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CHOICES OF VARIABLE STEPS OF THE CQ ALGORITHM FOR THE SPLIT FEASIBILITY PROBLEM

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Abstract. We consider the CQ algorithm, with choice of steps introduced by Yang (J. Math. Anal. Appl. 302 (2005), 166-179), for solving the split feasibility problem (SFP): find $x \in C$ such that $Ax \in Q$, where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ matrix. We convert the SFP to an equivalent convexly constrained nonlinear system of finding a zero in C of an inverse strongly monotone operator, which enables us to introduce new convergent iterative algorithms. Two restrictive conditions of Yang (i.e., the boundedness of Q and the full column rank of A) are completely removed in our new algorithms.
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1. Introduction

The problem under consideration in this paper is formulated as finding a point $\hat{x}$ satisfying the property:

$$\hat{x} \in C \quad \text{and} \quad A\hat{x} \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subset of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ matrix (i.e., a linear operator from $\mathbb{R}^n$ into $\mathbb{R}^m$). Problem (1.1) is called

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by Censor and Elfving [5] the split feasibility problem (SFP) and has been proved very useful in dealing with a variety of signal processing [4].

Various algorithms have been invented to solve the SFP (1.1) (see [2, 8, 9, 10, 11, 12, 13] and reference therein). In particular, Byrne introduced his CQ algorithm:

$$x_{k+1} = P_C(x_k - \tau A^t(I - P_Q)Ax_k), \tag{1.2}$$

where A^t is the transpose of A and the stepsize τ is a fixed real number in $(0, 2/\varrho(A^tA))$, with $\varrho(A^tA)$ being the spectral radius of A^tA. Compared with the original algorithm in [5], the CQ algorithm (1.2) is more easily performed because it dose not involve matrix inverses. However, to implement the CQ algorithm, one has to compute or estimate the value of $\varrho(A^tA)$, which is not always possible in practice. To overcome this drawback, Yang [13] suggested, instead of the constant-step, the following variable-step:

$$\tau_k = \frac{\varrho_k}{\|A^t(I - P_Q)Ax_k\|}, \tag{1.3}$$

where (ϱ_k) is a sequence of positive real numbers such that

$$\sum_{k=0}^{\infty} \varrho_k = \infty, \quad \sum_{k=0}^{\infty} \varrho_k^2 < \infty. \tag{1.4}$$

With this choice of the steps, the computation of $\varrho(A^*A)$ is avoided and hence one need not know *a prior* any information of the spectral radius of A^*A. Yang proved the convergence of the modified algorithm to a solution of the SFP provided that (i) Q is a bounded subset; and (ii) A is a matrix with full column rank.

However, Yang's conditions imposed on Q and A are restrictive and they may exclude many important cases. The purpose of this note is to relax these conditions by requiring no boundedness on Q nor full column rank of A. Our success of doing so is due to an equivalent formulation of the SFP (1.1) as finding a zero of a weakly co-coercive operator, which enables us to invent new ways to select variable-steps for the CQ algorithm (1.2), and in particular, we are able to remove Yang's boundedness condition on Q and the full column rank assumption on A.

The paper is structured as follows. In the next section, after the concept of projections, we introduce the class of weakly inverse strongly monotone operators and prove that the operator $f := A^t(I - P_Q)A$ is inverse strongly maximal monotone. Moreover, we state our novel idea to solve the SFP (1.1) which is, roughly speaking, to equivalently reformulate it as a convexly constrained nonlinear system of finding a zero of f in C. In Section 3, we introduce two new iterative algorithms and prove their convergence to a solution of the SFP (1.1) (if any). In particular, we completely remove Yang's conditions (i) and (ii) described above.

2. Preliminaries

Let C be a nonempty closed convex subset of $\mathbb{R}^n$. Denote by P_C the projection from $\mathbb{R}^n$ onto C; that is, $P_Cx = \arg\min_{y \in C} \|x - y\|^2, \quad x \in \mathbb{R}^n$. The projection operator has the following properties (see [6]).

Lemma 2.1. *Let P_C be the projection operator onto C. Then for any $x, y \in \mathbb{R}^n$,*

(i) *P_C is nonexpansive, i.e., $\|P_C x - P_C y\| \le \|x - y\|$,*
(ii) *P_C is firmly nonexpansive, i.e., $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle$,*
(iii) *$I - P_C$ is firmly nonexpansive.*

We need definitions of some classes of nonlinear mappings.

