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AN INVERSE SOURCE PROBLEM IN NEAR ZERO FREQUENCY SOUNDING OF LAYERED MEDIA

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Abstract. Geophysicists have long known that near-zero frequency data is insensitive to the variations of the index of refraction in a medium. Using the contraction mapping principle we are able to prove this fact for layered media. Moreover, the method of proof extracts from the data the part which is sensitive to the variation of the refractive index. We illustrate this sensitivity by reconstructing an approximate refractive index. The approximation is necessary due to the ill-posedness in the reconstruction problem.

Key Words and Phrases: Frequency sounding, contraction mapping principle, asymptotic expansion, inverse Riccati equations.

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1. Introduction

We consider an inverse source problem for the differential equation

$$\frac{\partial y}{\partial x}(x,s) + \frac{2s}{1+sx} y(x,s) = a(x), \quad 0 \le s \le \epsilon, \tag{1}$$

for some $\epsilon > 0$ small. The source $a(x)$ is unknown and the boundary data

$$y(0,s) = 0, \tag{2}$$

$$y(1,s) = \varphi(s) \tag{3}$$

are available for small values of s near zero. This problem occurs in the frequency sounding of layered media when we only have knowledge of near zero frequency data.

The method of frequency sounding was introduced in geophysical prospecting of the Earth's crust in the early fifties by Tikhonov [18] and Cagniard [6]. An explosion is set off at the surface and the response is recorded afterwards. One is to determine the sound speed of the inner structure of the Earth from these recordings. Due to its applications, the problem has been extensively studied both on the mathematical and on the engineering front and the reference below are by no means exhaustive. For example, by reducing the wave equation to the Helmholtz equation, the problem reduces to finding the refraction coefficient from boundary data, see [16], [5], [3], [4], [15],[9]. In [17] the frequency sounding problem is reduced to an inverse source problem for a first order quadratic differential equation. All of the above methods above use knowledge of high frequency data. By contrast we consider the problem for near zero frequency data starting from the model in [17].

In Sections 1 and 2 we describe the reduction of the frequency sounding problem to an inverse problem for (1). Namely, in Section 1 we describe the reduction from the acoustic wave model to an intermediate Riccati differential equation. In Section 2, by asymptotic methods (as frequency goes to zero), we show that near zero frequency data in the intermediate problem is not sensitive to variations of the sound speed.

In Section 4 we consider the linearized inverse problem (which is justified by the asymptotic result in Section 3) and point out the equivalence of the inverse source problem to the inversion of the Laplace transform, which guarantees uniqueness. However our inverse problem is severely ill-posed: In an example we perturb the coefficient of $y(x, \tilde{s})$ in (26), for which the corresponding inverse problem exhibits a large class (co-dimension three) non-uniqueness.

We regularize the problem by seeking its solution in a finite dimensional space of polynomials or wavelets (see Section 5). We show that the first $n-1^{st}$ derivatives of the data uniquely determines the best L^2- polynomial approximation of the source $a(x)$, see Theorem 5.1. In the end some numerical experiments are conducted with simulated data.

2. Connection with the inverse acoustic problem of layered media

This section briefly describe the steps in [17] to reduce an inverse acoustic problem in a layered media to an inverse Riccati equation.

We start with the pulsed acoustic wave equation in layered media

$$c^{-2}(z)U_{tt}(z,t) - U_{zz}(z,t) = 0,\ z > 0,\ t > 0, \tag{4}$$

$$U(z,0) = U_z(z,0) = 0, \tag{5}$$

$$U(0,t) = \delta(t). \tag{6}$$

We further assume that the medium is homogeneous past its Epstein layer ([7]): i.e. $c(z) = c_0$ for $z > L$, for some depth $L > 0$. Motivated by our technique, we work under the assumption that the pressure wave is bounded at all times $t > 0$:

$$|U(z,t)| \leq M\ \forall t > 0, \tag{7}$$

Assume the model given in (4) through (7), for a given the depth of the Epstein layer L, and a given speed $c(z) = c_0$ for $z > L$. *Knowing the derivative of the pressure at the surface $U_z(0,t) = \Psi(t),\ t > 0$ one is to find the sound speed $c(z)$ in $(0, L)$.*

