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STRONG CONVERGENCE THEOREMS BY GENERALIZED CQ METHOD IN HILBERT SPACES

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Abstract. Recently, CQ method has been investigated extensively. However, it is mainly applied to modify Mann, Ishikawa and Halpern iterations to get strong convergence. In this paper, we study the properties of CQ method and proposed a framework. Based on that, we obtain a series of strong convergence theorems. Some of them are the extensions of previous results. On the other hand, CQ method, monotone Q method, monotone C method and monotone CQ method, used to be given separately, have the following relations: CQ method TRUE ⇒ monotone Q method TRUE ⇒ monotone C method TRUE ⇔ monotone CQ method TRUE.
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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and T a self-mapping of C. Recall that T is said to be a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C. \tag{1.1}$$

T is said to be strictly pseudo-contractive if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \tag{1.2}$$

for all $x, y \in C$. For such cases, T is also said to be a κ-strictly pseudo-contractive mapping. It is also said to be pseudo-contractive if $\kappa = 1$ in (1.2). That is,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \tag{1.3}$$

for all $x, y \in C$. Clearly, the class of strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions.

It is very clear that, in a real Hilbert space H, (1.3) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \tag{1.4}$$

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Recall that three iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is Halpern's iteration process [1] which is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = (1-\alpha_n)x_0 + \alpha_n T x_n,\ n \geq 0 \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in the interval $[0, 1]$.

The second is known as Mann's iteration process [2] which is defined as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \geq 0 \tag{1.6}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

The third is referred to as Ishikawa's iteration process [3] which is defined recursively by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T z_n \end{cases},\ n \geq 0 \tag{1.7}$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$.

We know that (1.5) has strong convergence under certain conditions, but both (1.6) and (1.7) have only weak convergence, in general, even for nonexpansive mappings (see an example in [4]).

Recently, modifications of algorithm (1.5), (1.6) and (1.7) have been extensively investigated; see [5, 6, 7, 8, 9, 10] and the references therein. For instance, one of the most important methods was firstly introduced by Nakajo and Takahashi [6] in 2003.

Theorem 1.1 (see [6])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases} \tag{1.8}$$

where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.

Remark 1.1. It is also known as CQ method or CQ method. The purpose of the authors is to modify Mann's iteration process and obtain a strong convergent sequence. However, we can learn more from (1.8). In fact, (1.8) is equivalent to

$$\begin{cases} C_n = \{z \in C : \|z - ((1-\alpha_n)x_n + \alpha_n T x_n)\| \leq \|z - x_n\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases} \tag{1.9}$$

From (1.9) we can conclude that in each recursive step, the algorithms can be divided into two parts: (P_1) construct an appropriate set and (P_2) project the given fixed point onto the set.

According to this point view, the crux of CQ method is how to construct an appropriate set and (1.9) is just a special case:

(A_1) construct C_n based on iteration scheme (1.6) and the properties of the mapping T.

(A_2) construct Q_n by the property of the metric projection.

$C_n \cap Q_n$ is the appropriate set. Then, together with (P_2), we can yield (1.9), i.e., Theorem 1.1.

Actually, based on this idea we can accomplish (P1) in many ways and construct different kinds of appropriate sets based on scheme (1.5), (1.6), (1.7) and their combinations. And we name this method as *generalized CQ method*.

Motivated by Remark 1.1, we propose a CQ algorithm framework, which is the basic work in this paper. Then, based on this framework, we introduce a series of strong convergence theorems. Some of them, used to be given separately, have direct relations between each other.

In section 8, we study the relations among CQ method, monotone Q method, monotone C method and monotone CQ method.

2. Preliminaries and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let C be a closed convex subset of H. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$, where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping.

$x_n \to x$ means that $\{x_n\}$ converges strongly to x. $x_n \rightharpoonup x$ means x_n converges weakly to x.

We know that a Hilbert space H satisfies Opial's condition [11], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. We also know that H has the Kadec-Klee property, that is $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. In fact, from $\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2$, we get that a Hilbert space has Kadec-Klee property.

For a given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak limit set of $\{x_n\}$

Now we collect some lemmas which will be used in the proof of our main theorems.

Lemma 2.1 (see [5]). *Let H be a real Hilbert space. There hold the following identities:*
(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle,\ \forall x, y \in H$
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2,\ \forall \alpha \in [0, 1]$ *and* $x, y \in H$.

Lemma 2.2. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leq 0, \ \textit{for all } y \in C.$$

Lemma 2.3 (see [8])**.** *Let H be a real Hilbert space. Given a closed convex subset C and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is closed and convex.

Lemma 2.4 (see [8])**.** *Let C be a closed convex subset of real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\omega_w(x_n) \subset C$ and*

$$\|x_n - u\| \leq \|u - q\|$$

for all n, then $x_n \to q$.

Lemma 2.5 (see [10])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a demi-continuous pseudo-contractive self-mapping from C into itself. Then $F(T)$ is a closed convex subset of C and $I - T$ is demiclosed at zero.*

Lemma 2.6 (see [5])**.** *Let C be a nonempty closed convex subset of H and $T : C \to C$ a $\kappa-$strict pseudo-contraction for some $0 \leq \kappa < 1$. Then $F(T)$ is a closed convex subset of C and $I - T$ is demiclosed at zero.*

Lemma 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H and T an $L-$Lipschitz mapping from C into itself. Assume $F(T) \neq \emptyset$ is closed and convex, $L + 1 \leq \mu < \infty$ and $\theta \geq 0$. Let*

$$C_x = \{z \in C : \|x - Tx\| \leq \mu\|x - z\| + \theta\}, \ \forall x \in C.$$

Then, $F(T) \subset C_x$.

Proof. Let $p \in F(T)$, we have $\forall x \in C$

$$\begin{aligned}\|x - Tx\| &\leq \|x - p\| + \|p - Tx\| \\ &\leq \|x - p\| + L\|x - p\| \\ &= (L + 1)\|x - p\| \\ &\leq \mu\|x - p\| + \theta.\end{aligned}$$

Hence, $p \in C_x$, i.e., $F(T) \subset C_x$.
□

3. Main result

In this section, a strong convergence theorem is obtained by generalized CQ method.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and T an $L-$Lipschitz mapping from C into itself. Assume $F(T) \neq \emptyset$ is closed and convex, $I - T$ is demiclosed at zero, $\{\mu_n\}$ is a sequence such that $L + 1 \leq \mu_n \leq \mu < \infty$ and $\theta_n(z)$ is a nonnegative function on C. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ C_n = \{z \in C : \|x_n - Tx_n\| \leq \mu_n\|x_n - z\| + \theta_n(z)\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0, \end{cases} \tag{3.1}$$

where C_n^* *is a closed convex set with* $F(T) \subset C_n^* \subset C_n$*. Assume* $\lim_{n\to\infty} \theta_n(x_{n+1}) = 0$*. Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$*.*

Proof. According to the assumption, we see that $P_{F(T)}x_0$ is well defined. It is obvious that Q_n is closed and convex, hence, $C_n^* \cap Q_n$ is closed and convex.

