Fixed Point Theory, 12(2011), No. 2, 475-484
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINTS OF GENERALIZED ASYMPTOTIC CONTRACTIONS

SHYAM LAL SINGH*, S.N. MISHRA** AND RAJENDRA PANT***

*Govind Nagar, Rishikesh 249201 India.
E-mail: vedicmri@gmail.com

**Department of Mathematics, Walter Sisulu University
Mthatha 5117, South Africa.
E-mail: smishra@wsu.ac.za

***Department of Mathematics, Walter Sisulu University
Mthatha 5117, South Africa.
E-mail: pant.rajendra@gmail.com

Abstract. Following Suzuki's asymptotic contraction of Meir-Keeler type (ACMK), we obtain a coincidence theorem for a generalized ACMK for a pair of maps and derive some general fixed point theorems on metric spaces.
Key Words and Phrases: Coincidence point, fixed point, asymptotic contraction, asymptotic contraction of Meir-Keeler type, generalized asymptotic contraction of Meir-Keeler type.
2010 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

In 2003, Kirk [17] introduced the notion of *asymptotic contractions* on a metric space, and proved a fixed point theorem for this new class of maps.

Definition 1.1. ([17]). Let (X, d) be a metric space. A self-map T of X is an *asymptotic contraction* on X if there exists a continuous function φ from $[0, \infty)$ into itself and a sequence $\{\varphi_n\}$ of functions from $[0, \infty)$ into itself such that

- **(K1):** $\varphi(0) = 0$;
- **(K2):** $\varphi(r) < r$ for $r \in (0, \infty)$;
- **(K3):** $\{\varphi_n\}$ converges to φ uniformly on the range of d; and
- **(K4):** for x, $y \in X$ and $n \in \mathbb{N}$, $d(T^n x, T^n y) \le \varphi_n(d(x, y))$.

Theorem 1.1. ([17]). *Let (X, d) be a complete metric space and T a continuous asymptotic contraction on X with $\{\varphi_n\}$ and φ as in* Definition 1.1. *Assume that there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ of x is bounded, and that φ_n is continuous for $n \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover $\lim_n T^n x = z$ for all $x \in X$.*

Theorem 1.1 is an asymptotic version of Boyd and Wong [4] (see also [13], [17]). It has important outcomes in metric fixed point theory (see, for instance, [1], [2], [3], [5], [6], [7], [8], [9], [11], [12], [13], [15], [17], [19], [24], [27], [31], [32], [33], [34], [35], [36].)

Underlying the power and importance of this new class of maps, Briseid [5, 7] has observed that a continuous self-map of a compact metric space satisfying any one of the first 50 contractive conditions listed by Rhoades [25] is an asymptotic contraction. Wlodarczyk *et al.* [34] and [35] discussed some ideas for applications of the theory of asymptotic contractions in the analysis of set-valued dynamical systems.

The purpose of this paper is to present a brief review of numerous definitions and fixed point theorems which followed Kirk's asymptotic contractions. Further, we obtain coincidence and fixed point theorems for generalized asymptotic contractions of Meir-Keeler type on metric spaces.

2. Review of asymptotic contractions

Jachymski and Jòżwic [13] showed that the continuity of the map T is essential in Theorem 1.1. Further, they extended Theorem 1.1 and obtained a complete characterization of asymptotic contractions on a compact metric space.

Proof of Theorem 1.1 in [17] involves a sophisticated ultrapower technique. Aranđelović's attempt [1] (see also [2]) to provide a simple constructive proof of Theorem 1.1 was perfected by Jachymski [15].

Improving upon Theorem 1.1, Chen [9] obtained the following result under weaker assumptions. In this paper, R^+ denotes the set of all nonnegative real numbers.