Let

$$\tilde{u}(z,\sigma) = \int_0^\infty e^{-\sigma t} U(z,t)dt,\ z > 0,\ \sigma \geq 0,$$

be the Laplace transform of U solution of (4) through (7). By introducing the dimensionless variables $x = z/L,\ s = \sigma L/c_0$, and denoting $n(z) = c_0/c(z)$ and

$$u(x,s) = \tilde{u}(Lx, c_0 s/L),$$

we arrive to the dimensionless equation

$$\frac{\partial w}{\partial x}(x,s) = w^2(x,s) - s^2 n^2(x), \ 0 < x < 1 \tag{8}$$

$$w(1,s) = s, \ s \geq 0, \tag{9}$$

where the *admittance function* w is given by

$$w(x,s) = -\frac{\frac{\partial u}{\partial x}(x,s)}{u(x,s)}.$$

Note that solutions of (8) are positive. Otherwise, they have negative local minima, where they should also be concave down. Consequently, the admittance function w is well defined.

Following the normalization, the Laplace transform of the measured data

$$\varphi(\sigma) = \int_0^\infty e^{-\sigma t} U_x(0,t)dt,$$

gives

$$w(0,s) = \frac{\partial u}{\partial x}(0,s) = L\varphi(s).$$

In this paper we assume the data are available only for $0 \leq s < \epsilon$.

Finally, since we like to solve initial value problems in the positive x direction, for

$$g(x,s) = w(1-x,s),$$

we obtain

$$\begin{cases} \frac{\partial g}{\partial x}(x,s) = s^2 m^2(x) - g^2(x,s), \ 0 \leq x \leq 1 \\ g(0,s) = s, \end{cases} \tag{10}$$

where

$$m(x) = \frac{c_0}{c(L(1-x))}. \tag{11}$$

3. Asymptotic expansion as $s \to 0^+$

In this section we study the asymptotic expansion (for $s \to 0$) of solutions of (10) and show that the main order term is independent of $m(x)$ in (11).

Theorem 3.1. *Consider the problem (10), for some coefficient* m *integrable on* $[0,1]$.

There exists an $\bar{s} > 0$*, such that for each* $s \in [0, \overline{s}]$ *the solution* $x \to g(x,s)$ *of* (10) *exists throughout* $0 \leq x \leq 1$ *and the following asymptotic formula holds:*

$$g(x,s) = \frac{s}{1+sx} + s^2 y(x,s), \tag{12}$$

where $y \in C([0,1] \times [0,\bar{s}])$ *is given by*

$$y(x,s) = \int_0^x m^2(t)dt + O(s), \quad \text{as } s \to 0^+.$$

Proof. We seek solutions g of (10) by using the ansatz (12). The initial value problem for $y(x,s)$ is

$$\begin{cases} \frac{\partial y}{\partial x}(x,s) = m^2(x) - \frac{2s}{1+sx} y(x,s) - s^2 y^2(x,s), \ 0 \le x \le 1, \\ y(0,s) = 0. \end{cases} \tag{13}$$

The function $y(x,s)$ is a solution to (13) if and only if $g(x,s)$ solves the family of Riccati problems in (10).

For each s, the solution to (13) exists and is unique for x near 0. Using the Contraction Mapping Principle, we show that the solution to (13) exists on the entire $[0,1]$ for all $s \in [0,\bar{s}]$ and $\bar{s}$ sufficiently small. Let

$$\phi(x) = \int_0^x m^2(t)\,dt. \tag{14}$$

Note that $x \mapsto \phi(x)$ is increasing, hence $0 = \phi(0) \le \phi(x) \le \phi(1)$. By integrating both sides of (13) and taking into account (14), we obtain the equivalent Volterra integral equation

$$y(x,s) = V[y](x,s), \tag{15}$$

where

$$V[y](x,s) = \phi(x) - s \cdot \int_0^x \left(\frac{2}{1+st} y(t,s) + s y^2(t,s) \right) \, dt \tag{16}$$

is a Volterra operator acting on the space of continuous function $C([0,1]\times[0,\bar{s}])$ with the L^∞-norm, for some $\bar{s}$ to be specified later on. More precisely we restrict the domain of V to the closed ball $B(\phi;R)$ centered at ϕ with the radius R:

$$B(\phi;R) := \{ y \in C([0,1]\times[0,\bar{s}]) : \sup_{[0,1]\times[0,\bar{s}]} \|y(x,s) - \phi(x)\| \le R \}. \tag{17}$$

We will show that, for $\bar{s}$ sufficiently small, $V\colon\ B(\phi;R) \to B(\phi;R)$ is contractive. By the Contraction Mapping Principle V has a unique fixed point in $B(\phi;R)$, which is also a solution of (13).