Next, we show that $F(T) \subset C_n^* \cap Q_n$. From the assumption, $F(T) \subset C_n^*$, hence, it suffices to prove $F(T) \subset Q_n$. We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n^* \cap Q_n$, we have

$$\langle z - x_{n+1}, x_{n+1} - x_0\rangle \geq 0, \forall z \in C_n^* \cap Q_n.$$

As $F(T) \subset C_n^* \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence, $F(T) \subset Q_n$ holds for all $n \geq 0$ and $\{x_n\}$ is well defined.

From $x_n = P_{Q_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for all $y \in C_n^* \cap Q_n$. So, for $p \in F(T)$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - p\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - p\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \cdot \|x_0 - p\|. \end{aligned}$$

Hence,

$$\|x_0 - x_n\| \leq \|x_0 - p\| \tag{3.2}$$

for all $p \in F(T)$. This implies that $\{x_n\}$ is bounded.

From $x_n = P_{Q_n}x_0$ and $x_{n+1} = P_{C_n^* \cap Q_n}x_0 \in C_n^* \cap Q_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0.$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \cdot \|x_0 - x_{n+1}\|, \end{aligned}$$

therefore

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|,$$

which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Besides, by Lemma 2.1 we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned}$$

Let $n \to \infty$, we get $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Noticing $x_{n+1} = P_{C_n^* \cap Q_n} x_0 \subset C_n$, we have

$$\|x_n - Tx_n\| \le \mu_n \|x_n - x_{n+1}\| + \theta_n(x_{n+1}).$$

Combining with the assumption of $\{\mu_n\}$ and $\{\theta_n(x_{n+1})\}$, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Since $I - T$ is demiclosed, then every weak limit point of $\{x_n\}$ is a fixed point of T. That is, $\omega_w(x_n) \subset F(T)$. By Lemma 2.4, $x_n \to P_{F(T)} x_0$. □

4. Applications of the main result

First, we use Theorem 3.1 to prove Theorem 1.1.

Obviously, the following theorem can be easily verified by Theorem 3.1.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ {}^*C_n = \{z \in C : \|x_n - Tx_n\| \le \frac{2}{\alpha_n}\|x_n - z\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^{**} \cap Q_n} x_0 \end{cases}$$

where C_n^{**} is a closed convex set with $F(T) \subset C_n^{**} \subset {}^*C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \le \alpha_n \le 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

Let $C_n^{**} = C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}$ in Theorem 4.1. Easily, we can prove C_n is closed and convex with $F(T) \subset C_n \subset {}^*C_n$. So, Theorem 4.2 is valid.

Theorem 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^{**} \cap Q_n} x_0 \end{cases}$$

where $\{\alpha_n\}$ is chosen such that $0 < \alpha \le \alpha_n \le 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

Clearly, Theorem 4.2 is the same as Theorem 1.1.

Moreover, if C_n^* is a closed convex set satisfies $F(T) \subset C_n^* \subset C_n$, then, $F(T) \subset C_n^* \subset {}^*C_n$. Therefore, we obtain the following theorem.

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Theorem 4.3 is one of the generalized CQ algorithms in this paper. Likewise, we can yield many other similar algorithms for Mann, Ishikawa and Halpern iterations, respectively. They will be proposed in the following three sections. Some of them are the extensions of previous results. However, others are obtained directly based on the framework.

5. Generalized CQ algorithms for Mann's iteration process

In this section, we proposed some algorithms for Mann's iteration process. To prove the main theorems, we need the following lemmas.

Lemma 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudo-contractive mapping with Lipschitz constant $L \geq 1$. $\forall x \in C$, $\alpha \in (0, \frac{1}{L+1})$ and $\tau \in (0, 1]$, let*

$$y = (1-\alpha)x + \alpha Tx,$$

$$C_x = \{z \in C : \tau\alpha[1-(1+L)\alpha]\|x - Tx\|^2 \leq \langle x - z, y - Ty\rangle\}$$

and

$${}^*C_x = \{z \in C : \|x - Tx\| \leq \frac{(L+1)\alpha + 1}{\tau\alpha[1-(L+1)\alpha]}\|x - z\|\}.$$

Then, there holds C_x is a closed convex set with $F(T) \subset C_x \subset {}^*C_x$.

Proof. Obviously, C_x is closed and convex. From [7], we have

$$\alpha[1-(L+1)\alpha]\|x - Tx\|^2 \leq \langle x - p, y - Ty\rangle, \ \forall p \in F(T). \tag{5.1}$$

Since $\tau \in (0, 1]$, we obtain

$$\tau\alpha[1-(L+1)\alpha]\|x - Tx\|^2 \leq \langle x - p, y - Ty\rangle. \tag{5.2}$$

From (5.2), we can conclude that $F(T) \subset C_x$. Let $u \in C_x$, we have $\forall x \in C$

$$\begin{aligned} \tau\alpha[1-(L+1)\alpha]\|x - Tx\|^2 &\leq \langle x - u, y - Ty\rangle \\ &\leq \|x - u\|\|y - Ty\| \\ &\leq \|x - u\|[\|y - x\| + \|x - Tx\| + \|Tx - Ty\|] \\ &\leq \|x - u\|[(L+1)\|x - y\| + \|x - Tx\|] \\ &= [(L+1)\alpha + 1]\|x - u\|\|x - Tx\|. \end{aligned} \tag{5.3}$$

From the assumption of the coefficients we have

$$\|x - Tx\| \leq \frac{(L+1)\alpha + 1}{\tau\alpha[1-(L+1)\alpha]}\|x - u\| \tag{5.4}$$

which implies $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. $\square$

Lemma 5.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudo-contractive mapping with Lipschitz constant $L \geq 1$ and $F(T) \neq \emptyset$. $\forall x \in C$, $\alpha \in (0, \frac{1}{L+1})$ and $\tau \in (0, 1]$, let*

$$y = (1-\alpha)x + \alpha T x, \tag{5.5}$$

$$C_x = \{z \in C : \tau\|\alpha(I-T)y\|^2 \leq 2\alpha\langle x - z, (I-T)y\rangle\}.$$

and

$$^*C_x = \{z \in C : \|x - Tx\| \leq \frac{2}{\tau\alpha[1-(L+1)\alpha]}\|x - z\|\}.$$

Then, there holds C_x is a closed convex set with $F(T) \subset C_x \subset {}^*C_x$.

Proof. Obviously, C_x is closed and convex. From [9], we have

$$\|\alpha(I-T)y\|^2 \leq 2\alpha\langle x - p, (I-T)y\rangle, \ \forall p \in F(T).$$

since $\tau \in (0, 1]$, we obtain

$$\tau\|\alpha(I-T)y\|^2 \leq 2\alpha\langle x - p, (I-T)y\rangle.$$

which implies that $p \in C_x$, i.e., $F(T) \subset C_x$. Let $u \in C_x$, then $\forall x \in C$

$$\begin{aligned} \tau\|\alpha(I-T)y\|^2 &\leq 2\alpha\langle x - u, (I-T)y\rangle \\ &\leq 2\alpha\|x-u\|\|(I-T)y\|. \end{aligned} \tag{5.6}$$

It follows that

$$\|y - Ty\| \leq \frac{2}{\tau\alpha}\|x - u\|. \tag{5.7}$$

On the other hand, we have

$$\begin{aligned} \|x - Tx\| &\leq \|x - y\| + \|y - Ty\| + \|Ty - Tx\| \\ &\leq (L+1)\alpha\|x - Tx\| + \|y - Ty\|. \end{aligned} \tag{5.8}$$

Substitute (5.7) into (5.8), together with the assumption of coefficients, we get

$$\|x - Tx\| \leq \frac{2}{\tau\alpha[1-(L+1)\alpha]}\|x - u\| \tag{5.9}$$

which implies $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. □

Lemma 5.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a $\kappa-$strict pseudo-contraction of C into itself for some $0 \leq \kappa < 1$ with $F(T) \neq \emptyset$. $\forall x \in C$ and $\alpha \in (0, 1]$, let*

$$y = (1-\alpha)x + \alpha T x,$$

$$C_x = \{z \in C : \|y - z\|^2 \leq \|x - z\|^2 + \alpha(\kappa - (1-\alpha))\|x - Tx\|^2\}$$

and

$$^*C_x = \{z \in C : \|x - Tx\| \leq \frac{2}{1-\kappa}\|x - z\|\}.$$

Then, C_x is a closed convex subset of C with $F(T) \subset C_x \subset {}^*C_x$.

