and S are continuous maps,*
(d) *T is S-non-decreasing w.r.t. $\preceq$.*
Suppose that

$$\rho(Tx, Ty) \leq_B \rho(Sx, Sy) - \Phi(\rho(Sx, Sy)), \text{ for all } x, y \in \mathbf{X} \text{ for which } Sy \preceq Sx, \quad (2)$$

where $\Phi : \operatorname{int}(\mathbf{K}) \cup \{0_B\} \to \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ is continuous with
(e) *$\Phi(t) = 0_B \Leftrightarrow t = 0_B$,*
(f) *$\Phi(t) \ll t$ for all $t \in \operatorname{int}(\mathbf{K})$,*
(g) *either $\Phi(t) \leq_B \rho(x, y)$ or $\rho(x, y) \leq_B \Phi(t)$, for $t \in \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ and $x, y \in \mathbf{X}$.*
Suppose also that there exists $x_0 \in \mathbf{X}$ such that $Sx_0 \preceq Tx_0$. Then, T and S have a coincidence point $x^* \in \mathbf{X}$, i.e., $Tx^* = Sx^*$.

Proof. Let $x_0 \in \mathbf{X}$ such that $Sx_0 \preceq Tx_0$. By (a), there exists $x_1 \in \mathbf{X}$ such that $Sx_1 = Tx_0$. Again, from (a), there exists $x_2 \in \mathbf{X}$ such that $Sx_2 = Tx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ in $\mathbf{X}$ such that

$$Sx_{n+1} = Tx_n, \ \forall\, n \in \mathbb{N}. \quad (3)$$

Since T is S-non-decreasing, we have

$$Sx_0 \preceq Tx_0 = Sx_1 \Rightarrow Tx_0 = Sx_1 \preceq Tx_1 = Sx_2 \Rightarrow \cdots \Rightarrow Tx_n \preceq Tx_{n+1}.$$

Then

$$Sx_n \preceq Sx_{n+1}, \ \forall\, n \in \mathbb{N}. \quad (4)$$

If $Sx_n = Sx_{n+1}$ for some n, then trivially x_n is a coincidence point of T and S. Then, we assume that $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N}$.

Now, from (4), we can apply (2) for $x = x_{n-1}$ and $y = x_n$. We obtain:

$$\begin{aligned} \rho(Sx_n, Sx_{n+1}) &\leq_B \rho(Tx_{n-1}, Tx_n) \quad (5)\\ &\leq_B \rho(Sx_{n-1}, Sx_n) - \Phi(\rho(Sx_{n-1}, Sx_n)) \leq_B \rho(Sx_{n-1}, Sx_n). \end{aligned}$$

Then, the sequence $\{\rho(Sx_n, Sx_{n+1})\}$ is monotone decreasing. Since $\mathbf{K}$ is a regular cone and $0_B \leq_B \rho(Sx_n, Sx_{n+1})$ for all $n \in N$, there exists $r \geq_B 0_B$ such that

$$\rho(Sx_n, Sx_{n+1}) \to r \text{ as } n \to +\infty. \quad (6)$$

We claim that

$$r = 0_B. \quad (7)$$

By contradiction, suppose that (7) not holds, that is, $r \neq 0_B$. Since $\mathbf{K}$ is a regular cone, it is also a normal cone. Let us denote by $L > 0$ the normal constant of $\mathbf{K}$. Let H be the set defined by:

$$\mathcal{H} := \left\{ s \in \mathrm{int}(\mathbf{K}) \mid \|s\|_B < \frac{\|r\|_B}{L} \right\}.$$

For every real number a with $0 < a < \dfrac{\|r\|_B}{L}$ and $s \in \mathrm{int}(\mathbf{K})$, it is clear that $\dfrac{as}{\|s\|_B} \in H$. Then, H is a non-empty set. Now, we claim that for every $s \in H$,

$$\Phi(s) \leq_B \rho(Sx_n, Sx_{n+1}), \ \forall\, n \in \mathbb{N}. \tag{8}$$

If otherwise, from (g) there is $s_0 \in H$ such that $\rho(Sx_m, Sx_{m+1}) <_B \Phi(s_0)$, for some $m \in N$. Since the sequence $\{\rho(Sx_n, Sx_{n+1})\}$ is decreasing, we have:

$$\rho(Sx_n, Sx_{n+1}) \leq_B \rho(Sx_m, Sx_{m+1}) <_B \Phi(s_0), \ \forall\, n \geq m.$$