Definition 2.2. Let f be an operator from $\mathbb{R}^n$ into $\mathbb{R}^n$.

(i) f is said to be monotone if $\langle f(x) - f(y), x - y\rangle \ge 0$ for all $x, y \in \mathbb{R}^n$. Moreover, a monotone operator f is maximal monotone if its graph is not properly contained in the graph of any other monotone operator.
(ii) f is said to be inverse strongly monotone (ism, for short) (also known as co-coercive) if there exists a constant $\gamma > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \ge \gamma \|f(x) - f(y)\|^2, \quad x, y \in \mathbb{R}^n;$$

and in this case, f is also referred to as γ-ism.
(iii) f is said to be weakly inverse strongly monotone (wism, for short) (or weakly co-coercive) if there exists a continuous positive function $g : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$ such that

$$\langle f(x) - f(y), x - y\rangle \ge g(x, y)\|f(x) - f(y)\|^2, \quad x, y \in \mathbb{R}^n.$$

It is not hard to find that the co-coerciveness implies the weak co-coerciveness, while the latter implies the monotonicity. Moreover it is worth noting that these implications are proper.

Let Q be a nonempty closed convex subset of $\mathbb{R}^m$ and let A be an $m \times n$ matrix, and set

$$A^{-1}(Q) = \{x \in \mathbb{R}^n : Ax \in Q\} \quad \text{and} \quad f^{-1}(0) = \{x \in \mathbb{R}^n : f(x) = 0\}.$$

Our idea to solve the SFP (1.1) is to transform it equivalently to a problem of finding a zero of a maximal monotone operator. To this end, we define a mapping f by setting

$$f := A^t(I - P_Q)A. \tag{2.1}$$

Lemma 2.3. *The operator f defined in (2.1) is maximal monotone on $\mathbb{R}^n$ and is also $\frac{1}{\rho(A^tA)}$-ism.*

Proof. For all $x, y \in \mathbb{R}^n$, we have, as $I - P_Q$ is firmly nonexpansive, that

$$\begin{aligned}\langle f(x) - f(y), x - y\rangle &= \langle (I - P_Q)Ax - (I - P_Q)Ay, Ax - Ay\rangle \\ &\ge \|(I - P_Q)Ax - (I - P_Q)Ay\|^2 \\ &\ge \frac{1}{\|A\|^2}\|A^t(I - P_Q)Ax - A^t(I - P_Q)Ay\|^2 = \frac{1}{\rho(A^tA)}\|f(x) - f(y)\|^2.\end{aligned}$$

This shows that f is $\frac{1}{\rho(A^tA)}$-ism. f is also maximal monotone since it is Lipschitz continuous on $\mathbb{R}^n$. □

The following lemma is crucial to our main argument in the next section since it asserts that to solve the SFP (1.1) is equivalent to finding a zero of f in C, namely,

$$\text{Find } x \in C \text{ such that } f(x) = 0. \tag{2.2}$$

Lemma 2.4. *Let f be given as in (2.1). If $A^{-1}(Q)$ is nonempty, then $f^{-1}(0) = A^{-1}(Q)$.*

Proof. *It is straightforward that if $Ax \in Q$, then $f(x) = 0$. This verifies that $A^{-1}(Q) \subset f^{-1}(0)$. To see the converse relation $f^{-1}(0) \subset A^{-1}(Q)$, let $z \in A^{-1}(Q)$ be fixed and take $x \in f^{-1}(0)$. Since $I - P_Q$ is firmly nonexpansive, it follows that*

$$\begin{aligned}\|(I - P_Q)Ax\|^2 &= \|(I - P_Q)Ax - (I - P_Q)Az\|^2 \\ &\le \langle Ax - Az, (I - P_Q)Ax\rangle \\ &= \langle x - z, A^*(I - P_Q)Ax\rangle = \langle x - z, f(x)\rangle = 0.\end{aligned}$$

This implies that $Ax \in Q$ and the proof is complete. □

It is proved that f is co-coercive (see [3]), and hence weakly co-coercive. In other words, the SFP is a special case of the problem for finding a zero point of a weakly co-coercive operator onto a closed convex subset.