Theorem 2.1. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a self-map satisfying the conditions* (C1)-(C3)*:*

- **(C1):** $d(T^n x, T^n y) \le \varphi_n(d(x, y))$ *for all* $x, y \in X$*, where* $\phi_n : R^+ \to R^+$ *and* $\lim_{n\to\infty} \varphi_n = \varphi$ *uniformly on any bounded interval* $[0, b]$*. Suppose that* φ *is upper semicontinuous and* $\varphi(t) < t$ *for* $t > 0$*.*
- **(C2):** *There exists a positive integer* n^* *such that* φ_{n^*} *is upper semicontinuous and* $\varphi_{n^*}(0) = 0$*.*
- **(C3):** *There exists* $x_0 \in X$ *such that the orbit* $O(x_0)$ *is bounded.*

Then T *has a unique fixed point* $x^* \in X$ *such that* $\lim_{n\to\infty} T^n x = x^*$ *for all* $x \in X$*.*

Aranđelović [2] presented a fixed point theorem of Kirk type unifying and generalizing fixed point theorems of Kirk [17], Jachymski and Jòżwic [13] and Chen [9].

As Theorem 2.1 does not guarantee the uniform convergence of the iterates of the map T (see also [3]), Reich and Zaslavski [24] obtained a convergence theorem [3, Theorem 1.3] for asymptotic contctions with some additional hypotheses. Further, modifying the condition (C2) a slightly, Sastry *et al.* [27] obtained effectively another version of Theorem 2.1.

Gerhardy [11] obtained a quantitative version of Theorem 1.1 by using the techniques of proof-mining (for proof-mining technique, one may refer to Kohlenbach and

Oliva [20]). Indeed, he introduced the following alternative definition of asymptotic contraction and obtained a fixed point theorem for the same.

Definition 2.1. ([11]). Let (X, d) be a metric space and T a self-map on X. The map T is called a *G-asymptotic contraction* on X if for each $b > 0$ there exists moduli $\eta^b : (0, b] \to (0, 1)$ and $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ and the following hold:

(G1): there exists a sequence of functions $\phi_n^b : (0, \infty) \to (0, \infty)$ such that for all $x, y \in X$, for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$,

$$b > d(x, y) \geq \varepsilon \Rightarrow d(T^n x, T^n y) \leq \phi_n^b(\varepsilon).d(x, y);$$

(G2): for each $0 < l \leq b$ the function $\beta_l^b := \beta^b(l, .)$ is a modulus of uniform convergence for each ϕ_n^b on $[l, b]$, i.e.,

$$\forall \varepsilon > 0 \; \forall s \in [l, b] \; \forall m, n \geq \beta_l^b(\varepsilon), \quad (|\phi_m^b(s) - \phi_n^b(s)| \leq \varepsilon); \text{ and}$$

(G3): defining $\phi^b := \lim_{n\to\infty} \phi_n^b$, then for each $0 < \varepsilon \leq b$, we have $\phi^b(s) + \eta^b(\varepsilon) \leq 1$ for each $s \in [\varepsilon, b]$.

Theorem 2.2. [11]. *Let (X, d) be a complete metric space and $T : X \to X$ a continuous G-asymptotic contraction, where $b > 0$ and η, β are given. If for some $x_0 \in X$, the sequence $\{x_n\}$ is bounded by b then T has a unique fixed point z and $\{x_n\}$ converges to z.*

Since the convergence to the fixed point of a continuous G-asymptotic contraction need not be monotone (see also [5, p. 367], [6, p. 18], [13]), Theorem 2.2 does not provide a full rate of convergence. Briseid [5] gives an explicit rate of convergence for the iteration sequence for modified G-asymptotic contractions, expressed in the relevant moduli and a bound on the sequence. Further, a characterization of Kirk's asymptotic contractions (cf. Definition 1.1), G-asymptotic contractions (cf. Definition 2.1) and modified G-asymptotic contractions [5, Definition 2.2] on bounded complete metric spaces has been discussed in [5] (see also [6], [7] and [8]).

Recently Kirk [18] introduced the following notion of an asymptotic pointwise contraction.