Proof. By Lemma 2.3, C_x is closed and convex. Let $p \in F(T)$, for any $x \in C$ we have

$$\begin{aligned}\|y-p\|^2 &= \|(1-\alpha)(x-p)+\alpha(Tx-p)\|^2\\ &= (1-\alpha)\|x-p\|^2+\alpha\|Tx-p\|^2-\alpha(1-\alpha)\|x-Tx\|^2\\ &\le (1-\alpha)\|x-p\|^2+\alpha(\|x-p\|^2+\kappa\|x-Tx\|^2)\\ &\quad -\alpha(1-\alpha)\|x-Tx\|^2\\ &= \|x-p\|^2+\alpha(\kappa-(1-\alpha))\|x-Tx\|^2.\end{aligned} \tag{5.10}$$

Hence, $F(T) \subset C_x$. Let $u \in C_x$, then $\forall x \in C$, we obtain

$$\|y-u\|^2 \le \|x-u\|^2+\alpha(\kappa-(1-\alpha))\|x-Tx\|^2 \tag{5.11}$$

Besides, we have

$$\|y-u\|^2 = (1-\alpha)\|x-u\|^2+\alpha\|Tx-u\|^2-\alpha(1-\alpha)\|x-Tx\|^2. \tag{5.12}$$

Substitute (5.11) into (5.12) to get

$$\alpha\|Tx-u\|^2 \le \alpha\|x-u\|^2+\alpha\kappa\|x-Tx\|^2. \tag{5.13}$$

Since $\alpha > 0$, we have

$$\|Tx-u\|^2 \le \|x-u\|^2+\kappa\|x-Tx\|^2. \tag{5.14}$$

On the other hand, we compute

$$\|Tx-u\|^2 = \|Tx-x\|^2+2\langle Tx-x, x-u\rangle+\|x-u\|^2. \tag{5.15}$$

Combining (5.14) and (5.15) yields

$$(1-\kappa)\|x-Tx\|^2 \le 2\langle x-Tx, x-u\rangle \le 2\|x-Tx\|\|x-u\|. \tag{5.16}$$

Since $\kappa < 1$, then

$$\|x-Tx\| \le \frac{2}{1-\kappa}\|x-u\| \tag{5.17}$$

which implies $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. □

Using Lemma 5.1, we obtain the following theorem.

Theorem 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and T a Lipschitz pseudo-contraction from C into itself with the Lipschitz constant $L \ge 1$ and $F(T) \ne \emptyset$. Assume sequence $\{\tau_n\} \subset [\tau, 1]$ with $\tau \in (0, 1]$ and sequence $\{\alpha_n\} \subset [a, b]$ with $a, b \in (0, \frac{1}{L+1})$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}\\ y_n = (1-\alpha_n)x_n+\alpha_n Tx_n\\ C_n = \{z \in C : \tau_n\alpha_n[1-(1+L)\alpha_n]\|x_n-Tx_n\|^2 \le \langle x_n-z, y_n-Ty_n\rangle\}\\ Q_n = \{z \in C : \langle z-x_n, x_n-x_0\rangle \ge 0\}\\ x_{n+1} = P_{C_n^*\cap Q_n}x_0\end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \le \frac{(L+1)\alpha_n+1}{\tau_n\alpha_n[1-(L+1)\alpha_n]}\|x_n - z\|\}$, then using Lemma 5.1, we obtain $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. From the assumption, $\frac{(L+1)\alpha_n+1}{\tau_n\alpha_n[1-(L+1)\alpha_n]} \le \frac{(L+1)b+1}{\tau a[1-(L+1)b]} < \infty$. By Lemma 2.5 and Theorem 3.1, we can prove $x_n \to P_{F(T)}x_0$. □

We can prove the following theorem based on Lemma 5.2.

Theorem 5.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping such that $L \ge 1$ and $F(T) \neq \emptyset$. Assume sequence $\{\tau_n\} \subset [\tau, 1]$ with $\tau \in (0, 1]$ and sequence $\{\alpha_n\} \subset [a, b]$ with $a, b \in (0, \frac{1}{L+1})$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Tx_n \\ C_n = \{z \in C : \tau_n\|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)y_n\rangle\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n}x_0 \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. From the assumption, we have $\frac{2}{\tau_n\alpha_n[1-(L+1)\alpha_n]} \le \frac{2}{\tau a[1-(L+1)b]} < \infty$. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \le \frac{2}{\tau_n\alpha_n[1-(L+1)\alpha_n]}\|x_n - z\|\}$, using Lemma 5.2, we can conclude $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, by Lemma 2.5 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. □

Remark 5.1. In fact, it is easily to prove Theorem 5.2 by Theorem 5.1 directly.

By Lemma 5.3, the following theorem is valid.

Theorem 5.3. *Let C be a nonempty closed convex subset of a Hilbert space H and T a $\kappa-$strict pseudo-contraction of C into itself for some $0 \le \kappa < 1$ with $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Tx_n \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\kappa - (1-\alpha_n))\|x_n - Tx_n\|^2\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n}x_0 \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \le \alpha_n \le 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. Clearly, $\frac{2}{1-\kappa} < \infty$. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \le \frac{2}{1-\kappa}\|x - z\|\}$, then using Lemma 5.3, $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, using Lemma 2.6 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. □

Corollary 5.4. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Tx_n \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n}x_0 \end{cases}$$

where C_n^* *is a closed convex set with* $F(T) \subset C_n^* \subset C_n$ *and* $\{\alpha_n\}$ *is chosen such that* $0 < \alpha \leq \alpha_n \leq 1$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Remark 5.2. Corollary 5.4 is a deduced result of Theorem 5.3. In these two theorems, set $C_n^* = C_n$, then we obtain two algorithms which were also proposed in [5]. Corollary 5.4 is also the deduced result of Theorem 4.3.

6. Generalized CQ algorithms for Ishikawa's iteration process

In this section, we introduce some algorithms for Ishikawa's iteration process. To prove the main theorems, we need the following lemmas.