3. Iterative algorithms and their convergence analysis

By Lemma 2.4, we know that solving the SFP (1.1) is equivalent to solving the system (2.2), i.e., finding a zero of f in C. We will propose two iterative algorithms to approximate a solution of problem (2.2). However, for more generality, we consider the convexly constrained nonlinear equation

$$f(x) = 0, \quad x \in C, \tag{3.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is weakly inverse strongly monotone (wism) and C is a nonempty closed convex subset of $\mathbb{R}^n$. We assume that (3.1) is consistent (i.e., solvable).

Algorithm 3.1. Choose an arbitrary initial guess x_1.

Step 1. Given x_k, if $f(x_k) = 0$, stop; otherwise compute:

$$x_{k+1} = P_C(x_k - \tau_k f(x_k)), \tag{3.2}$$

where τ_k is defined as

$$\tau_k := \frac{\varrho_k}{\|f(x_k)\|}. \tag{3.3}$$

Step 2. If $x_{k+1} = x_k$, then stop; otherwise go to step 1.

Remark 3.2. Algorithm 3.1 was originally proposed by Auslender [1] for solving variational inequalities and recently reinvestigated by Yang [13].

We next present another variable-step for approximating solutions of problem (2.2).

Algorithm 3.3. Choose an arbitrary initial guess x_1.

Step 1. Given x_k, compute the next iteration:

$$x_{k+1} = P_C(x_k - \varrho_k f(x_k)), \tag{3.4}$$

where ϱ_k is a positive number to be selected appropriately.

Step 2. If $x_{k+1} = x_k$, then stop; otherwise go to step 1.

Remark 3.4. It is not hard to check that if the iteration above terminates within finite steps, then the current iteration must be a solution of the problem. So without loss of generality we assume that both algorithms generate an infinite iterative sequence.

The following elementary lemma play an important role in our convergence analysis.

Lemma 3.5. [7] *Let (ϵ_k) and (s_k) be positive real sequences. Assume that $\sum_k \epsilon_k < \infty$. If either (i) $s_{k+1} \leq (1+\epsilon_k)s_k$, or (ii) $s_{k+1} \leq s_k + \epsilon_k$, then the limit of (s_k) exists.*

Theorem 3.6. *If the condition (1.4) holds and $f : \mathbb{R}^n \to \mathbb{R}^n$ is weakly inverse strongly monotone, then the sequence (x_k) generated by Algorithm 3.1 converges to a solution of Eq. (3.1), that is, a point in $C \cap f^{-1}(0)$.*

Proof. Abbreviate $f_k = f(x_k)$ and let $z \in C \cap f^{-1}(0)$ be fixed. Using the fact that P_C is nonexpansive and that $z \in C$, we have from (3.2)

$$\|x_{k+1} - z\|^2 \leq \|(x_k - z) - \tau_k f_k\|^2. \tag{3.5}$$

Expanding the right-hand side of (3.5) yields

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 - 2\tau_k \langle f_k, x_k - z\rangle + \varrho_k^2. \tag{3.6}$$

Let us now estimate the second term of the right-hand side of (3.6). Since $f(z) = 0$ and f is monotone, it follows that $\langle f_k, x_k - z\rangle \geq \langle f(z), x_k - z\rangle = 0$. Hence, (3.6) implies that

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + \varrho_k^2. \tag{3.7}$$

We therefore can apply Lemma 3.5 to (3.7) to conclude that $(\|x_k - z\|)$ is convergent and hence $(\|x_k\|)$ is bounded.

We next show that $\lim_k f_k = 0$. The boundedness of (x_k) implies that there exists a closed ball $B_M = \{x \in \mathbb{R}^n : \|x\| \leq M\}$ containing z and (x_k). Since f is wism, there exists a continuous positive function $g : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$ such that

$$\langle f(x) - f(y), x - y\rangle \geq g(x, y)\|f(x) - f(y)\|^2, \quad x, y \in \mathbb{R}^n. \tag{3.8}$$

Since g is continuous and positive, there is $\delta > 0$ such that $g(x, y) \geq \delta$, for all $x, y \in C \cap B_M$. Consequently,

$$\langle f_k, x_k - z\rangle = \langle f_k - f(z), x_k - z\rangle \geq \delta\|f_k\|^2. \tag{3.9}$$

Combining (3.9) and (3.6) yields

$$2\delta\|f_k\| \leq \|x_k - z\|^2 - \|x_{k+1} - z\|^2 + \varrho_k^2,$$

which immediately implies that

$$2\delta \sum_{\ell=1}^{k} \varrho_\ell \|f_\ell\| \leq \|x_1 - z\|^2 + \sum_{\ell=1}^{n} \varrho_\ell^2.$$