Definition 2.2. Let K be a weakly compact convex subset of a Banach space X. A map $T : K \to K$ is an asymptotic pointwise contraction if there exists a function $\alpha : K \to [0, 1)$ such that, for each integer $n \geq 1$,

$$||T^n x - T^n y|| \leq \alpha_n(x)||x - y|| \text{ for each } x, \; y \in K,$$

where $\alpha_n \to \alpha$ pointwise on K.

Kirk [18] also obtained a fixed point theorem for asymptotic pointwise contractions defined on a bounded closed convex subset of a super reflexive Banach space. The proof outlined there involves a sophisticated ultrapower technique, and also requires some additional assumptions.

Kirk and Xu [19] give a simple and elementary proof (in the sense that ultrapower methods are not needed) of the fact that an asymptotic pointwise contraction defined on a weakly compact convex set has a unique fixed point (with convergence of iterates). In conjunction with the concept of asymptotic pointwise contractions, Kirk and Xu

[19] introduced the the notion of pointwise asymptotically nonexpansive maps and pointwise eventually nonexpansive maps.

Definition 2.3. (cf. [19]). A map $T : K \to K$ is pointwise eventually nonexpansive if for each $x \in K$, there exists $n(x) \in \mathbb{N}$ such that $n \geq n(x)$,

$$||T^n x - T^n y|| \leq ||x - y||, \ x, \ y \in K.$$

Kirk and Xu [19] posed two questions regarding the existence of fixed points of eventually nonexpansive maps in reflexive Banach spaces having the fixed point property for nonexpansive maps.

Hussain and Khamsi [12] extend Definition 2.2 to a metric space as follows.

Definition 2.4. Let (X, d) be a metric space. A self-map $T : X \to X$ is an asymptotic pointwise map if there exists a map $\alpha_n : X \to [0, \infty)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x) d(x, y) \text{ for any } y \in X.$$

(i) If $\{\alpha_n\}$ converges pointwise to $\alpha : M \to [0, 1)$, then T is called asymptotic pointwise contraction.
(ii) If $\limsup_{n\to\infty} \alpha_n(x) \leq 1$, then T is called asymptotic pointwise nonexpansive.
(iii) If $\limsup_{n\to\infty} \alpha_n(x) \leq k$, with $0 < k < 1$, then T is called strongly asymptotic pointwise nonexpansive.

They [12] also extend the main results of Kirk [18] to metric spaces and discuss the case of multivalued maps as well.

Generalizing Banach contraction principle, Meir-Keeler [22] obtained the following theorem.

Theorem 2.3. *Let (X, d) be a complete metric space and T a self-map of X. Assume that for every $\varepsilon > 0$, there exists $\delta > 0$ such that*
(MK) $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$
for all x, $y \in X$. Then T has a unique fixed point.

Cho *et al.* [10], Jachymski [13], Lim [21], Park and Rhoades [23], Rus [26], Singh and Kumar [28], Suzuki [31, 32, 33] and many others obtained various generalizations of Theorem 2.3. For a good bibliography on the development of the condition (MK), one may refer to [26] and [28].

In [31], Suzuki combined the two ideas of Meir-Keeler contraction (MKC) and Kirk asymptotic contraction (KAC) and introduced the following notion of *asymptotic contraction of Meir-Keeler type* (see also [32], [33]).

Definition 2.5. Let (X, d) be a metric space. A self-map T of X is an *asymptotic contraction of Meir-Keeler type* (*ACMK* for short) if there exists a sequence φ_n of functions from $[0, \infty)$ into itself satisfying the following conditions:

- **(S1):** $\limsup \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;
- **(S2):** for each $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\varphi_\nu(t) \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$;
- **(S3):** $d(T^n x, T^n y) < \varphi_n(d(x, y))$ for all $n \in \mathbb{N}$ and x, $y \in X$ with $x \neq y$.

Inspired by Jachymski and Jòżwic [13, Lemma 4], Suzuki [31] obtained the following result.

Theorem 2.4. *Let (X, d) be a complete metric space and T an ACMK on X. Assume that T^l is continuous for some $l \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover, $\lim_n T^n x = z$ for all $x \in X$.*

Recently Singh and Pant [30] obtained the following generalization and extension of Theorems 1.1 and 2.4. In all that follows Y denotes an arbitrary nonempty set.