Lemma 6.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be a Lipschitz pseudo-contractive mapping with Lipschitz constant* $L \geq 1$ *and* $F(T) \neq \emptyset$. $\forall x \in C$ *and* $\alpha, \beta \in (0,1)$ *such that* $0 < \beta \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$, *let*

$$v = (1-\alpha)x + \alpha Tx$$
$$y = (1-\beta)x + \beta Tv,$$
$$C_x = \{z \in C : \|y-z\|^2 \leq \|x-z\|^2 - \alpha\beta(1-2\alpha-L^2\alpha^2)\|x-Tx\|^2\}$$

and

$${}^*C_x = \{z \in C : \|x-Tx\| \leq \frac{2(1+L\alpha)}{\alpha(1-2\alpha-L^2\alpha^2)}\|x-z\|\}.$$

Then, there holds C_x *is a closed convex set with* $F(T) \subset C_x \subset {}^*C_x$.

Proof. Obviously, by Lemma 2.3, we can conclude C_x is closed and convex. From [10], we can easily obtain $F(T) \subset C_x$. Taking $u \in C_x$, $\forall x \in C$, we get

$$\|y-u\|^2 \leq \|x-u\|^2 - \alpha\beta(1-2\alpha-L^2\alpha^2)\|x-Tx\|^2. \tag{6.1}$$

On the other hand,

$$\|y-u\|^2 = \|y-x\|^2 + 2\langle y-x, x-u\rangle + \|x-u\|^2. \tag{6.2}$$

Combining (6.1) and (6.2), we have

$$\alpha(1-2\alpha-L^2\alpha^2)\|x-Tx\|^2 \leq 2\langle x-Tv, x-u\rangle \tag{6.3}$$

It follows that,

$$\begin{aligned} \alpha(1-2\alpha-L^2\alpha^2)\|x-Tx\|^2 &\leq 2\langle x-Tv, x-u\rangle \\ &\leq 2\|x-u\|\|x-Tv\| \\ &\leq 2\|x-u\|(\|x-Tx\| + \|Tx-Tv\|) \\ &\leq 2(1+L\alpha)\|x-u\|\|x-Tx\|. \end{aligned} \tag{6.4}$$

Noting that the function $f(t) = 1-2t-L^2t^2$ is strictly decreasing in $t \in (0,1)$, we infer that

$$1-2\alpha-L^2\alpha^2 > 0.$$

Then, from (6.4), we have

$$\|x-Tx\| \leq \frac{2(1+L\alpha)}{\alpha(1-2\alpha-L^2\alpha^2)}\|x-u\| \tag{6.5}$$

which implies $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. □

Lemma 6.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a $\kappa-$strict pseudo-contractive mapping for some $0 \le \kappa < 1$ with $F(T) \neq \emptyset$. $\forall x \in C$ and $\alpha, \beta \in [0,1]$ such that $0 < \alpha < \frac{2}{\sqrt{4\kappa L^2 + (\kappa+1)^2} + (\kappa+1)}$ and $0 < \beta \le \kappa\alpha + (1-\kappa)$, let*

$$v = (1-\alpha)x + \alpha Tx$$
$$y = (1-\beta)x + \beta Tv,$$
$$C_x = \{z \in C : \|y-z\|^2 \le \|x-z\|^2 - \alpha\beta[1-(\kappa+1)\alpha - \kappa L^2\alpha^2]\|x-Tx\|^2\}$$

and

$${}^*C_x = \{z \in C : \|x - Tx\| \le \frac{2(1+L\alpha)}{\alpha[1-(\kappa+1)\alpha-\kappa L^2\alpha^2]}\|x-z\|\}.$$

Then, there holds C_x is a closed convex set with $F(T) \subset C_x \subset {}^*C_x$.

Proof. Obviously, by Lemma 2.3, we can conclude C_x is closed and convex. Let $p \in F(T)$. We have,

$$\begin{aligned}
\|v-p\|^2 &= \|(1-\alpha)(x-p) + \alpha(Tx-p)\|^2 \\
&= (1-\alpha)\|x-p\|^2 + \alpha\|Tx-p\|^2 - \alpha(1-\alpha)\|x-Tx\|^2 \\
&\le (1-\alpha)\|x-p\|^2 + \alpha(\|x-p\|^2 + \kappa\|x-Tx\|^2) \\
&\quad - \alpha(1-\alpha)\|x-Tx\|^2 \\
&= \|x-p\|^2 + \alpha[\kappa - (1-\alpha)]\|x-Tx\|^2
\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}
\|v-Tv\|^2 &= \|(1-\alpha)(x-Tv) + \alpha(Tx-Tv)\|^2 \\
&= (1-\alpha)\|x-Tv\|^2 + \alpha\|Tx-Tv\|^2 - \alpha(1-\alpha)\|x-Tx\|^2 \\
&\le (1-\alpha)\|x-Tv\|^2 + L^2\alpha\|x-v\|^2 - \alpha(1-\alpha)\|x-Tx\|^2 \\
&= (1-\alpha)\|x-Tv\|^2 + L^2\alpha^3\|x-Tx\|^2 - \alpha(1-\alpha)\|x-Tx\|^2 \\
&= (1-\alpha)\|x-Tv\|^2 + \alpha(L^2\alpha^2 + \alpha - 1)\|x-Tx\|^2
\end{aligned} \tag{6.7}$$

and also,

$$\begin{aligned}
\|y-p\|^2 &= \|(1-\beta)(x-p) + \beta(Tv-p)\|^2 \\
&= (1-\beta)\|x-p\|^2 + \beta\|Tv-p\|^2 - \beta(1-\beta)\|x-Tv\|^2 \\
&\le (1-\beta)\|x-p\|^2 + \beta(\|v-p\|^2 + \kappa\|v-Tv\|^2) - \beta(1-\beta)\|x-Tv\|^2
\end{aligned} \tag{6.8}$$

Substituting (6.6) and (6.7) in (6.8), we yield

$$\begin{aligned}
\|y-p\|^2 &\le (1-\beta)\|x-p\|^2 + \beta\|x-p\|^2 + \beta\alpha[\kappa-(1-\alpha)]\|x-Tx\|^2 \\
&\quad + \beta\kappa(1-\alpha)\|x-Tv\|^2 + \beta\kappa\alpha(L^2\alpha^2 + \alpha - 1)\|x-Tx\|^2 \\
&\quad - \beta(1-\beta)\|x-Tv\|^2 \\
&= \|x-p\|^2 + \beta[\kappa(1-\alpha) - (1-\beta)]\|x-Tv\|^2 \\
&\quad + \alpha\beta[\kappa L^2\alpha^2 + (\kappa+1)\alpha - 1]\|x-Tx\|^2.
\end{aligned}$$

Since $\beta \le \kappa\alpha + (1-\kappa)$, then,

$$\begin{aligned}\|y-p\|^2 &\le \|x-p\|^2 + \alpha\beta[\kappa L^2\alpha^2 + (\kappa+1)\alpha - 1]\|x-Tx\|^2\\ &= \|x-p\|^2 - \alpha\beta[1-(\kappa+1)\alpha - \kappa L^2\alpha^2]\|x-Tx\|^2.\end{aligned}$$

Therefore, $p \in C_x$, i.e., $F(T) \subset C_x$. Taking $u \in C_x$, $\forall x \in C$, we get

$$\|y-u\|^2 \le \|x-u\|^2 - \alpha\beta[1-(\kappa+1)\alpha - \kappa L^2\alpha^2]\|x-Tx\|^2. \tag{6.9}$$