We first show that $V\{B(\phi;R)\} \subset B(\phi;R)$. Let $y \in B(\phi;R)$ be arbitrary. Then for all $(x,s) \in [0,1]\times[0,\bar{s}]$ we get

$$|y(t,s)| \le R + M, \tag{18}$$

where for simplicity we denote $M = \phi(1)$. For all $(x,s) \in [0,1]\times[0,\bar{s}]$ the following estimate holds:

$$\begin{aligned} |\ V[y](x,s) - \phi(x)\ | = &\ s\Big| \int_0^x \frac{2}{1+st} y(t,s) + s y^2(t,s)\ dt\ \Big| \\ \le &\ s\Big[\int_0^x \frac{2}{1+st} |y(t,s)|\ dt + s \int_0^x |y(t,s)|^2\ dt \Big] \\ \le &\ s\Big[2\int_0^1 |y(t,s)|\ dt + s \int_0^1 (|y(t,s)|)^2\ dt \Big]. \end{aligned} \tag{19}$$

It follows from (19) and (18) that

$$|\ V[y](x,s) - \phi(x)\ | \le s \cdot [2(R+M) + s(R+M)^2] \tag{20}$$

The right hand side of (20) can be bounded by R,

$$s \cdot [2(R+M) + s(R+M)^2] \leq R, \tag{21}$$

provided that $0 \leq s \leq \sigma$, where

$$\sigma = \frac{R}{(R+M)(\sqrt{R+1}+1)}. \tag{22}$$

Next, we show that V is contractive. Letting $\tilde{y} \in B(\phi; R)$ and taking into account (18) and (21), we have

$$\begin{aligned} |V[y](x,s) - V[\tilde{y}](x,s)| &\leq s \int_0^x \frac{2 + s|y(t,s)| + s|\tilde{y}(t,s)|}{1+st} |y(t,s) - \tilde{y}(t,s)| dt \\ &\leq 2s[1 + s(R+M)] \int_0^x |y(t,s) - \tilde{y}(t,s)| dt \\ &\leq 2s\sqrt{R+1} \cdot \int_0^x |y(t,s) - \tilde{y}(t,s)| dt. \end{aligned} \tag{23}$$

By taking the maximum in $x \in [0,1]$ and $s \in [0, \sigma]$, we get

$$\|V[y] - V[\tilde{y}]\|_\infty \leq 2s\sqrt{R+1} \cdot \|y - \tilde{y}\|_\infty \tag{24}$$

By further constraining s so that $s < \bar{\sigma}$ where

$$\bar{\sigma} = \frac{1}{2\sqrt{R+1}} \tag{25}$$

we get that V is a contraction. Thus, let $h = min\{\sigma, \bar{\sigma}\}$, then $0 < \bar{s} < h$. Moreover, let $\hat{y}$ be a fixed point of (15), then by (18) we get

$$\hat{y}(x,s) = \int_0^x m^2(t) dt + O(s), \quad as\ s \to 0^+. \qquad \square$$

4. The linearized inverse problem

Since solutions $y(x,s)$ as in (12) are bounded as $s \to 0^+$, we get that, for $\tilde{s} = 2s$ sufficiently small, solutions of (1) are asymptotically close to solution of the linear equation

$$\begin{cases} \frac{\partial y}{\partial x}(x,\tilde{s}) + \tilde{s} y(x,\tilde{s}) = a(x), \ 0 \leq x \leq 1, \\ y(0,\tilde{s}) = 0. \end{cases} \tag{26}$$