Theorem 2.5. *Let (X, d) be a complete metric space and T a self-map satisfying the conditions* (S1), (S2*) and the following:*

(S4): *$d(T^n x, T^n y) < \varphi_n(m(x, y))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$, where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.*

If T^k is continuous for some $k \in \mathbb{N}$ then T has a unique fixed point $z \in X$.

Moreover, $\lim_n T^n x = z$ for all $x \in X$.

Theorem 2.6. *Let (X, d) be a metric space and $T, f : Y \to X$ satisfying the conditions* (S1) *and the following:*

(S5): *$T(Y) \subseteq f(Y)$;*

(S6): *for each $\varepsilon > 0$ there exists $\delta > 0$ and $\mu \in \mathbb{N}$ such that $\varphi_\mu(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$;*

(S7): *$d(T^n x, T^n y) < \varphi_n(d(fx, fy))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute just at a coincidence point.

3. Generalized asymptotic contractions of Meir-Keeler type

First we present an extension of Suzuki's definition of ACMK (cf. Definition 2.2) for a pair of maps on an arbitrary nonempty set with values in a metric space.

Definition 3.1. Let (X, d) be a metric space and $T, f : Y \to X$. The map T will be called *a generalized asymptotic contraction of Meir-Keeler type* (in short, GACMK) with respect to f if the following hold:

(P1) $\limsup_n \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;

(P2) for each $\varepsilon > 0$ there exists $\delta > 0$ and $\mu \in \mathbb{N}$ such that $\varphi_\mu(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$;

(P3) $d(T^n x, T^n y) < \varphi_n(M(x, y))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $M(x, y) > 0$, where $M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\}$.

The following result for GACMK extends Theorems 2.4-2.6.

Theorem 3.1. *Let (X, d) be a metric space and $T, f : Y \to X$ such that $TY \subseteq fY$. Let T be a GACMK with respect to f. If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence point.*

Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute at a coincidence point.

Proof. Pick $x_0 \in Y$. Define a sequence $\{y_n\}$ by $y_n = Tx_n = fx_{n+1}$, $n = 0, 1, 2, ...$ We can do so since the range of f contains the range of T. First we show that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{3.1}$$

It initially holds if $x_0 = x_1$. In the other case of $x_0 \neq x_1$, we assume that $\alpha := \limsup_n d(Tx_n, Tx_{n+1}) > 0$. From the condition (P2), we can choose $\mu_1 \in \mathbb{N}$ satisfying $\varphi_{\mu_1}(d(Tx_0, Tx_1)) \leq d(Tx_0, Tx_1)$. By (P3) and (P1),

$$d(y_{\mu_1+1}, y_{\mu_1+2}) = d(Tx_{\mu_1+1}, Tx_{\mu_1+2}) < \varphi_{\mu_1}(M(x_1, x_2)) \leq M(x_1, x_2).$$

Then proceeding as in [30],

$$\begin{aligned}
\alpha : &= \lim_{n\to\infty} \sup d(Tx_{\mu_1+n}, Tx_{\mu_1+n+1}) \qquad (3.2)\\
&\leq \lim_{n\to\infty} \sup \varphi_n(M(Tx_{\mu_1}, Tx_{\mu_1+1})) \leq M(Tx_{\mu_1}, Tx_{\mu_1+1})\\
&= \max\{d(fx_{\mu_1+1}, fx_{\mu_1+2}), d(fx_{\mu_1+1}, Tx_{\mu_1+1}), d(fx_{\mu_1+2}, Tx_{\mu_1+2}),\\
&\quad \frac{1}{2}[d(fx_{\mu_1+1}, Tx_{\mu_1+2}) + d(fx_{\mu_1+2}, Tx_{\mu_1+1})\}\\
&= \max\{d(Tx_{\mu_1}, Tx_{\mu_1+1}), d(Tx_{\mu_1}, Tx_{\mu_1+1}), d(Tx_{\mu_1+1}, Tx_{\mu_1+2}),\\
&\quad \frac{1}{2}[d(Tx_{\mu_1}, Tx_{\mu_1+2}) + 0)\}\\
&= \max\{d(Tx_{\mu_1}, Tx_{\mu_1+1}), d(Tx_{\mu_1+1}, Tx_{\mu_1+2}),\\
&\quad \frac{1}{2}[d(Tx_{\mu_1}, Tx_{\mu_1+1}) + d(Tx_{\mu_1+1}, Tx_{\mu_1+2})\}\\
&= \max\{d(Tx_{\mu_1}, Tx_{\mu_1+1}), d(Tx_{\mu_1+1}, Tx_{\mu_1+2})\}.
\end{aligned}$$