This implies that $\rho(Sx_n, Sx_{n+1}) <_B \Phi(s_0), \ \forall\, n \geq m$. Letting $n \to +\infty$ in the above inequality and using (f) and (6), we get

$$0_B \leq_B r \leq_B \Phi(s_0) \ll s_0.$$

This implies from the normality of $\mathbf{K}$ that $\|r\|_B \leq L\|s_0\|_B$, i.e., $\|s_0\|_B \geq \frac{\|r\|_B}{L}$. This is a contradiction since $s_0 \in \mathcal{H}$. Then, (8) holds. Now, letting $n \to +\infty$ in (8), we obtain $\Phi(s) \leq_B r$, $\forall\, s \in \mathcal{H}$. Therefore, for every $s \in \mathcal{H}$, there exists $p(s) \in P$ such that $r = \Phi(s) + p(s)$. Now, since $\Phi(s) \in \mathrm{int}(\mathbf{K})$ (by (e) and since $s \neq 0_B$), we can write $0_B \leq_B p(s) \ll p(s) + \Phi(s)$, which implies that $0_B \ll p(s) + \Phi(s) = r$. This gives that $r \in \mathrm{int}(\mathbf{K})$. Then, $\Phi(r)$ is well defined. Letting $n \to +\infty$ in (5) and using the continuity of Φ, we obtain $r \leq_B r - \Phi(r)$, i.e., $-\Phi(r) \in \mathbf{K}$. Since we also have that $\Phi(r) \in \mathbf{K}$, by the definition of a cone, we get $\Phi(r) = 0_B$.

By (e), we obtain $r = 0_B$: a contradiction. We deduce that (7) holds. Therefore,

$$\rho(Sx_n, Sx_{n+1}) \to 0_B \text{ as } n \to +\infty. \tag{9}$$

Now, let us prove that $\{Sx_n\}$ is a Cauchy sequence in the K-metric space $(\mathbf{X}, \rho)$. We proceed by contradiction. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. By (f), there exists $c \in \mathbf{B}$ with $0_B \ll c$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \ \rho(Sx_{m(k)}, Sx_{n(k)}) >_B \Phi(c), \ \rho(Sx_{m(k)}, Sx_{n(k)-1}) \leq_B \Phi(c).$$

Now, we have:

$$\begin{aligned} \Phi(c) \quad & <_B \rho(Sx_{m(k)}, Sx_{n(k)}) \leq_B \rho(Sx_{m(k)}, Sx_{n(k)-1}) + \rho(Sx_{n(k)-1}, Sx_{n(k)}) \\ & \leq_B \Phi(c) + \rho(Sx_{n(k)-1}, Sx_{n(k)}), \end{aligned}$$

that is

$$\Phi(c) <_B \rho(Sx_{m(k)}, Sx_{n(k)}) \leq_B \Phi(c) + \rho(Sx_{n(k)-1}, Sx_{n(k)}).$$

Letting $k \to +\infty$ in the above inequality and using (9), we obtain:

$$\lim_{k \to +\infty} \rho(Sx_{m(k)}, Sx_{n(k)}) = \Phi(c). \tag{10}$$

On the other hand, we have

$$\begin{aligned}&\rho(Sx_{m(k)}, Sx_{n(k)})\\ &\leq_B \rho(Sx_{m(k)}, Sx_{m(k)+1}) + \rho(Sx_{m(k)+1}, Sx_{n(k)+1}) + \rho(Sx_{n(k)+1}, Sx_{n(k)})\end{aligned}$$

and

$$\begin{aligned}&\rho(Sx_{m(k)+1}, Sx_{n(k)+1})\\ &\leq_B \rho(Sx_{m(k)+1}, Sx_{m(k)}) + \rho(Sx_{m(k)}, Sx_{n(k)}) + \rho(Sx_{n(k)}, Sx_{n(k)+1}).\end{aligned}$$