We consider the linearized inverse source problem: *Assume that* $y(x,\tilde{s})$ *solves* (26) *for some unknown* $a(x)$*. Given* $y(1,\tilde{s})$ *find* $a(x)$*.*

We note first that the inverse source problem formulated above is equivalent to the inversion of the Laplace transform for $b(x)$ where:

$$b(1-x) = a(x), \ 0 \leq x \leq 1. \tag{27}$$

Indeed, the solution to the 1st order differential equation is

$$y(x,\tilde{s}) = \int_0^x a(t) e^{-\tilde{s}(x-t)}\, dt \tag{28}$$

so that the data

$$y(1,\tilde{s}) = \int_0^\infty \tilde{b}(t)e^{-\tilde{s}t}dt \tag{29}$$

is exactly the Laplace transform of $\tilde{b}(x)$, where

$$\tilde{b}(x) = \begin{cases} b(x), & x \in [0,1] \\ 0, & x > 1 \end{cases} = \begin{cases} a(1-x), & x \in [0,1] \\ 0 & x > 1. \end{cases}$$

Therefore the inverse source problem has unique solution, solution given by the inversion of its Laplace transform. Since we only know the Laplace transform for $s \in [0, \bar{s}]$ with $\bar{s}$ small, this is a problem of analytic continuation which is severely ill-posed.

To further emphasize the ill-posedness character of our inverse problem for (26) consider the following example.

Example. Let $y(x,s)$ be the solution of the problem

$$\begin{cases} \frac{\partial y}{\partial x}(x,s) + q(x,s)y(x,s) = a(x), \ s \in [0,\epsilon), \\ y(0,s) = 0, \end{cases} \tag{30}$$

where the coefficient $q(x,s) = \frac{2s}{1+sx}$ is $O(\epsilon^2)$-close to $2s$:

$$|q(x,s) - 2s| = \left| \frac{2s}{1+sx} - 2s \right| \le \frac{2\epsilon^2}{1-\epsilon},$$

for sufficiently small $\epsilon > 0$ and $x \in [0,1]$. The data $y(1,s)$ satisfies:

$$(1+s)^2 y(1,s) = \int_0^1 a(t)dt + 2s\int_0^1 a(t)tdt + s^2 \int_0^1 a(t)t^2 dt.$$

Therefore, the problem is to find the source $a(x)$ from knowing its first three moments $\int_0^1 a(t)t^j dt$, for $j = 0,1,2$. This example exhibits a co-dimension three non-uniqueness.

5. Regularized solutions

We regularize the problem by restricting the class of sources $a(x)$ to some finite dimensional spaces: polynomials or finite expansion in a wavelet base.

5.1. **The best least square polynomial approximations and the Hilbert matrix.**

Recall that $y(x,s)$ denote the solutions to the differential equation (26):

$$\begin{cases} \frac{\partial y}{\partial x}(x,\tilde{s}) + \tilde{s}y(x,\tilde{s}) = a(x), \ 0 \le x \le 1, \\ y(0,s) = 0, \end{cases}$$

where for simplicity we denote s for $\tilde{s}$.

We show next that the first n-th derivatives of the data $y(1,s)$ at frequency $s = 0$ determine the best least square polynomial approximation of $a(x)$.

Theorem 5.1. *The first n derivatives $\frac{\partial^k y}{\partial s^k}(1,s)|_{s=0}$, ($k = 0,1,\ldots n$) of the data uniquely determine the best least square n-th degree polynomial approximation to $a(x)$.*

Proof. It is easy to see that the solution y to (26) satisfies

$$e^{sx}y(x,s) = \int_0^x a(t)e^{st}dt.$$

In particular for $x = 1$ the function

$$e^{s}y(1,s) = \int_0^1 a(t)e^{st}dt. \tag{31}$$

is analytic in s. Let b_k denote the k^{th} Taylor coefficient, i.e.