If

$$\max\{d(Tx_{\mu_1}, Tx_{\mu_1+1}), d(Tx_{\mu_1+1}, Tx_{\mu_1+2})\} = d(Tx_{\mu_1+1}, Tx_{\mu_1+2}),$$

we have a contradiction. Therefore

$$\max\{d(Tx_{\mu_1}, Tx_{\mu_1+1}), d(Tx_{\mu_1+1}, Tx_{\mu_1+2})\} = d(Tx_{\mu_1}, Tx_{\mu_1+1})$$

and we conclude that $M(Tx_{\mu_1}, Tx_{\mu_1+1}) = d(Tx_{\mu_1}, Tx_{\mu_1+1})$.
By (3.2),

$$\begin{aligned}
d(Tx_{\mu_1+1}, Tx_{\mu_1+2}) &< \varphi_{\mu_1}(M(x_1, x_2)) \leq M(x_1, x_2)\\
&= \max\{d(fx_1, fx_2), d(fx_1, Tx_1), d(fx_2, Tx_2),\\
&\quad \frac{1}{2}[d(fx_1, Tx_2) + d(fx_2, Tx_1)]\}\\
&= \max\{d(Tx_0, Tx_1), d(Tx_0, Tx_1), d(Tx_1, Tx_2),\\
&\quad \frac{1}{2}[d(Tx_0, Tx_2) + 0]\}\\
&= \max\{d(Tx_0, Tx_1), d(Tx_1, Tx_2), \frac{1}{2}[d(Tx_0, Tx_1) + d(Tx_1, Tx_2]\}\\
&= d(Tx_0, Tx_1).
\end{aligned}$$

So $\alpha < d(Tx_0, Tx_1)$. By a similar argument, we obtain $\alpha < d(Tx_k, Tx_{k+1})$ for all $k \in \mathbb{N} \cup \{0\}$. Hence $\{d(Tx_n, Tx_{n+1}\}$ converges to α.

Since $0 < \alpha < d(Tx_0, Tx_1) < \infty$, there exists $\delta_2 > 0$ and $\mu_2 \in \mathbb{N}$ such that

$$\varphi_{\mu_2}(t) \leq \alpha \text{ for all } t \in [\alpha, \alpha + \delta_2].$$

We choose $\mu_3 \in \mathbb{N}$ with $d(Tx_{\mu_3}, Tx_{\mu_3+1}) < \alpha + \delta_2$. Then we have

$$d(Tx_{\mu_2+\mu_3}, Tx_{\mu_2+\mu_3+1}) < \varphi_{\mu_2} d(Tx_{\mu_3}, Tx_{\mu_3+1}) \leq \alpha,$$

a contradiction. This proves that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Now we show that $\{y_n\}$ is Cauchy sequence. Suppose $\{y_n\}$ is not Cauchy. Then there exists $\beta > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers, such that for all $n \leq m_k < n_k$,

$$d(y_{m_k}, y_{n_k}) \geq \beta \textit{ and } d(y_{m_k}, y_{n_k-1}) < \beta.$$

By the triangle inequality,

$$d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}).$$

Making $k \to \infty$, $d(y_{m_k}, y_{n_k}) < \beta$. Thus $d(y_{m_k}, y_{n_k}) \to \beta$ as $k \to \infty$. By (P2),