On the other hand,

$$\|y-u\|^2 = \|y-x\|^2 + 2\langle y-x, x-u\rangle + \|x-u\|^2. \tag{6.10}$$

Combining (6.9) and (6.10), we have

$$\alpha[1-(\kappa+1)\alpha - \kappa L^2\alpha^2]\|x-Tx\|^2 \le 2\langle x-Tv, x-u\rangle \tag{6.11}$$

It follows that,

$$\begin{aligned}\alpha[1-(\kappa+1)\alpha - \kappa L^2\alpha^2]\|x-Tx\|^2 &\le 2\langle x-Tv, x-u\rangle\\ &\le 2\|x-u\|\|x-Tv\|\\ &\le 2\|x-u\|(\|x-Tx\| + \|Tx-Tv\|)\\ &\le 2(1+L\alpha)\|x-u\|\|x-Tx\|.\end{aligned} \tag{6.12}$$

Noting that the function $f(t) = 1-(\kappa+1)t - \kappa L^2t^2$ is strictly decreasing in $t \in (0,1)$, we infer that

$$1-(\kappa+1)\alpha - \kappa L^2\alpha^2 > 0.$$

Then, from (6.12), we have

$$\|x-Tx\| \le \frac{2(1+L\alpha)}{\alpha[1-(\kappa+1)\alpha - \kappa L^2\alpha^2]}\|x-u\| \tag{6.13}$$

which implies $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. □

Lemma 6.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself with $F(T) \ne \emptyset$. $\forall x \in C$, $0 < \beta \le 1$ and $0 \le \alpha \le 1$ let*

$$v = (1-\alpha)x + \alpha Tx$$
$$y = (1-\beta)x + \beta Tv,$$
$$C_x = \{z \in C : \|z-y\|^2 \le \|z-x\|^2 + \beta(\|v\|^2 - \|x\|^2 + 2\langle x-v, z\rangle)\}$$

and

$${}^*C_x = \{z \in C : \beta(1-\alpha)\|x-Tx\| \le 3\|x-z\| + \alpha\|Tx-z\|\}.$$

Then, C_x is a closed convex subset with $F(T) \subset C_x \subset {}^*C_x$.

Proof. By Lemma 2.3, we see that C_x is closed and convex. Let $p \in F(T)$, for any $x \in C$ we have,

$$\begin{aligned}\|y-p\|^2 &= \|(1-\beta)(x-p) + \beta(Tv-p)\|^2\\ &\le (1-\beta)\|x-p\|^2 + \beta\|v-p\|^2\\ &= \|x-p\|^2 + \beta(\|v-p\|^2 - \|x-p\|^2)\\ &= \|x-p\|^2 + \beta(\|v\|^2 - \|x\|^2 + 2\langle x-v, p\rangle)\end{aligned}$$

Hence, $F(T) \subset C_x$. Let $u \in C_x$, then we get

$$\begin{aligned} \|y-u\|^2 &\leq \|x-u\|^2 + \beta(\|v\|^2 - \|x\|^2 + 2\langle x-v, u\rangle) \\ &= \|x-u\|^2 + \beta(\|v-x\|^2 + 2\langle x-v, u-x\rangle) \\ &\leq \|x-u\|^2 + \beta\|v-u\|^2 \\ &\leq [\|x-u\| + \sqrt{\beta}\|v-u\|]^2. \end{aligned} \tag{6.14}$$

It follows that,

$$\begin{aligned} \|y-u\| &\leq \|x-u\| + \sqrt{\beta}\|v-u\| \\ &= \|x-u\| + \sqrt{\beta}\|(1-\alpha)(x-u) + \alpha(Tx-u)\| \\ &\leq \|x-u\| + \sqrt{\beta}(1-\alpha)\|x-u\| + \sqrt{\beta}\alpha\|Tx-u\| \\ &\leq 2\|x-u\| + \alpha\|Tx-u\|. \end{aligned} \tag{6.15}$$

Besides,

$$\begin{aligned} \|x-Tx\| &\leq \|x-Tv\| + \|Tv-Tx\| \\ &\leq \frac{1}{\beta}\|x-y\| + \|v-x\| \\ &\leq \frac{1}{\beta}[\|x-u\| + \|y-u\|] + \alpha\|x-Tx\|. \end{aligned} \tag{6.16}$$

Combining (6.15) and (6.16), we obtain

$$\begin{aligned} \beta(1-\alpha)\|x-Tx\| &\leq \|x-u\| + \|y-u\| \\ &\leq 3\|x-u\| + \alpha\|Tx-u\|. \end{aligned} \tag{6.17}$$

From (6.17), we can conclude $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. □

Lemma 6.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. $\forall x \in C$, $0 \leq \alpha < 1$ and $0 \leq \frac{\alpha}{1-\alpha} < \beta \leq 1$, let*

$$v = (1-\alpha)x + \alpha Tx$$
$$y = (1-\beta)x + \beta Tv,$$

$$C_x = \{z \in C : \|z-y\|^2 \leq \|z-x\|^2 + \beta(\|v\|^2 - \|x\|^2 + 2\langle x-v, z\rangle)\}$$

and

$${}^*C_x = \{z \in C : \|x-Tx\| \leq \frac{4}{\beta(1-\alpha)-\alpha}\|x-z\|\}.$$

Then, C_x is a closed convex subset with $F(T) \subset C_x \subset {}^*C_x$.

Proof. By Lemma 2.3, we see that C_x is closed and convex and similarly, we can prove $F(T) \subset C_x$. Let $u \in C_x$, then we get

$$\begin{aligned} \|y-u\|^2 &\leq \|x-u\|^2 + \beta(\|v\|^2 - \|x\|^2 + 2\langle x-v, u\rangle) \\ &\leq [\|x-u\| + \sqrt{\beta}\|v-u\|]^2. \end{aligned} \tag{6.18}$$

It follows that,

$$\begin{aligned}\|y-u\| &\le \|x-u\| + \sqrt{\beta}\|v-u\| \\ &= \|x-u\| + \sqrt{\beta}\|(1-\alpha)(x-u)+\alpha(Tx-u)\| \\ &\le \|x-u\| + \sqrt{\beta}(1-\alpha)\|x-u\| + \sqrt{\beta}\alpha\|Tx-u\| \\ &\le 2\|x-u\| + \alpha\|Tx-u\| \\ &\le 2\|x-u\| + \alpha\|Tx-x\| + \alpha\|x-u\| \\ &\le 3\|x-u\| + \alpha\|x-Tx\|.\end{aligned} \tag{6.19}$$

Besides,

$$\begin{aligned}\|x-Tx\| &\le \|x-Tv\| + \|Tv-Tx\| \\ &\le \frac{1}{\beta}\|x-y\| + \|v-x\| \\ &\le \frac{1}{\beta}[\|x-u\| + \|y-u\|] + \alpha\|x-Tx\|.\end{aligned} \tag{6.20}$$

Combining (6.19) and (6.20), we obtain

$$\begin{aligned}\beta(1-\alpha)\|x-Tx\| &\le \|x-u\| + \|y-u\| \\ &\le 4\|x-u\| + \alpha\|x-Tx\|.\end{aligned} \tag{6.21}$$

Hence,

$$\|x-Tx\| \le \frac{4}{\beta(1-\alpha)-\alpha}\|x-u\|. \tag{6.22}$$

From (6.22), we can conclude $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$ □

Theorem 6.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudo-contraction such that $L \ge 1$ and $F(T) \ne \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ satisfying the conditions:*
(C_1) $0 < \beta_n \le \alpha_n$, $\forall n \ge 0$;
(C_2) $\liminf_{n\to\infty} \alpha_n \ge \alpha' > 0$;
(C_3) $\limsup_{n\to\infty} \alpha_n \le \alpha < \frac{1}{\sqrt{1+L^2}+1}, \forall n \ge 0$.
Let a sequence $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ v_n = (1-\alpha_n)x_n + \alpha_n Tx_n \\ y_n = (1-\beta_n)x_n + \beta_n Tv_n \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n\beta_n(1-2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then, $\{x_n\}$ converges strongly to a fixed point $P_{F(T)}x_0$.