$$e^{s}y(1,s) = \sum_{k=0}^{\infty} \frac{b_k}{k!}s^k. \tag{32}$$

Using the left hand side of (31) we have

$$b_k = \int_0^1 a(t)t^k dt. \tag{33}$$

We show next that if $a(x)$ were a polynomial, say

$$a(x) = \alpha_0 + \alpha_1 x \ + \ldots + \alpha_n x^n, \tag{34}$$

then its coefficients are uniquely determined by the first n derivatives of the data at $s = 0$, by solving (for the unknown $\alpha' s$) the linear system

$$\sum_{j=0}^{n} \frac{\alpha_j}{k+j+1} = b_k. \tag{35}$$

Indeed, if we plug (32) into the left hand side of (31) and (34) into the right hand side of (31) we get

$$\sum_{k=0}^{\infty} \frac{b_k}{k!}s^k = \sum_{k=0}^{n} \alpha_k \int_0^1 t^k e^{st}dt. \tag{36}$$

By identifying the coefficients of the first n powers of s we are lead to the linear system (35).

Note that $H = (\frac{1}{j+k-1})_{1\le k,j\le n+1}$ is the Hilbert matrix and (35) can be written as a matrix equation:

$$H \cdot \vec{\alpha} = \vec{b}$$

where $\vec{\alpha} = (\alpha_0, ..., \alpha_n)^T, \vec{b} = (b_0, ..., b_n)^T$.

On the other hand let us consider the best least square polynomial approximation of the function $a(x)$:

$$\min_{\alpha_0,\ldots,\alpha_n\in\mathbb{R}} \int_0^1 |a(x) - \sum_{j=0}^{n} \alpha_j x^j|^2 dx.$$

The quadratic form

$$Q(\alpha_0,...,\alpha_n) = \int_0^1 |a(x) - \sum_{j=0}^{n} \alpha_j x^j|^2 dx = \int_0^1 a^2(x)dx - 2\sum_{j=0}^{n} \alpha_j b_j \cdots$$
$$+ \sum_{j=0}^{n} \frac{\alpha_j^2}{2j+1} + 2 \sum_{0\le i<j\le n} \frac{\alpha_i \alpha_j}{i+j+1}$$

has a minimum where $\frac{\partial Q}{\partial \alpha_k}(\alpha_0, ..., \alpha_n) = 0$. A simple calculation leads to the linear system for $\alpha_0, ..., \alpha_n$:

$$\sum_{j=0}^{n} \frac{\alpha_j}{j+k+1} = \int_0^1 a(x)x^k dx \tag{37}$$

By comparing the right hand side of (37) with (33) we see that $\alpha_0, ..., \alpha_n$ also solves the system $H \cdot \vec{\alpha} = \vec{b}$. □

5.2. **Finite expansion in a wavelet basis.** In this method, $a(x)$ is expanded in a wavelet basis and the approximation truncates the expansion. We determine one scale $a_k(x)$ at a time for $x \in [0, 1]$, i.e.

$$a(x) = \sum_{i=0}^{K} a_i(x) = \sum_{i=0}^{K} \sum_{j=1}^{2^i} \alpha_{i,j} g_{i,j}(x),$$

where

$$g_{i,j}(x) = \begin{cases} 1, \frac{j-1}{2^i} \le x \le \frac{j}{2^i}, \\ 0, elsewhere. \end{cases} \tag{38}$$

The choice of K pre-determines the number of intervals for a piecewise approximation of $a(x)$, see Figure 2.

6. Numerical Results

In this section we illustrate the source reconstructions from simulated data using the two approximation methods described in sections 5.1 and, respectively, 5.2. In both cases we simulated the data for the source $a(x)$ given by

$$\begin{cases} 4.5|x - \frac{1}{3}| + 0.25 & if\ x \in [\frac{1}{6}, \frac{1}{2}] \\ 1 & otherwise. \end{cases} \tag{39}$$

By using the polynomial approximation method in section 5.1, for the cubic respectively quartic polynomials, we find:

$$P_3(x) = 1.2381 - 4.5064x + 10.1806x^2 - 6.0176x^3,$$

and

$$P_4(x) = 1.2824 - 5.3927x + 14.1690x^2 - 12.2217x^3 + 3.1021x^4.$$

In the Figure 1 we show the source $a(x)$ versus the two polynomial approximations $w(x)$.