$$\begin{aligned} d(y_{m_k+n}, y_{n_k+n}) &= d(Tx_{m_k+n}, Tx_{n_k+n}) \\ &< \varphi_n(M(x_{m_k}, x_{n_k})) \\ &= \varphi_n(\max\{d(fx_{m_k}, fx_{n_k}), d(fx_{m_k}, Tx_{m_k}), d(fx_{n_k}, Tx_{n_k}), \\ &\quad \frac{1}{2}[d(fx_{m_k}, Tx_{n_k}) + d(fx_{n_k}, Tx_{m_k})]\}) \\ &= \varphi_n(\max\{d(y_{m_k-1}, y_{n_k-1}), d(y_{m_k-1}, y_{m_k}), d(y_{n_k-1}, y_{n_k}), \\ &\quad \frac{1}{2}[d(y_{m_k-1}, y_{n_k}) + d(y_{n_k-1}, y_{m_k})]\}). \end{aligned}$$

Making $k \to \infty$, $\beta \leq \varphi_n(\beta) < \beta$, a contradiction, and the sequence $\{y_n\}$ is Cauchy.

Suppose $f(Y)$ is complete. Then $\{y_n\}$ being contained in $f(Y)$ has a limit in $f(Y)$. Call it z. Let $u \in f^{-1}z$. Then $fu = z$. Using (P3),

$$\begin{aligned} d(Tu, Tx_n) &\leq \varphi(M(u, x_n)) \\ &= \varphi(\max\{d(fu, fx_n), d(fu, Tu), d(fx_n, Tx_n), \\ &\quad \frac{1}{2}[d(fu, Tx_n) + d(fx_n, Tu)]\}). \end{aligned}$$

Making $n \to \infty$, $d(Tu, z) \leq \varphi(d(Tu, z)) < d(Tu, z)$. Therefore $Tu = z = fu$. Now if $Y = X$ and the pair (T, f) commutes just at u then $Tfu = fTu$ and $TTu = Tfu = fTu = ffu$. In view of (A2), it follows that

$$\begin{aligned} d(Tu, TTu) &\leq \varphi(M(u, Tu)) \\ &= \varphi(\max\{d(fu, fTu), d(fu, Tu), d(fTu, TTu), \\ &\quad \frac{1}{2}[d(fu, TTu) + d(fTu, Tu)]\}) \\ &< d(Tu, TTu), \end{aligned}$$

a contradiction. Therefore $TTu = Tu$ and $fTu = TTu = Tu = z$.

In case $T(Y)$ is a complete subspace of X, the condition $TY \subseteq fY$ implies that the sequence $\{y_n\}$ converges in $f(Y)$, and the previous argument works. The uniqueness of the common fixed point follows easily. □

The following result is evident as the condition (S7) implies (P3).

Corollary 3.1. *Theorem* 2.6.

Corollary 3.2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a map satisfying* (S1), (S6) *and the following:*
(P4) $d(T^n x, T^n y) < \varphi_n(M_1(x, y))$
for all $n \in \mathbb{N}$ *and* $x, y \in X$ *with* $M_1(x, y) > 0$,
where, $M_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$.
If T^k *is continuous for some* $k \in \mathbb{N}$, *then* T *has a unique fixed point* $z \in X$.

Proof. It comes from Theorem 3.1 when $Y = X$ and f is the identity map. □

The following example illustrate the usefulness of Theorem 3.1.