Letting $k \to +\infty$ in the above inequalities and using (9) and (10), we obtain:

$$\lim_{k\to+\infty} \rho(Sx_{m(k)+1}, Sx_{n(k)+1}) = \Phi(c). \tag{11}$$

Putting $x = x_{m(k)}$ and $y = x_{n(k)}$ and applying the contractive condition (2), we get

$$\begin{aligned}\rho(Sx_{m(k)+1}, Sx_{n(k)+1}) &\leq_B \rho(Tx_{m(k)}, Tx_{n(k)})\\ &\leq_B \rho(Sx_{m(k)}, Sx_{n(k)}) - \Phi(\rho(Sx_{m(k)}, Sx_{n(k)})).\end{aligned}$$

Letting $k \to +\infty$ in the above inequality and using (10), (11) and the continuity of Φ, we obtain:

$$\Phi(c) \leq_B \Phi(c) - \Phi(\Phi(c)).$$

This implies that $\Phi(\Phi(c)) = 0_B$. From (e), we deduce that $\Phi(c) = 0_B$. Again, from (e), we deduce that $c = 0_B$, which is a contradiction, since c is supposed to be an element of $\mathrm{int}(\mathbf{K})$. Hence, we obtain that $\{Sx_n\}$ is Cauchy in the K-metric space $(\mathbf{X}, \rho)$.

Now, since $(\mathbf{X}, \rho)$ is a complete K-metric space, there exists $x^* \in \mathbf{X}$ such that

$$\lim_{n\to+\infty} Sx_n = x^*. \tag{12}$$

From (12) and the continuity of S, we get:

$$\lim_{n\to+\infty} S(Sx_n) = Sx^*. \tag{13}$$

Now, by the triangular inequality, we obtain:

$$\rho(Sx^*, Tx^*) \leq_B \rho(Sx^*, S(Sx_{n+1})) + \rho(S(Tx_n), T(Sx_n)) + \rho(T(Sx_n), Tx^*). \tag{14}$$

On the other hand, we have:

$$Sx_n \to x^*, \quad Tx_n \to x^* \quad \text{as } n \to +\infty.$$

Since S and T are compatible maps, this implies that

$$\lim_{n\to+\infty} \rho(S(Tx_n), T(Sx_n)) = 0_B. \tag{15}$$

Using the continuity of T, we obtain from (12):

$$\lim_{n\to+\infty} \rho(T(Sx_n), Tx^*) = 0_B. \tag{16}$$

Now, combining (13), (15) and (16) and letting $n \to +\infty$ in (14), we get:

$$\rho(Sx^*, Tx^*) \leq_B 0_B,$$

i.e., $\rho(Sx^*, Tx^*) = 0_B$. This implies that $Sx^* = Tx^*$ and x^* is a coincidence point of T and S. This makes end to the proof. $\square$

In the next theorem, we omit the continuity and the compatibility hypotheses of T and S and we consider other conditions that assure the existence of a coincidence point.

At first we introduce the following definition.