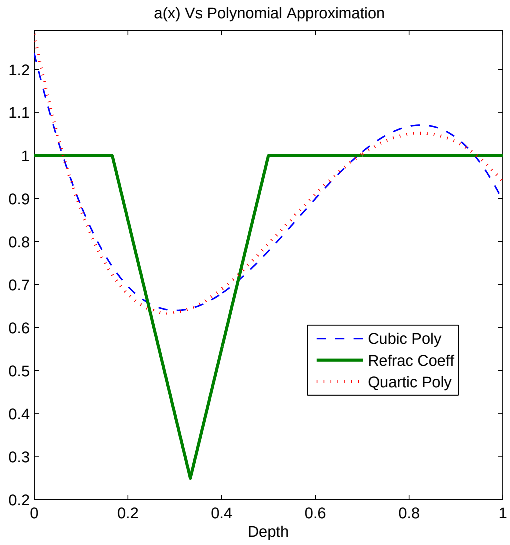

FIGURE 1. Targeted refraction $a(x)$ (*Solid*) and its reconstructed polynomial approximations (*dashed and dotted*).

For the same $a(x)$ as in (39) we use the wavelet approximation method in section 5.2 to get the approximation

$$w(x) = \sum_{i=0}^{3} \sum_{j=1}^{2^i} \alpha_{i,j} g_{i,j}(x),$$

where $g_{i,j}$'s are the basis functions in (38) and the computed coefficients are:

$\alpha_{0,1} = 0.8740,$
$\alpha_{1,1} = -0.0864, \quad \alpha_{1,2} = 0.0864,$
$\alpha_{2,1} = 0.1715, \quad \alpha_{2,2} = -0.3086, \quad \alpha_{2,3} = 0.1090, \quad \alpha_{2,4} = 0.0282,$
$\alpha_{3,1} = 0.0215, \quad \alpha_{3,2} = 0.0268, \quad \alpha_{3,3} = -0.2487, \quad \alpha_{3,4} = 0.3290,$
$\alpha_{3,5} = -0.0962, \quad \alpha_{3,6} = -0.0505, \quad \alpha_{3,7} = 0.0061, \quad \alpha_{3,8} = 0.0121.$

The Figure 2 below shows the wavelet expansion approximation $w(x)$ versus the source $a(x)$.

7. CONCLUSIONS

We considered an inverse source problem for a first order Riccati equation. This problem has been motivated by the geophysics problem of recovering the refractive index of a layered medium from the near zero frequency data.

The near zero frequency data in the geophysics problem turned out to be insensitive to the variation of the refractive index. We studied first the asymptotic as frequency $s \to 0$ in order to extract the sensitive part. Furthermore, the asymptotic showed that it is sufficient to work with the linear model.

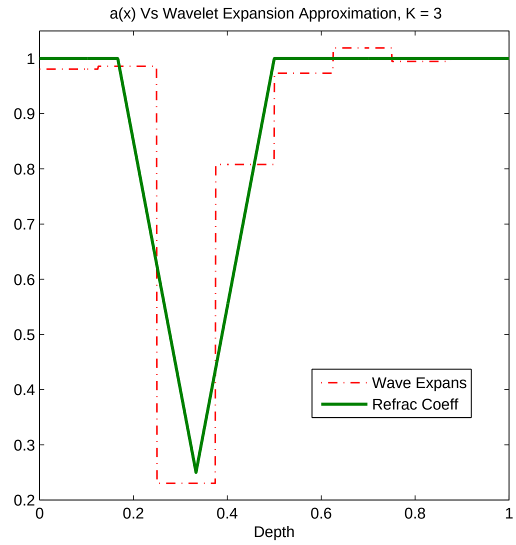

FIGURE 2. Targeted refraction $a(x)$ (*solid*) and its wavelet expansion approximation (*dashed-dotted*).

The inverse problem in the linear model is equivalent to inverting a Laplace transform from near zero frequency. This is a problem of unique analytic continuation which is severely ill-posed. We regularized the inverse problem by seeking an approximate source in the finite dimensional space of polynomials of a given degree or as a finite wavelet expansion.

When looking at polynomial approximations, we showed that the first n derivatives of the data uniquely determined the best least square n-th degree polynomial approximation of the unknown coefficient.

We perform a series of numerical experiments with simulated data. The results confirmed the severity of the ill-posedness in the problem.

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