Example 3.1. Let $X = [0, 1]$ be endowed with the usual metric d. Let $T : X \to X$ be such that

$$Tx = \begin{cases} 1 & \text{if } x = 0, \\ \frac{x}{2} & \text{if } x \in (0, 1]. \end{cases}$$

Jachymski [13] has shown that T satisfies all the conditions of Theorem 1.1 except the continuity with an appropriate choice of φ_n. Since the map T is without fixed point, it does not satisfy the hypotheses of Theorem 2.4 as well. We consider a map $f : X \to X$ such that

$$fx = x^2.$$

Assume that

$$\varphi_n(t) = \frac{t}{2^{n-1}}, \quad t > 0.$$

Notice that $TX \subset fX$ and fX is obviously complete. Further, this is not difficult to see that the pair of maps T and f satisfy the condition (P3) for all $x, y \in X$.
Evidently $T(\frac{1}{2}) = f(\frac{1}{2}) = \frac{1}{4}$, and T and f are not commuting at the coincidence point $\frac{1}{2}$.

The following example shows the superiority of Theorem 3.1 over Theorems 2.4-2.6.

Example 3.2. Let $X = \{1, 2, 3, 4\}$ be endowed with the usual metric d. Let $f, T : X \to X$ be such that $T1 = T3 = T4 = 1$, $T2 = 2$ and $\varphi_n(t) = \frac{9t}{10}$ for all $t > 0$ (or any other choice of φ_n with $\varphi_n(t) < t$). Then it can be easily seen that Theorems 2.4 -2.6 are not applicable to this map T. Indeed, as $d(T^n x, T^n y) = 1$ for $x = 1$ and $y = 2$, none of the conditions (S3), (S4) and (S7) is satisfied. However, if we take $f1 = 1$, $f2 = f3 = 4$, $f4 = 2$ then Theorem 3.1 is applicable to these maps. Notice that $TX \subset fX$ and $f1 = T1 = 1$.

The following result generalizes Theorems 2.4 and 2.5.

Theorem 3.2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a self-map satisfying* (S1), (S2) *and* (P4). *If* T^k *is continuous for some* $k \in \mathbb{N}$ *then* T *has a unique fixed point* $z \in X$.

Proof. It may be completed following the proof of [30, Theorem 2.1]. □

Acknowledgement. The authors thank the referee and Professor Marcel-Adrian Serban for their comments and suggestions to improve upon the original typescript. The second author (SNM) is supported by the Directorate of Research Development, Walter Sisulu University.

References

[1] I.D. Aranđelović, *On a fixed point theorem of Kirk,* J. Math. Anal. Appl. **301**(2005), 384-385.
[2] I.D. Aranđelović, *Note on asymptotic contractions*, App. Anal. Disc. Math., **1**(2007), 211-216.
[3] M. Arav, F.E. Castillo Santos, S. Reich and A.J. Zaslavski, *A note on asymptotic contractions*, Fixed Point Theory Appl., 2007, Article ID **39465**, 6 pp.
[4] D.W. Boyd and J.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 458-464.
[5] E.M. Briseid, *A rate of convergence for asymptotic contractions. J. Math. Anal. Appl.*, **330**(2007), 364-376.
[6] E.M. Briseid, *Some results on Kirk's asymptotic contractions* Fixed Point Theory, **8**(2007), 17-27.
[7] E. M. Briseid, *Fixed points of generalized contractive mappings*, J. Nonlinear Convex Anal. **9**(2008), 181-204.
[8] E.M. Briseid, *A new uniformity for asymptotic contractions in the sense of Kirk*, Int. J. Math. Stat., **6**(2010), 2-13.
[9] Y.-Z. Chen, *Asymptotic fixed points for nonlinear contractions*, Fixed Point Theory and Applications, **2**(2005), 213-217.
[10] Y.J. Cho, P.P. Murthy and G. Jungck, *A theorem of Meer-Keeler type revisited*, Internat J. Math. Math. Sci., **23**(2000), 507-511.
[11] P. Gerhardy, *A quantitative version of Kirks fixed point theorem for asymptotic contractions*, J. Math. Anal. Appl., **316**(2006), 339-345.
[12] N. Hussain and M.A. Khamsi, *On asymptotic pointwise contractions in metric spaces*, Nonlinear Anal., **71**(2009), 4423–4429.
[13] J. Jachymski, *Equivalent conditions and the Meir-keeler type theorems*, J. Math. Anal. Appl., **194**(1995), 293-303.
[14] J.R. Jachymski and I. Jóźwic, *On Kirk's asymptotic contractions,* J. Math. Anal. Appl., **300**(2004), 147-159.
[15] J.R. Jachymski, *A note on a paper of I.D. Aranđelović on asymptotic contractions*, J. Math. Anal. Appl., **358**(2009), 491-492.
[16] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, **83**(1976), 261-263.
[17] W.A. Kirk, *Fixed points of asymptotic contractions,* J. Math. Anal. Appl., **277**(2003), 645-650.
[18] W.A. Kirk, *Asymptotic pointwise contractions*, in: Plenary Lecture, the 8th International Conference on Fixed Point Theory and Its Applications, Chiang Mai University, Thailand, July 16-22, 2007.
[19] W.A. Kirk, Hong-Kun Xu, *Asymptotic pointwise contractions*, Nonlinear Anal., **69**(2008), 4706-4712.
[20] U. Kohlenbach and P. Oliva, *Proof Mining: A Systematic Way of Analyzing Proofs in Mathematics*, Proc. Steklov Inst. Math., **242**(2003), 136164.
[21] T.C. Lim, *On characterization of Meir-Keeler contractive maps*, Nonlinear Anal., **46**(2001), 113-120.
[22] A. Meir and E. Keeler, *A theorem on contraction mappings,* J. Math. Anal. Appl., **28**(1969), 326-329.
[23] S. Park and B.E. Rhoades, *Meir-Keeler type contractive conditions*, Math. Japon., **26**(1981), 13-20.
[24] S. Reich and A.J. Zaslavski, *A convergence theorem for asymptotic contractions*, J. Fixed Point Theory Appl., **4**(2008), 27-33.