Proof. From the assumption, we have $\frac{2(1+L\alpha_n)}{\alpha_n(1-2\alpha_n-L^2\alpha_n^2)} \le \frac{2(1+L\alpha)}{\alpha'(1-2\alpha-L^2\alpha^2)} < \infty$. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \le \frac{2(1+L\alpha_n)}{\alpha_n(1-2\alpha_n-L^2\alpha_n^2)}\|x_n - z\|\}$, then using Lemma 6.1, $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, by Lemma 2.5 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. □

Remark 6.1. In this theorem, let $C_n^* = C_n$, then we yield a theorem which was also introduced in [10].

Theorem 6.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and T a Lipschitz pseudo-contraction from C into itself with the Lipschitz constant $L \geq 1$. Assume sequence $\{\tau_n\} \subset [\tau, 1]$ with $\tau \in (0,1]$, sequence $\{\alpha_n\} \subset [a,b]$ with $a, b \in (0, \frac{1}{L+1})$ and sequence $\{\beta_n\} \subset (0,1)$ satisfies that $\beta_n \leq \alpha_n$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ v_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n = (1-\beta_n)x_n + \beta_n T v_n \\ C_n' = \{z \in C : \tau_n \alpha_n [1-(1+L)\alpha_n]\|x_n - Tx_n\|^2 \leq \langle x_n - z, v_n - Tv_n \rangle\} \\ C_n'' = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n \beta_n (1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\} \\ C_n = C_n' \cap C_n'' \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0 \rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases} \tag{6.23}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. Obviously, $F(T) \subset C_n^* \subset C_n \subset C_n'$. Then, by Theorem 5.1, we can obtain $x_n \to P_{F(T)}x_0$. □

Theorem 6.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a $\kappa-$strict-pseudo-contraction for some $0 \leq \kappa < 1$ such that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $0 < \alpha' \leq \alpha_n \leq \alpha < \frac{2}{\sqrt{4\kappa L^2 + (\kappa+1)^2} + (\kappa+1)}$ and $0 < \beta_n \leq \kappa\alpha_n + (1-\kappa)$. Suppose sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ v_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n = (1-\beta_n)x_n + \beta_n T v_n \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1-(\kappa+1)\alpha_n - \kappa L^2\alpha_n^2)\|x_n - Tx_n\|^2\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0 \rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then, $\{x_n\}$ converges strongly to a fixed point $P_{F(T)}x_0$.

Proof. From the assumption, we have $\frac{2(1+L\alpha_n)}{\alpha_n[1-(\kappa+1)\alpha_n - \kappa L^2\alpha_n^2]} \leq \frac{2(1+L\alpha)}{\alpha'[1-(\kappa+1)\alpha - \kappa L^2\alpha^2]} < \infty$. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \leq \frac{2(1+L\alpha_n)}{\alpha_n[1-(\kappa+1)\alpha_n - \kappa L^2\alpha_n^2]}\|x_n - z\|\}$, then using Lemma 6.2, $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, by Lemma 2.6 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. □

Theorem 6.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a $\kappa-$strict-pseudo-contraction for some $0 \leq \kappa < 1$ such that*

$F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $0 < \alpha \leq \alpha_n \leq 1$ and $0 \leq \beta_n \leq \kappa\alpha_n + (1-\kappa)$. Let a sequence $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ v_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n = (1-\beta_n)x_n + \beta_n T v_n \\ C'_n = \{z \in C : \|v_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n(\kappa - (1-\alpha_n))\|x_n - Tx_n\|^2\} \\ C''_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1-(\kappa+1)\alpha_n - \kappa L^2\alpha_n^2)\|x_n - Tx_n\|^2\} \\ C_n = C'_n \cap C''_n \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. It is clear that $F(T) \subset C_n^* \subset C_n \subset C'_n$. Hence, by Theorem 5.3, $x_n \to P_{F(T)}x_0$. □

Using our method, we can yield at least four different CQ algorithms for Ishikawa's iteration process for nonexpansive mappings.

Theorem 6.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $0 < \alpha' \leq \alpha_n \leq \alpha < 1$ and $0 < \beta_n \leq 1$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ v_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n = (1-\beta_n)x_n + \beta_n T v_n \\ C_n = \{z \in C : \|z - y_n\|^2 \leq \|z - x_n\|^2 - \alpha_n\beta_n(1-\alpha_n)\|x_n - Tx_n\|^2\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then, $\{x_n\}$ converges strongly to a fixed point $P_{F(T)}x_0$.

Proof. By Theorem 6.3, we can prove the conclusion. □

Theorem 6.6. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $0 < \beta \leq \beta_n \leq 1$ and $\alpha_n \to 0$. Define a sequence $\{x_n\}$ in C by algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ v_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n = (1-\beta_n)x_n + \beta_n T v_n \\ C_n = \{z \in C : \|z - y_n\|^2 \leq \|z - x_n\|^2 + \beta_n(\|v_n\|^2 - \|x_n\|^2 + 2\langle x_n - v_n, z\rangle)\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. First observing that $\alpha_n \to 0$, we can conclude $\alpha_n \neq 1$ since n is sufficient big. So, without losing generality, we can assume $0 \leq \alpha_n \leq \alpha < 1$. Combining with the assumption of β_n, we have $\frac{3}{\beta_n(1-\alpha_n)} \leq \frac{3}{\beta(1-\alpha)} < \infty$. Easily, we can prove $\|Tx_n - x_{n+1}\|$ is bounded. Then, $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n(1-\alpha_n)}\|Tx_n - x_{n+1}\| = 0$. Let ${}^*C_n = \{z \in C : \beta_n(1-\alpha_n)\|x_n - Tx_n\| \leq 3\|x_n - z\| + \alpha_n\|Tx_n - z\|\}$, then using Lemma 6.3, $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, by Lemma 2.6 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. $\square$

Remark 6.2. In this theorem, let $C_n^* = C_n$, then we obtain an algorithm which was also proposed in [8].