Letting $k \to \infty$, we arrive at

$$\sum_{k=1}^{\infty} \varrho_k \|f_k\| < \infty. \tag{3.10}$$

Since $\sum_k \varrho_k = \infty$, it follows that $\liminf_k \|f_k\| = 0$. Thus to see $\lim_k \|f_k\| = 0$, it suffices to verify that the limit of $(\|f_k\|)$ exists. Taking into account the fact that f is $(1/\delta)$-Lipschitz continuous on $C \cap B_M$, we deduce that

$$\|f_{k+1} - f_k\| \leq \frac{1}{\delta}\|x_k - x_{k+1}\| = \frac{1}{\delta}\|x_k - P_C(x_k - \tau_k f_k)\| \leq \frac{\tau_k}{\delta}\|f_k\| = \frac{\varrho_k}{\delta},$$

where the last inequity uses the nonexpansiveness of the projection and the fact that $x_k = P_C x_k$. It turns out that

$$\begin{aligned}\|f_{k+1}\|^2 &= \|f_k\|^2 + \|f_{k+1} - f_k\|^2 + 2\langle f_k, f_{k+1} - f_k\rangle \\ &\leq \|f_k\|^2 + \|f_{k+1} - f_k\|^2 + 2\|f_k\|\|f_{k+1} - f_k\| \\ &\leq \|f_k\|^2 + \frac{\varrho_k^2}{\delta^2} + \frac{2\varrho_k}{\delta}\|f_k\| = \|f_k\|^2 + \sigma_k,\end{aligned}$$

where $\sigma_k = (\varrho_k/\delta)^2 + (2/\delta)\varrho_k\|f_k\|$. In view of (3.10) and (1.4), we have that $\sum_k \sigma_k < \infty$. Consequently, an application of Lemma 3.5 guarantees that the $\lim_k \|f_k\|$ exists; hence we must have $\lim_k f_k = 0$.

Finally we show that every cluster point of (x_k) is in the set $C \cap f^{-1}(0)$. So suppose that a subsequence (x_{k_j}) of (x_k) converges to a point $\hat{x}$. It is readily seen that $\hat{x} \in C \cap B_M$. Since f is continuous on $C \cap B_M$, we have that $f(\hat{x}) = \lim_{k\to\infty} f_{k_j} = \lim_{k\to\infty} f_k = 0$, that is, $\hat{x} \in f^{-1}(0)$. Note that $\lim \|x_k - z\|$ exists for all $z \in C \cap f^{-1}(0)$. In particular, we have that $\lim \|x_k - \hat{x}\|$ exists. Since, however, the subsequence (x_{k_j}) converges to $\hat{x}$, we must have $\lim \|x_k - \hat{x}\| = 0$. Therefore $x_k \to \hat{x}$. □

Theorem 3.7. *If the condition* (1.4) *holds and* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is inverse strongly monotone (ism), then the sequence* (x_n) *generated by Algorithm 3.3 converges to a solution of Eq. (3.1), namely, a point in* $C \cap f^{-1}(0)$.

Proof. Let again $f_k := f(x_k)$. We first show that $(\|x_k - z\|)$ is convergent for any fixed $z \in C \cap f^{-1}(0)$. Indeed, similarly to the derivation of (3.6), we have

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 - 2\varrho_k\langle f_k, x_k - z\rangle + \varrho_k^2\|f_k\|^2. \tag{3.11}$$

Since now f is ism, there exists $\delta > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \geq \delta\|f(x) - f(y)\|^2, \quad x, y \in \mathbb{R}^n. \tag{3.12}$$

It is clear that f is also $(1/\delta)$-Lipschitz continuous, so that

$$\|f_k\| = \|f_k - f(z)\| \leq (1/\delta)\|x_k - z\|. \tag{3.13}$$

Note that $\langle f_k, x_k - z\rangle \geq 0$. Substituting (3.13) into (3.11) yields

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + \left(\frac{\varrho_k}{\delta}\right)^2 \|x_k - z\|^2 = (1 + \sigma_k)\|x_k - z\|^2,$$

where $\sigma_k = (\varrho_k/\delta)^2$. It is readily seen that $\sum \sigma_k < \infty$ due to (1.4). By Lemma 3.5, we conclude that the sequence $(\|x_k - z\|)$ is convergent; in particular, (x_k) is bounded.

