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A FIXED POINT APPROACH TO THE STABILITY OF ADDITIVE FUNCTIONAL INEQUALITIES IN RN-SPACES

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Abstract. Using the fixed point method, we prove the generalized Hyers-Ulam stability of the Cauchy additive functional inequality and of the Cauchy-Jensen additive functional inequality in random normed spaces.
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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [50] concerning the stability of group homomorphisms. Hyers [21] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [40] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [40] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [18] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. Important contributions for *generalized Hyers-Ulam stability* was obtained by Forti [16]. For Jensen functional equation stability, significant generalizations were given by Jung [24] and successively, by Lee and Jun [28] by using direct method (Hyers-Ulam method).

A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [49] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 12, 13, 17, 20], [25]–[27], [41]–[43]).

In the sequel, we adopt the usual terminology and notations of the theory of random normed spaces, as in [10, 30, 31, 47, 48]. Throughout this paper Δ^+ is the space of distribution functions, i.e., the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x, that is, $l^-f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. ([47]) A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm).

Definition 1.2. ([48]) A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:
(RN_1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN_2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \ne 0$;
(RN_3) $\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and all $t,s \ge 0$.

Every normed space $(X, \|.\|)$ defines a random normed space (X,μ,T_M), where

$$\mu_x(t) = \frac{t}{t+\|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3. Let (X,μ,T) be an RN-space.
(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1-\lambda$ whenever $n \ge N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1-\lambda$ whenever $n \ge m \ge N$.
(3) An RN-space (X,μ,T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([47]) *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.5. ([4, 14, 46]) *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [23] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. Starting with 2003, the fixed point alternative was applied to investigate the generalized Hyers-Ulam stability for Jensen functional equation in [4, 7, 39], as well as for the Cauchy functional equation in [5] (see also [32] for quadratic functional equations, [6] for monomial functional equations and [45] for operatorial equations etc). The proofs of the main theorems, i.e., Theorems 2.1, 2.3, 3.1 and 3.3, follow the techniques from the above papers. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4]–[8], [30, 32, 36, 37, 39]).

Gilányi [19] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [44]. Fechner [15] and Gilányi [20] proved the generalized Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [38] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + f(2z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the generalized Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Throughout this paper, let X be a real vector space and (Y, μ, T) a complete RN-space.

The generalized Hyers-Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [29]–[35]. They are completed with the recent paper [9], which contains some stability results for functional equations in probabilistic metric and random normed spaces.

This paper is organized as follows: In Section 2, we prove the generalized Hyers-Ulam stability of the Cauchy additive functional inequality (1.2) in complete RN-spaces. In Section 3, we prove the generalized Hyers-Ulam stability of the Cauchy-Jensen additive functional inequality (1.3) in complete RN-spaces.

2. Stability of the Cauchy additive functional inequality

In this section, using the fixed point method, we prove the generalized Hyers-Ulam stability of the Cauchy additive functional inequality (1.2) in complete RN-spaces.

Theorem 2.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \leq \frac{L}{2}\varphi(2x,2y,2z)$$

for all $x,y,z \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{f(x)+f(y)+f(z)}(t) \geq \min\left\{\mu_{f(x+y+z)}\left(\frac{t}{2}\right), \frac{t}{t+\varphi(x,y,z)}\right\} \tag{2.1}$$

for all $x,y,z \in X$ and all $t > 0$. Then $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x,-2x)} \tag{2.2}$$

for all $x \in X$ and all $t > 0$.

Proof. Since f is odd, $f(0) = 0$. So $\mu_{f(0)}\left(\frac{t}{2}\right) = 1$. Letting $y = x$ and replacing z by $-2x$ in (2.1), we get

$$\mu_{f(2x)-2f(x)}(t) \geq \frac{t}{t+\varphi(x,x,-2x)} \tag{2.3}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\nu \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x,-2x)},\ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete. (See the proof of Lemma 2.1 in [31]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t+\varphi(x,x,-2x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)}(L\varepsilon t) \\
&= \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{2}\varepsilon t\right) \\
&\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}, -x\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, x, -2x)} \\
&= \frac{t}{t + \varphi(x, x, -2x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.3) that

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}\left(\frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.4}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.4) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(\nu t) \geq \frac{t}{t + \varphi(x, x, -2x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$\begin{aligned}&\mu_{2^n\left(f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+f\left(\frac{z}{2^n}\right)\right)}\left(2^n t\right)\\&\qquad\ge \min\left\{\mu_{2^n f\left(\frac{x+y+z}{2^n}\right)}\left(2^{n-1}t\right), \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)}\right\}\end{aligned}$$

for all $x, y, z \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{2^n\left(f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+f\left(\frac{z}{2^n}\right)\right)}(t) \ge \min\left\{\mu_{2^n f\left(\frac{x+y+z}{2^n}\right)}\left(\frac{t}{2}\right), \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y,z)}\right\}$$

for all $x, y, z \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y,z)} = 1$ for all $x, y, z \in X$ and all $t > 0$,

$$\mu_{A(x)+A(y)+A(z)}(t) \ge \mu_{A(x+y+z)}\left(\frac{t}{2}\right)$$

for all $x, y, z \in X$ and all $t > 0$. By Definition 1.2, the mapping $A : X \to Y$ is Cauchy additive, as desired. □