Theorem 6.7. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $0 \leq \alpha_n \leq \alpha < 1$ and $0 \leq \frac{\alpha_n}{1-\alpha_n} \leq \frac{\alpha}{1-\alpha} < \beta \leq \beta_n \leq 1$. Define a sequence $\{x_n\}$ in C by algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ v_n = (1-\alpha_n)x_n + \alpha_n Tx_n \\ y_n = (1-\beta_n)x_n + \beta_n Tv_n \\ C_n = \{z \in C : \|z - y_n\|^2 \leq \|z - x_n\|^2 + \beta_n(\|v_n\|^2 - \|x_n\|^2 + 2\langle x_n - v_n, z\rangle)\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. From the assumption, we have $\frac{4}{\beta_n(1-\alpha_n)-\alpha_n} \leq \frac{4}{\beta(1-\alpha)-\alpha} < \infty$. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \leq \frac{4}{\beta_n(1-\alpha_n)-\alpha_n}\|x_n - z\|\}$, then using Lemma 6.4, $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, by Lemma 2.6 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. $\square$

Theorem 6.8. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} v_n = (1-\alpha_n)x_n + \alpha_n Tx_n \\ y_n = (1-\beta_n)x_n + \beta_n Tv_n \\ C_n' = \{z \in C : \|z - v_n\| \leq \|z - x_n\|\} \\ C_n'' = \{z \in C : \|z - y_n\|^2 \leq \|z - x_n\|^2 + \beta_n(\|v_n\|^2 - \|x_n\|^2 + 2\langle x_n - v_n, z\rangle)\} \\ C_n = C_n' \cap C_n'' \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen such that $0 < \alpha \leq \alpha_n \leq 1$ and $0 \leq \beta_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. It is obvious that $F(T) \subset C_n^* \subset C_n \subset C_n'$. Hence, by Theorem 4.3, $x_n \to P_{F(T)}x_0$. $\square$

7. Generalized CQ algorithms for Halpern' iteration process

In this section, we give some algorithms for Halpern's iteration process. To prove the main theorems, we need the following lemmas.

Lemma 7.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudo-contractive mapping with Lipschitz constant $L \geq 1$. $\forall x, x_0 \in C$ and $\alpha \in [0, 1]$, let*

$$y = (1 - \alpha)x_0 + \alpha Tx$$

and

$$C_x = \{z \in C : \|y - z\|^2 \leq \|x - z\|^2 + 2(1 - \alpha)\langle x - x_0, z\rangle + \theta\}$$

where $\theta = (1 - \alpha)(\|x_0\|^2 - \|x\|^2) + \alpha\|x - Tx\|^2 - \alpha(1 - \alpha)\|x_0 - Tx\|^2$. Then, there holds C_x is a closed convex set with $F(T) \subset C_x$.

Proof. Obviously, by Lemma 2.3, we can conclude C_x is closed and convex. Let $p \in F(T)$. We have,

$$\begin{aligned}
\|y - p\|^2 &= \|(1 - \alpha)(x_0 - p) + \alpha(Tx - p)\|^2 \\
&= (1 - \alpha)\|x_0 - p\|^2 + \alpha\|Tx - p\|^2 - \alpha(1 - \alpha)\|x_0 - Tx\|^2 \\
&\leq (1 - \alpha)\|x_0 - p\|^2 + \alpha(\|x - p\|^2 + \|x - Tx\|^2) - \alpha(1 - \alpha)\|x_0 - Tx\|^2 \\
&= \|x - p\|^2 + (1 - \alpha)(\|x_0 - p\|^2 - \|x - p\|^2) \\
&\quad + \alpha\|x - Tx\|^2 - \alpha(1 - \alpha)\|x_0 - Tx\|^2 \\
&= \|x - p\|^2 + (1 - \alpha)(\|x_0\|^2 - \|x\|^2 + 2\langle x - x_0, p\rangle) \\
&\quad + \alpha\|x - Tx\|^2 - \alpha(1 - \alpha)\|x_0 - Tx\|^2 \\
&= \|x - p\|^2 + 2(1 - \alpha)\langle x - x_0, p\rangle + (1 - \alpha)(\|x_0\|^2 - \|x\|^2) \\
&\quad + \alpha\|x - Tx\|^2 - \alpha(1 - \alpha)\|x_0 - Tx\|^2
\end{aligned}$$

Let $\theta = (1 - \alpha)(\|x_0\|^2 - \|x\|^2) + \alpha\|x - Tx\|^2 - \alpha(1 - \alpha)\|x_0 - Tx\|^2$. Then,

$$\|y - p\|^2 \leq \|x - p\|^2 + 2(1 - \alpha)\langle x - x_0, p\rangle + \theta$$

Therefore, $p \in C_x$, i.e., $F(T) \subset C_x$. $\square$

Lemma 7.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a $\kappa-$strict pseudo-contractive mapping for some $0 \leq \kappa < 1$ with $F(T) \neq \emptyset$. $\forall x, x_0 \in C$ and $\alpha \in [0, 1]$, let*

$$y = (1 - \alpha)x_0 + \alpha Tx,$$

$$C_x = \{z \in C : \|y - z\|^2 \leq \|x - z\|^2 + 2(1 - \alpha)\langle x - x_0, z\rangle + \theta\}$$

and

$$^*C_x = \{z \in C : \|x - Tx\| \leq \frac{2}{1 - \sqrt{\alpha\kappa}}\|x - z\| + \frac{\sqrt{1 - \alpha}}{1 - \sqrt{\alpha\kappa}}(\|x_0 - Tx\| + \|x_0 - z\|)\}$$

where $\theta = (1 - \alpha)(\|x_0\|^2 - \|x\|^2) + \alpha\kappa\|x - Tx\|^2 - \alpha(1 - \alpha)\|x_0 - Tx\|^2$. Then, there holds C_x is a closed convex set with $F(T) \subset C_x \subset {}^*C_x$.

Proof. Obviously, by Lemma 2.3, we can conclude C_x is closed and convex. Let $p \in F(T)$. We have,

$$\begin{aligned}
\|y-p\|^2 &= \|(1-\alpha)(x_0-p)+\alpha(Tx-p)\|^2 \\
&= (1-\alpha)\|x_0-p\|^2+\alpha\|Tx-p\|^2-\alpha(1-\alpha)\|x_0-Tx\|^2 \\
&\le (1-\alpha)\|x_0-p\|^2+\alpha(\|x-p\|^2+\kappa\|x-Tx\|^2)-\alpha(1-\alpha)\|x_0-Tx\|^2 \\
&= \|x-p\|^2+(1-\alpha)(\|x_0-p\|^2-\|x-p\|^2) \\
&\quad +\alpha\kappa\|x-Tx\|^2-\alpha(1-\alpha)\|x_0-Tx\|^2 \\
&= \|x-p\|^2+(1-\alpha)(\|x_0\|^2-\|x\|^2+2\langle x-x_0,p\rangle) \\
&\quad +\alpha\kappa\|x-Tx\|^2-\alpha(1-\alpha)\|x_0-Tx\|^2 \\
&= \|x-p\|^2+2(1-\alpha)\langle x-x_0,p\rangle+(1-\alpha)(\|x_0\|^2-\|x\|^2) \\
&\quad +\alpha\kappa\|x-Tx\|^2-\alpha(1-\alpha)\|x_0-Tx\|^2
\end{aligned}$$