[25] B.E. Rhoades, *A comparison of various definitions of contracting mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-290.

[26] I.A. Rus, A. Petrusel and G. Petrusel, *Fixed Point Theory:* 1950-2000 *Romanian Contributions*, House of Book of Science, Cluj-Napoca, 2002.

[27] K.P.R. Sastry, G.V.R. Babu, S. Ismail and M. Balaiah *A fixed point theorem on asymptotic contractions*, Mathematical Communications, **12(**2007), 191-194.

[28] S.L. Singh, Ashish Kumar and Y.J. Cho, Fixed points of Meir-Keeler type hybrid contractions, Panamer. Math. J., **16**(2006), No. 4, 35-54.

[29] S.L. Singh and B.D. Pant, *Coincidence and fixed point theorems for a family of mappings of Menger spaces and extension to uniform spaces* Math. Japon., **33**(1988), No. 6, 957-973.

[30] S.L. Singh and Rajendra Pant, *Remarks on some recent fixed point theorems*, C.R. Math. Rep. Acad. Sci. Canada, **30**(2008), No. 2, 56-63.

[31] T. Suzuki, *Fixed-point theorem for asymptotic contractions of Meir- Keeler type in complete metric spaces*, Nonlinear Anal., **64**(2006), 971-978.

[32] T. Suzuki, *A definitive result on asymptotic contractions*, J. Math. Anal. Appl., **335**(2007) 707-715.

[33] T. Suzuki, *Fixed point theorems for more generalized contractions in complete metric space*, Demons. Math., **1**(2007), 219-227.

[34] K. Wlodarczyk, D. Klim, R. Plebaniak, *Existence and uniqueness of endpoints of closedset-valued asymptotic contractions in metric spaces*, J. Math. Anal. Appl., **328**(2007), 4657.

[35] K. Wlodarczyk, R. Plebaniak, C. Obczyński, *Endpoints set-valued dynamical systems of asymptotic contractions of Meir-Keeler type and strict contractions in uniform spaces*, Nonlinear Anal., **67**(2007), No. 6, 1668-1679.

[36] H.K. Xu, *Asymptotic and weakly asymptotic contractions*, Indian J. Pure Appl. Math., **36**(2005), 145-150.

Received: February 16, 2010; Accepted: October 14, 2010.