Definition 2.4. *Let $(\mathbf{X}, \preceq, \rho)$ be a partially ordered K-metric space. We say that $(\mathbf{X}, \preceq, \rho)$ is regular if and only if the following condition holds:*
If $\{x_n\}$ is a non-decreasing sequence in $\mathbf{X}$ w.r.t. $\preceq$ such that $\rho(x_n, x) \to 0_B$ as $n \to +\infty$, $x \in \mathbf{X}$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 2.5. *Let $(\mathbf{X}, \preceq)$ be a partially ordered set and suppose that there exists a K-metric ρ in $\mathbf{X}$ such that the K-metric space $(\mathbf{X}, \rho)$ is complete and $\mathbf{K}$ is a regular cone such that $\rho(x, y) \in \operatorname{int}(\mathbf{K})$ for all $x, y \in \mathbf{X}$, $x \neq y$. Suppose that $(\mathbf{X}, \preceq, \rho)$ is regular. Let $T, S : \mathbf{X} \to \mathbf{X}$ be such that*
(a) *$T\mathbf{X} \subseteq S\mathbf{X}$,*
(b) *$S\mathbf{X}$ is a closed subspace of $(\mathbf{X}, \rho)$,*
(c) *T is S-non-decreasing w.r.t. $\preceq$.*
Suppose that $\rho(Tx, Ty) \leq_B \rho(Sx, Sy) - \Phi(\rho(Sx, Sy))$, for all $x, y \in \mathbf{X}$ for which $Sy \preceq Sx$, where $\Phi : \operatorname{int}(\mathbf{K}) \cup \{0_B\} \to \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ is continuous with
(d) *$\Phi(t) = 0_B \Leftrightarrow t = 0_B$,*
(e) *$\Phi(t) \ll t$ for all $t \in \operatorname{int}(\mathbf{K})$,*
(f) *either $\Phi(t) \leq_B \rho(x, y)$ or $\rho(x, y) \leq_B \Phi(t)$, for $t \in \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ and $x, y \in \mathbf{X}$.*
Suppose also that there exists $x_0 \in \mathbf{X}$ such that $Sx_0 \preceq Tx_0$. Then, T and S have a coincidence point $x^* \in \mathbf{X}$, i.e., $Tx^* = Sx^*$.

Proof. Following the proof of Theorem 2.3, since $\{Sx_n\}$ is a Cauchy sequence in the closed subspace $S\mathbf{X}$, there exists $y^* = Sx^*$, $x^* \in \mathbf{X}$ such that

$$Sx_n \to y^* = Sx^* \text{ as } n \to +\infty. \tag{17}$$

On the other hand, from the proof of Theorem 2.3, we know that $\{Sx_n\}$ is a non-decreasing sequence w.r.t. $\preceq$. Since $(\mathbf{X}, \preceq, \rho)$ is regular, it follows from (17) that

$$Sx_n \preceq Sx^*, \ \forall\, n \in \mathbb{N}.$$

Then, we can apply the considered contractive condition for $x = x^*$ and $y = x_n$. We get:

$$\rho(Tx^*, Sx_{n+1}) = \rho(Tx^*, Tx_n) \leq_B \rho(Sx^*, Sx_n) - \Phi(\rho(Sx^*, Sx_n)).$$

Letting $n \to +\infty$ in the above inequality and using (17) and the continuity of Φ, we obtain:

$$\rho(Tx^*, Sx^*) \leq_B -\Phi(0_B) = 0_B.$$

This implies that $\rho(Tx^*, Sx^*) = 0_B$, i.e., $Tx^* = Sx^*$. Hence x^* is a coincidence point of T and S. This makes end to the proof. $\square$

Now, we give existence and uniqueness theorem of a common fixed point of T and S.

Theorem 2.6. *In addition to the hypotheses of Theorem 2.3, suppose that for every $(x, y) \in \mathbf{X} \times \mathbf{X}$, there exists $u \in \mathbf{X}$ such that $Tx \preceq Tu$ and $Ty \preceq Tu$. Then, T and S have a unique common fixed point, that is, there exists a unique $z \in \mathbf{X}$ such that $z = Tz = Sz$.*

Proof. From Theorem 2.3, the set of coincidence points is non-empty. We shall show if $x^*$and $y^*$are coincidence points, that is, if $Sx^* = Tx^*$and $Sy^* = Ty^*$, then

$$Sx^* = Sy^*. \tag{18}$$

By assumption, there is $u_0 \in \mathbf{X}$ such that

$$Tx^* \preceq Tu_0 \quad \text{and} \quad Ty^* \preceq Tu_0. \tag{19}$$

Choose $u_1 \in \mathbf{X}$ such that $Su_1 = Tu_0$. Then, similarly as in the proof of Theorem 2.3, we can inductively define the sequence $\{Su_n\}$ by:

$$Su_{n+1} = Tu_n,\ \forall\, n \in \mathbb{N}. \tag{20}$$

Now,

$$Tx^* \preceq Tu_0 \Rightarrow Sx^* \preceq Su_1 \Rightarrow Tx^* \preceq Tu_1 \Rightarrow Sx^* \preceq Su_2 \Rightarrow \cdots$$