We next prove that $\lim_k \|f_k\| = 0$. Take $M > 0$ so that $\|x_k - z\| \leq M$ for all $k \in \mathbb{N}$. According to (3.13), we have

$$\|f_k\| \leq \frac{1}{\delta}\|x_k - z\| \leq \frac{M}{\delta}. \tag{3.14}$$

On the other hand, we deduce from (3.12) that

$$\langle f_k, x_k - z\rangle = \langle f_k - f(z), x_k - z\rangle \geq \delta\|f_k\|^2. \tag{3.15}$$

From (3.15), (3.11) and (3.14), it follows that

$$\begin{aligned} 2\delta\varrho_k\|f_k\|^2 &\leq 2\varrho_k\langle f_k, x_k - z\rangle \leq \|x_k - z\|^2 - \|x_{k+1} - z\|^2 + \varrho_k^2\|f_k\|^2 \\ &\leq \|x_k - z\|^2 - \|x_{k+1} - z\|^2 + \left(\frac{M}{\delta}\right)^2 \varrho_k^2, \end{aligned}$$

which immediately implies that

$$2\delta\sum_{j=1}^{k} \varrho_j\|f_j\|^2 \leq \|x_1 - z\|^2 + \left(\frac{M}{\delta}\right)^2 \sum_{j=1}^{k} \varrho_j^2.$$

Taking the limit by letting $k \to \infty$ in the last relation yields

$$\sum_{k=1}^{\infty} \varrho_k\|f_k\|^2 < \infty. \tag{3.16}$$

This together with the assumption $\sum_k \varrho_k = \infty$ particularly implies that $\liminf_k \|f_k\| = 0$. So to prove that $\lim_k \|f_k\| = 0$, it suffices to verify the existence of the $\lim_k \|f_k\|$. Actually, we have

$$\|f_{k+1} - f_k\| \leq \frac{1}{\delta}\|x_k - x_{k+1}\| = \frac{1}{\delta}\|x_k - P_C(x_k - \varrho_k f_k)\| \leq \frac{\varrho_k}{\delta}\|f_k\|,$$

where the last inequality uses the nonexpansiveness of projections and the fact that $x_k = P_C x_k$. By using the last inequalities, we have

$$\begin{aligned} \|f_{k+1}\|^2 &= \|f_k\|^2 + \|f_{k+1} - f_k\|^2 + 2\langle f_k, f_{k+1} - f_k\rangle \\ &\leq \|f_k\|^2 + \|f_{k+1} - f_k\|^2 + 2\|f_k\|\|f_{k+1} - f_k\| \\ &\leq \|f_k\|^2 + (\frac{\varrho_k}{\delta})^2\|f_k\|^2 + \frac{2\varrho_k}{\delta}\|f_k\|^2 \leq \|f_k\|^2 + \frac{M^2}{\delta^4}\varrho_k^2 + \frac{2\varrho_k}{\delta}\|f_k\|^2, \end{aligned}$$

where the last inequality uses (3.14). Setting $\sigma_k = (M/\delta^2)^2\varrho_k^2 + (2\varrho_k\|f_k\|^2)/\delta$, we have

$$\|f_{k+1}\|^2 \leq \|f_k\|^2 + \sigma_k. \tag{3.17}$$

It is clear that $\sum_n \sigma_n < \infty$ due to (3.16) and (1.4). We can therefore apply Lemma 3.5 to (3.17) to get the existence of the $\lim_k \|f_k\|$ and the proof is complete. □

The theorem below is an immediate consequence of Theorems 3.6 and 3.7 together with Lemma 2.4. We completely remove the boundedness of Q and the full column rank of A in Yang's result [13], as described in the Introduction.

Theorem 3.8. *Let $f = A^*(I - P_Q)A$ and let the condition* (1.4) *hold. Then the sequence (x_k) generated either by Algorithm 3.1 or by Algorithm 3.3 converges to a solution of the SFP* (1.1) *whenever its solution set is nonempty.*

Remark 3.9. To implement our algorithms, the closed convex subsets should be so simple that the projections onto them are easily calculated. So whether or not the convergence still holds for the relaxed version (see [12]) of the proposed algorithms is a subject deserving research.

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