Corollary 2.2. *Let $\theta \ge 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{f(x)+f(y)+f(z)}(t) \ge \min\left\{\mu_{f(x+y+z)}\left(\frac{t}{2}\right), \frac{t}{t+\theta(\|x\|^p+\|y\|^p+\|z\|^p)}\right\} \tag{2.5}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \frac{(2^p-2)t}{(2^p-2)t+(2+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p+\|y\|^p+\|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 2.3. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.1). Then $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \frac{(2-2L)t}{(2-2L)t+\varphi(x,x,-2x)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g\left(2x\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(x,x,-2x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{2}g(2x)-\frac{1}{2}h(2x)}\left(L\varepsilon t\right) \\
&= \mu_{g(2x)-h(2x)}\left(2L\varepsilon t\right) \\
&\geq \frac{2Lt}{2Lt + \varphi\left(2x,2x,-4x\right)} \geq \frac{2Lt}{2Lt + 2L\varphi(x,x,-2x)} \\
&= \frac{t}{t + \varphi(x,x,-2x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq L\varepsilon$. This means that

$$d(Jg,Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.3) that

$$\mu_{f(x)-\frac{1}{2}f(2x)}\left(\frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x,x,-2x)}$$

for all $x \in X$ and all $t > 0$. So $d(f,Jf) \leq \frac{1}{2}$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(2x\right) = 2A(x) \tag{2.7}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\nu \in (0,\infty)$ satisfying

$$\mu_{f(x)-A(x)}(\nu t) \geq \frac{t}{t + \varphi(x,x,-2x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \le \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,A) \le \frac{1}{2-2L}.$$

This implies that the inequality (2.6) holds.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Corollary 2.4. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(x)-A(x)}(t) \ge \frac{(2-2^p)t}{(2-2^p)t + (2+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. $\square$

3. Stability of the Cauchy-Jensen additive functional inequality

In this section, using the fixed point method, we prove the generalized Hyers-Ulam stability of the Cauchy-Jensen additive functional inequality (1.3) in complete RN-spaces.

Theorem 3.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \le \frac{L}{2}\varphi(2x,2y,2z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{f(x)+f(y)+f(2z)}(t) \ge \min\left\{\mu_{2f\left(\frac{x+y}{2}+z\right)}\left(\frac{2t}{3}\right), \frac{t}{t+\varphi(x,y,z)}\right\} \tag{3.1}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x,-x)} \tag{3.2}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x = -z$ in (3.1), we get

$$\mu_{f(2x)-2f(x)}(t) \ge \frac{t}{t+\varphi(x,x,-x)} \tag{3.3}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\nu \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x,-x)}, \ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete. (See the proof of Lemma 2.1 in [31].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t+\varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)}(L\varepsilon t) \\
&= \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{2}\varepsilon t\right) \\
&\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2}+\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2}+\frac{L}{2}\varphi(x,x,-x)} \\
&= \frac{t}{t+\varphi(x,x,-x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.3) that

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}\left(\frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{3.4}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (3.4) such that there exists a $\nu \in (0,\infty)$ satisfying

$$\mu_{f(x)-A(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x,-x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{L}{2-2L}.$$

This implies that the inequality (3.2) holds.

The rest of proof is similar to the proof of Theorem 2.1. $\square$

Corollary 3.2. *Let $\theta \ge 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{f(x)+f(y)+f(2z)}(t) \ge \min\left\{\mu_{f\left(\frac{x+y}{2}+z\right)}\left(\frac{2t}{3}\right), \frac{t}{t+\theta(\|x\|^p + \|y\|^p + \|z\|^p)}\right\} \tag{3.5}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \frac{(2^p-2)t}{(2^p-2)t + 3\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. $\square$

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.1). Then $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \frac{(2-2L)t}{(2-2L)t + \varphi(x, x, -x)} \tag{3.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t+\varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{2}g(2x)-\frac{1}{2}h(2x)}(L\varepsilon t) \\
&= \mu_{g(2x)-h(2x)}(2L\varepsilon t) \\
&\geq \frac{2Lt}{2Lt+\varphi(2x,2x,-2x)} \geq \frac{2Lt}{2Lt+2L\varphi(x,x,-x)} \\
&= \frac{t}{t+\varphi(x,x,-x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.3) that

$$\mu_{f(x)-\frac{1}{2}f(2x)}\left(\frac{1}{2}t\right) \geq \frac{t}{t+\varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{2}$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(2x) = 2A(x) \tag{3.7}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (3.7) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x,-x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{2-2L}.$$

This implies that the inequality (3.6) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an odd mapping satisfying (3.5). Then $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2-2^p)t}{(2-2^p)t + 3\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

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