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FIXED POINTS FOR PERTURBED CONTRACTIONS

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Abstract. It is proved a fixed point result for an operator of the form $T - F$, where T is a strict contraction and F is a nonlinear Lipschitz monotone operator.
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1. Introduction

Let E be a Banach space with the norm $\|\cdot\|_E$. If for an operator $T : E \to E$ there exist a real number $\alpha \in (0,1)$ such that

$$||T(x) - T(y)||_E \leq \alpha ||x - y||_E$$

for every $x,\ y \in H$, then will we say that T is a $\alpha-$contraction. Using the Banach Fixed Point Theorem, it is easy to observe that the following result holds:
Theorem 1.1. *Let $\alpha \in (0,1)$ and $T : E \to E$ be a $\alpha-$contraction. If $\beta \in (0, 1-\alpha)$ and $V : E \to E$ is a $\beta-$contraction, then the operator $T \pm V$ has an unique fixed point.*

Starting from this result, we are interested in finding operators $V : E \to E$, which are not strict contractions, but the perturbations $T - V$ or $T + V$ could have "the property of the unique fixed point". In this paper we give a positive answer to the previous problem, considering the case of strict contractions $T : H \to H$, where H is a real Hilbert space.

We prove that if $T : H \to H$ is a $\alpha-$contraction and $F : H \to H$ is a monotone Lipschitz operator, then the operator $T - F$ has an unique fixed point, giving a direct proof which uses the fact that $I - T + F$ is a strongly monotone Lipschitz operator(I is the identity of H).

The result can be regarded as a consequence of Theorem 3.9 (A. Granas and J. Dugundji [5], pp.62), which affirm that *if $f : H \to H$ is a Lipschitzian and strongly monotone map, then f is a homeomorphism of H onto itself.*

Also, the result follows from Theorem 3.6 (V. Berinde [1], pp. 73): *If K is a nonempty closed convex subset of a real Hilbert space and $T : K \to K$ is a generalized pseudocontractive Lipschitz operator, then T has a unique fixed point.*

2. The result

Let H be a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $T : H \to H$ be a $\alpha-$contraction, i.e.,

$$||T(x) - T(y)|| \leq \alpha||x - y|| \text{ for all } x, y \in H \text{ with } \alpha \in (0, 1).$$

Using the Schwartz inequality, we have

$$< (I - T)x - (I - T)y, x - y >= ||x - y||^2 - < T(x) - T(y), x - y >\geq$$
$$\geq ||x - y||^2 - ||T(x) - T(y)|| \cdot ||x - y||$$

for all $x, y \in H$.

Consequently $I - T$ is strongly monotone, i.e.,

$$< (I - T)x - (I - T)y, x - y >\geq (1 - \alpha)||x - y||^2 \text{ for all } x, y \in H, \tag{1}$$

with $1 - \alpha > 0$.

By a $L-$Lipschitz operator we mean an operator $F : H \to H$ for which there exists $L > 0$ such that

$$||F(x) - F(y)|| \leq L||x - y||,$$

for all $x,\ y \in H$. We say that $F : H \to H$ is a monotone operator if

$$< F(x) - F(y), x - y >\geq 0 \text{ for all } x, y \in H.$$

The main result of this paper is the following

Theorem 2.1. *Let $T : H \to H$ be a $\alpha-$contraction and $F : H \to H$ be a monotone, $L-$Lipschitz operator. Then the operator $T - F$ has an unique fixed point.*

Proof. Let us define the operator $A : H \to H$ by

$$Au = (I - T + F)u.$$

First, for every $x,\ y \in H$,we have

$$||A(x) - A(y)|| \leq ||x - y|| + ||T(x) - T(y)|| + ||F(x) - F(y)|| \leq$$
$$\leq (1 + \alpha + L)||x - y||,$$

so A is a $(1+\alpha+L)-$Lipschitz operator. Now, by using the strong monotony of $I-T$ from (1) and the monotony of F, we have

$$< A(x) - A(y), x - y >=$$
$$=< (I - T)x - (I - T)y, x - y > + < F(x) - F(y), x - y >\geq$$
$$\geq< (I - T)x - (I - T)y, x - y >\geq (1 - \alpha)||x - y||^2,$$

so A is strongly monotone as the sum of a strongly monotone operator with a monotone one. Now let us define, for $\gamma > 0$, the operator

$$S_\gamma : H \to H$$

given by

$$S_\gamma u = (I - \gamma A)u.$$

(The construction of the operator S_γ is usually. For example, see the proof of the Stampacchia Theorem in [2], chapter V). We have