By induction, we have:

$$Sx^* \preceq Su_n,\ \forall\, n \in \mathbb{N}^*.$$

The same inequality holds if we replace x^* by y^*. Then, for all $n \in \mathbb{N}^*$, we have:

$$Sx^* \preceq Su_n, \quad Sy^* \preceq Su_n. \tag{21}$$

Now, from (21) and the considered contractive condition, we obtain:

$$\begin{aligned} \rho(Su_{n+1}, Sx^*) \qquad &\leq_B \rho(Tu_n, Tx^*) \\ &\leq_B \rho(Su_n, Sx^*) - \Phi(\rho(Su_n, Sx^*)) \leq_B \rho(Su_n, Sx^*). \end{aligned} \tag{22}$$

This proves that $\{\rho(Su_{n+1}, Sx^*)\}$ is a decreasing sequence. Since $\mathbf{K}$ is a regular cone, there is $r \geq 0_B$ such that

$$\rho(Su_{n+1}, Sx^*) \to r \text{ as } n \to +\infty. \tag{23}$$

Now, suppose that $r \neq 0_B$. As in the proof of Theorem 2.3, one can show that $r \in \operatorname{int}(\mathbf{K})$. Then, $\Phi(r)$ is well defined. Letting $n \to +\infty$ in (22) and using (23) and the continuity of Φ, we obtain $r \leq_B r - \Phi(r)$. This implies that $\Phi(r) = 0_B$ and thus $r = 0_B$. The same case holds if we replace x^* by y^*. Then, we have

$$\rho(Su_{n+1}, Sx^*) \to 0_B \quad \text{ and } \quad \rho(Su_{n+1}, Sy^*) \to 0_B. \tag{24}$$

From (24) and the uniqueness of the limit, it follows that $Sx^* = Sy^*$. Then, (18) holds.

Since $Su_n \to Sx^*$ and $Tu_n = Su_{n+1} \to Sx^*$ as $n \to +\infty$, from the compatibility of T and S, we obtain:

$$\lim_{n\to+\infty} \rho(T(Su_n), S(Tu_n)) = 0_B. \tag{25}$$

Now, let us denote $z := Sx^*$. By the triangular inequality, we have:

$$\rho(Sz, Tz) \leq_B \rho(Sz, S(Tu_n)) + \rho(S(Tu_n), T(Su_n)) + \rho(T(Su_n), Tz).$$

Letting $n \to +\infty$ in the above inequality, using (25) and the continuity of T and S, we obtain that $\rho(Sz, Tz) \leq_B 0_B$, that is, $Sz = Tz$ and z is a coincidence point of T and S. With $y^* = z$ and from (18), we have $z = Sx^* = Sz = Tz$. This proves that z is a common fixed point of T and S.

Now, let us prove that such a point is unique. Suppose that p is another common fixed point of T and S, i.e., $p = Sp = Tp$. This implies that p is a coincidence point of T and S. From (18), we have that $Sp = Sz$. Hence, we get $p = Sp = Sz = z$. Then, the uniqueness of the common fixed point is proved. This makes end to the proof. □

Now, by considering $S = I_X : \mathbf{X} \to \mathbf{X}$ the identity mapping ($I_X x = x$ for all $x \in \mathbf{X}$), we obtain immediately the following results.

The following result is an immediate consequence of Theorem 2.3.

Corollary 2.7. *Let $(\mathbf{X}, \preceq)$ be a partially ordered set and suppose that there exists a K-metric ρ in $\mathbf{X}$ such that the K-metric space $(\mathbf{X}, \rho)$ is complete and $\mathbf{K}$ is a regular cone such that $\rho(x, y) \in \operatorname{int}(\mathbf{K})$ for all $x, y \in \mathbf{X}$, $x \neq y$. Let $T : \mathbf{X} \to \mathbf{X}$ be such that*
(a) *T is a continuous mapping,*
(b) *T is non-decreasing w.r.t. $\preceq$.*
Suppose that

$$\rho(Tx, Ty) \leq_B \rho(x, y) - \Phi(\rho(x, y)),$$

for all $x, y \in \mathbf{X}$ for which $y \preceq x$, where $\Phi : \operatorname{int}(\mathbf{K}) \cup \{0_B\} \to \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ is continuous with
(c) *$\Phi(t) = 0_B \Leftrightarrow t = 0_B$,*
(d) *$\Phi(t) \ll t$ for all $t \in \operatorname{int}(\mathbf{K})$,*
(e) *either $\Phi(t) \leq_B \rho(x, y)$ or $\rho(x, y) \leq_B \Phi(t)$, for $t \in \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ and $x, y \in \mathbf{X}$.*
Suppose also that there exists $x_0 \in \mathbf{X}$ such that $x_0 \preceq Tx_0$. Then, T has a fixed point $x^* \in \mathbf{X}$, i.e., $Tx^* = x^*$.

The next result is an immediate consequence of Theorem 2.5.

Corollary 2.8. *Let $(\mathbf{X}, \preceq)$ be a partially ordered set and suppose that there exists a K-metric ρ in $\mathbf{X}$ such that the K-metric space $(\mathbf{X}, \rho)$ is complete and $\mathbf{K}$ is a regular cone such that $\rho(x, y) \in \operatorname{int}(\mathbf{K})$ for all $x, y \in \mathbf{X}$, $x \neq y$. Suppose that $(\mathbf{X}, \preceq, \rho)$ is regular. Let $T : \mathbf{X} \to \mathbf{X}$ be a non-decreasing mapping w.r.t. $\preceq$. Suppose that*

$$\rho(Tx, Ty) \leq_B \rho(x, y) - \Phi(\rho(x, y)),$$

for all $x, y \in \mathbf{X}$ for which $y \preceq x$, where $\Phi : \operatorname{int}(\mathbf{K}) \cup \{0_B\} \to \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ is continuous with
(a) *$\Phi(t) = 0_B \Leftrightarrow t = 0_B$,*
(b) *$\Phi(t) \ll t$ for all $t \in \operatorname{int}(\mathbf{K})$,*
(c) *either $\Phi(t) \leq_B \rho(x, y)$ or $\rho(x, y) \leq_B \Phi(t)$, for $t \in \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ and $x, y \in \mathbf{X}$.*
Suppose also that there exists $x_0 \in \mathbf{X}$ such that $x_0 \preceq Tx_0$. Then, T has a fixed point $x^* \in \mathbf{X}$, i.e., $Tx^* = x^*$.

Finally, the following result follows immediately from Theorem 2.6.

Corollary 2.9. *In addition to the hypotheses of Corollary 2.7, suppose that for every $(x, y) \in \mathbf{X} \times \mathbf{X}$, there exists $u \in \mathbf{X}$ such that $Tx \preceq Tu$ and $Ty \preceq Tu$. Then, T has a unique fixed point, that is, there exists a unique $x^* \in \mathbf{X}$ such that $x^* = Tx^*$.*

3. An application

Consider the integral equation:

$$u(x) = \int_0^T F(x, s, u(s))\, ds + g(x),\ x \in [0, T], \tag{26}$$

where $T > 0$.

The purpose of this section is to give an existence theorem for the solution of (26) by using the obtained result given by Corollary 2.8.

Previously, we consider the space $\mathbf{X} = C(I, \mathbb{R})(I = [0, T])$ of continuous functions defined in I and taking values in $\mathbb{R}$. Let $\mathbf{B} = \mathbb{R}^2$ and $\mathbf{K} \subset \mathbf{B}$ be the cone defined by:

$$\mathbf{K} = \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0, y \geq 0\}.$$

We endow $\mathbf{X}$ with the K-metric $\rho : \mathbf{X} \times \mathbf{X} \to \mathbf{B}$ defined by:

$$\rho(u, v) = \left(\sup_{x \in I} |u(x) - v(x)|, \sup_{x \in I} |u(x) - v(x)| \right),\ \forall\, u, v \in \mathbf{X}.$$

It is clear that $(\mathbf{X}, \rho)$ is a complete K-metric space. Now, we endow $\mathbf{X}$ with the partial order $\preceq$ given by:

$$u, v \in \mathbf{X}, \quad u \preceq v \Leftrightarrow u(x) \leq v(x),\ \forall\, x \in I.$$