$$||S_\gamma x - S_\gamma y||^2 =< x - \gamma Ax - (y - \gamma Ay), x - \gamma Ax - (y - \gamma Ay) >=$$
$$= ||x - y||^2 - 2\gamma < Ax - Ay, x - y > +\gamma^2 ||Ax - Ay||^2 \leq$$
$$\leq [1 - 2\gamma(1 - \alpha) + \gamma^2(1 + \alpha + L)^2] \cdot ||x - y||^2,$$

so

$$||S_\gamma x - S_\gamma y|| \leq \sqrt{1 - 2\gamma(1 - \alpha) + \gamma^2(1 + \alpha + L)^2} \cdot ||x - y||,$$

for all $x,\ y \in H$. Further, remark that if

$$\gamma \in \left(0, \frac{2(1 - \alpha)}{(1 + \alpha + L)^2}\right),$$

then S_γ is a strict contraction, because

$$\sqrt{1 - 2\gamma(1 - \alpha) + \gamma^2(1 + \alpha + L)^2} < 1$$

and consequently, S_γ has an unique fixed point in H. In other words, there exists an unique element $u^* \in H$ such that

$$u^* = S_\gamma u^*,$$

which is successive equivalent with

$$u^* = (I - \gamma A)u^* \Leftrightarrow u^* = u^* - \gamma Au^* \Leftrightarrow Au^* = 0.$$

Further,

$$Au^* = 0 \Leftrightarrow (I - T + F)u^* = 0 \Leftrightarrow u^* = (T - F)u^*,$$

thus u^* is the fixed point of $T - F$ and the proof of the theorem is complete.□

3. Remarks

1. In the proof of the Theorem 2.1 we construct for $\gamma > 0$ the operator

$$S_\gamma = I - \gamma A = I - \gamma(I - T + F) = (1 - \gamma)I + \gamma(T - F).$$

An operator $B : H \to H$ is called the *averaged operator* associated to an operator $N : H \to H$ if $B = (1 - \alpha)\, I + \alpha N$ for some $\alpha \in (0, 1)$. Consequently S_γ can be regarded as the averaged operator associated to $T - F$.

The notion of averaged operator appears firstly in works [6] of M.A. Krasnoselskii, [3] of F.E. Browder as the averaged operator associated to a nonexpansive operator and is developed as instrument in the study of the operators having multiple fixed points by R.E. Bruck and S. Reich in work [4]. Motivated by Krasnoselskii-Mann Theorem, the averaged operators are considered in the study of the operators having multiple fixed points.

2. In work [7], I.A. Rus presents some theoretical and applicative aspects regarding "the theory" of a metrical fixed point theorem. In the context of the paper [7], we study in the next part of the paper some aspects regarding the equation

$$(T - F)(x) = x.$$

Firstly, it is important to observe that for every $\gamma \in \left(0, \frac{2(1-\alpha)}{(1+\alpha+L)^2}\right)$, we have

$$u_n = [(1-\gamma)I + \gamma(T-F)]^n(x) \to u^*$$

(in the metric generated by the norm of H) for all $x \in H$, i.e. $S_\gamma = (1-\gamma)I + \gamma(T-F)$ is a Picard operator. Moreover

$$\|u_n - u^*\| = \left\|S_\gamma^n(x) - u^*\right\| = \|[(1-\gamma)I + \gamma(T-F)]^n(x) - u^*\| \leq$$
$$\frac{\theta^n}{1-\theta}\|x - \gamma x + \gamma(T-F)(x) - x\| = \frac{\gamma\theta^n}{1-\theta}\|(I - T + F)(x)\|$$

for all $x \in H$ and $n \geq 1$, where $\theta = \sqrt{1 - 2\gamma(1-\alpha) + \gamma^2(1+\alpha+L)^2} < 1$.

Now let x^* be the unique solution of the equation $T(x) = x$. We have

$$\|u^* - x^*\| = \|(T-F)(u^*) - T(x^*)\| \leq$$
$$\|T(u^*) - T(x^*)\| + \|F(u^*)\| \leq \alpha\|u^* - x^*\| + L\|u^*\| + \|F(0)\|.$$

Therefore we obtained the estimation

$$\|u^* - x^*\| \leq \frac{L}{1-\alpha}\|u^*\| + \frac{1}{1-\alpha}\|F(0)\|$$

and, consequently, if $F(0) = 0$ and $u^* \neq 0$, then

$$\frac{\|u^* - x^*\|}{\|u^*\|} \leq \frac{L}{1-\alpha}.$$

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