It is easy to check the following properties:

- $\mathbf{K}$ is regular,
- $\rho(u, v) \in \operatorname{int}(\mathbf{K}),\ \forall\, u, v \in \mathbf{X},\ u \neq v.$

Let $\varphi : \operatorname{int}(\mathbf{K}) \cup \{0_B\} \to [0, +\infty)$ be a function satisfying the following properties:

- φ is continuous,
- $\varphi(t) = 0 \Leftrightarrow t = 0_B$,
- $\varphi(t) < \min(t_1, t_2)$, for all $t = (t_1, t_2) \in \operatorname{int}(\mathbf{K})$.

Now, the function $\Phi : \operatorname{int}(\mathbf{K}) \cup \{0_B\} \to \operatorname{int}(\mathbf{K}) \cup \{0_B\}$ defined by:

$$\Phi(t) = (\varphi(t), \varphi(t)),\ \forall\, t \in \operatorname{int}(\mathbf{K}) \cup \{0_B\},$$

satisfies all the required hypotheses by Corollary 2.8.

Note that in this considered case, it is proved in [32] that $(\mathbf{X}, \preceq, \rho)$ is regular.

Now, we are ready to prove the following theorem.

Theorem 3.1. *Suppose that the following hypotheses hold:*
(i) $F : I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous;*
(ii) *for all* $a, b \in \mathbb{R}$, *we have:*

$$a \leq b \Rightarrow F(x, s, a) \leq F(x, s, b),\ \forall\, x, s \in I;$$

(iii) *for all* $a, b \in \mathbb{R}$ *with* $a \leq b$,

$$|F(x, s, b) - F(x, s, a)| \leq \frac{1}{T}[|a - b| - \varphi(|a - b|, |a - b|)],$$

for all $x, s \in I$;
(iv) *there exists $u_0 \in C(I, \mathbb{R})$ such that for all $x \in I$, we have:*

$$u_0(x) \leq \int_0^T F(x, s, u_0(s))\, ds + g(x).$$

Then, the integral equation (26) admits a solution $u^* \in C(I, \mathbb{R})$.

Proof. Let $T : \mathbf{X} \to \mathbf{X}$ be the mapping defined by:

$$Tu(x) = \int_0^T F(x, s, u(s))\, ds + g(x),\ \forall\, x \in I,$$

for all $u \in \mathbf{X}$. From (ii), it follows immediately that T is a non-decreasing mapping w.r.t. $\preceq$. Now, for all $u, v \in \mathbf{X}$ with $v \preceq u$, from (iii), for all $x \in I$, we have:

$$\begin{aligned}
|Tu(x) - Tv(x)| &= \left| \int_0^T [F(x, s, u(s)) - F(x, s, v(s))]\, ds \right| \\
&\leq \int_0^T |F(x, s, u(s)) - F(x, s, v(s))|\, ds \\
&\leq \frac{1}{T} \int_0^T [|u(s) - v(s)| - \varphi(|u(s) - v(s)|, |u(s) - v(s)|)]\, ds \\
&\leq \frac{1}{T} \int_0^T [\|u - v\|_\infty - \varphi(\|u - v\|_\infty, \|u - v\|_\infty)]\, ds \\
&= \|u - v\|_\infty - \varphi(\|u - v\|_\infty, \|u - v\|_\infty),
\end{aligned}$$

where $\|\alpha - \beta\|_\infty = \sup_{x \in I} |\alpha(x) - \beta(x)|$ for all $\alpha, \beta \in \mathbf{X}$. This implies that

$$\|Tu - Tv\|_\infty \leq \|u - v\|_\infty - \varphi(\rho(u, v)).$$

Hence, we obtain:

$$\rho(Tu, Tv) \leq_B \rho(u, v) - \Phi(\rho(u, v)),$$

for all $u, v \in \mathbf{X}$ with $v \preceq u$. Moreover, from (iv), there exists $u_0 \in \mathbf{X}$ such that $u_0 \preceq Tu_0$. Then, all the required hypotheses by Corollary 2.8 are satisfied and T admits a fixed point $u^* \in \mathbf{X}$, that is, u^* is a solution to (26). □

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