Let $\theta=(1-\alpha)(\|x_0\|^2-\|x\|^2)+\alpha\kappa\|x-Tx\|^2-\alpha(1-\alpha)\|x_0-Tx\|^2$. Then,

$$\|y-p\|^2 \le \|x-p\|^2+2(1-\alpha)\langle x-x_0,p\rangle+\theta$$

Therefore, $p \in C_x$, i.e., $F(T) \subset C_x$. Let $u \in C_x$, then $\forall x \in C$

$$\begin{aligned}
\|y-u\|^2 &\le \|x-u\|^2+2(1-\alpha)\langle x-x_0,u\rangle+(1-\alpha)(\|x_0\|^2-\|x\|^2) \\
&\quad +\alpha\kappa\|x-Tx\|^2-\alpha(1-\alpha)\|x_0-Tx\|^2 \\
&= (1-\alpha)\|x_0-u\|^2+\alpha(\|x-u\|^2+\kappa\|x-Tx\|^2) \\
&\quad -\alpha(1-\alpha)\|x_0-Tx\|^2 \\
&\le (1-\alpha)\|x_0-u\|^2+\alpha(\|x-u\|^2+\kappa\|x-Tx\|^2) \\
&\le (1-\alpha)\|x_0-u\|^2+\alpha(\|x-u\|+\sqrt{\kappa}\|x-Tx\|)^2 \\
&\le [\sqrt{1-\alpha}\|x_0-u\|+\sqrt{\alpha}(\|x-u\|+\sqrt{\kappa}\|x-Tx\|)]^2.
\end{aligned} \tag{7.1}$$

It follows that,

$$\|y-u\| \le \sqrt{1-\alpha}\|x_0-u\|+\sqrt{\alpha}\|x-u\|+\sqrt{\alpha\kappa}\|x-Tx\|. \tag{7.2}$$

Besides,

$$\begin{aligned}
\|x-Tx\| &\le \|x-y\|+\|y-Tx\| \\
&\le \|x-u\|+\|y-u\|+(1-\alpha)\|x_0-Tx\|
\end{aligned} \tag{7.3}$$

Substitute (7.2) into (7.3) can yield

$$(1-\sqrt{\alpha\kappa})\|x-Tx\| \le 2\|x-u\|+\sqrt{1-\alpha}(\|x_0-Tx\|+\|x_0-u\|). \tag{7.4}$$

From the assumption of the coefficients, we have

$$\|x-Tx\| \le \frac{2}{1-\sqrt{\alpha\kappa}}\|x-u\|+\frac{\sqrt{1-\alpha}}{1-\sqrt{\alpha\kappa}}(\|x_0-Tx\|+\|x_0-u\|) \tag{7.5}$$

which implies $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. □

Lemma 7.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. $\forall x_0, x \in C$ and $0 \leq \alpha \leq 1$, let*

$$y = (1-\alpha)x_0 + \alpha T x,$$

$$C_x = \{z \in C : \|z-y\|^2 \leq \|z-x\|^2 + (1-\alpha)(\|x_0\|^2 + 2\langle x - x_0, z\rangle)\}$$

and

$${}^*C_x = \{z \in C : \|x - Tx\| \leq 2\|x-z\| + \sqrt{1-\alpha}[4\|x_0\| + \|x_0 - Tx\| + \|x - x_0\| + \|x_0 - z\|]\}$$

Then, C_x is a closed convex subset with $F(T) \subset C_x \subset {}^*C_x$.

Proof. By Lemma 2.3, we see that C_x is closed and convex. For any $p \in F(T)$, we have

$$\begin{aligned} \|y-p\|^2 &= \|(1-\alpha)(x_0 - p) + \alpha(Tx - p)\|^2 \\ &\leq (1-\alpha)\|x_0 - p\|^2 + \alpha\|x-p\|^2 \\ &= \|x-p\|^2 + (1-\alpha)(\|x_0-p\|^2 - \|x-p\|^2) \\ &= \|x-p\|^2 + (1-\alpha)(\|x_0\|^2 + 2\langle x - x_0, p\rangle). \end{aligned}$$

Hence, $F(T) \subset C_x$. Let $u \in C_x$, then $\forall x \in C$

$$\begin{aligned} \|y-u\|^2 &\leq \|x-u\|^2 + (1-\alpha)(\|x_0\|^2 + 2\langle x - x_0, u\rangle) \\ &\leq \|x-u\|^2 + (1-\alpha)(\|x_0\|^2 + \|x - x_0 + u\|^2) \\ &\leq \|x-u\|^2 + (1-\alpha)[\|x_0\| + \|x - x_0 + u\|]^2 \\ &\leq [\|x-u\| + \sqrt{1-\alpha}(2\|x_0\| + \|x+u\|)]^2. \end{aligned} \tag{7.6}$$

From (7.6), we obtain

$$\|y-u\| \leq \|x-u\| + \sqrt{1-\alpha}[2\|x_0\| + \|x+u\|]. \tag{7.7}$$

We also have,

$$\begin{aligned} \|x - Tx\| &\leq \|x-u\| + \|y-u\| + \|y - Tx\| \\ &= \|x-u\| + \|y-u\| + (1-\alpha)\|x_0 - Tx\|. \end{aligned} \tag{7.8}$$

Combining (7.7) and (7.8), we get

$$\begin{aligned} \|x - Tx\| &\leq \|x-u\| + (1-\alpha)\|x_0 - Tx\| + \|y-u\| \\ &\leq \|x-u\| + (1-\alpha)\|x_0 - Tx\| + \|x-u\| \\ &\quad + \sqrt{1-\alpha}[2\|x_0\| + \|x+u\|] \\ &\leq 2\|x-u\| + \sqrt{1-\alpha}[4\|x_0\| + \|x_0 - Tx\| + \|x - x_0\| + \|u - x_0\|]. \end{aligned} \tag{7.9}$$

From (7.9), we can conclude $u \in {}^*C_x$. So, $F(T) \subset C_x \subset {}^*C_x$. □

Theorem 7.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and T a Lipschitz pseudo-contraction from C into itself with the Lipschitz constant $L \geq 1$ and $F(T) \neq \emptyset$. Assume sequence $\{\tau_n\} \subset [\tau, 1]$ with $\tau \in (0,1]$, sequence*

$\{\alpha_n\} \subset [a,b]$ with $a,b \in (0, \frac{1}{L+1})$ and sequence $\{\beta_n\}$ satisfies that $\beta_n \in [0,1]$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = (1-\beta_n)x_0 + \beta_n T x_n \\ v_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_n' = \{z \in C : \tau_n \alpha_n [1-(1+L)\alpha_n] \|x_n - Tx_n\|^2 \le \langle x_n - z, v_n - Tv_n\rangle\} \\ C_n'' = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + 2(1-\beta_n)\langle x_n - x_0, z\rangle + \theta_n\} \\ C_n = C_n' \cap C_n'' \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases} \tag{7.10}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$ and $\theta_n = (1-\beta_n)(\|x_0\|^2 - \|x_n\|^2) + \beta_n \|x_n - Tx_n\|^2 - \beta_n(1-\beta_n)\|x_0 - Tx_n\|^2$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. It is obvious that $F(T) \subset C_n^* \subset C_n \subset C_n'$. Hence, by Theorem 5.1, $x_n \to P_{F(T)}x_0$. $\square$

Theorem 7.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a $\kappa-$strict-pseudo-contraction for some $0 \le \kappa < 1$ such that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfies that $\lim_{n\to\infty} \alpha_n = 1$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = (1-\alpha_n)x_0 + \alpha_n T x_n \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + 2(1-\alpha_n)\langle x_n - x_0, z\rangle + \theta_n\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} \end{cases}$$

where C_n^* is a closed convex set with $F(T) \subset C_n^* \subset C_n$ and $\theta_n = (1-\alpha_n)(\|x_0\|^2 - \|x_n\|^2) + \alpha_n \kappa \|x_n - Tx_n\|^2 - \alpha_n(1-\alpha_n)\|x_0 - Tx_n\|^2$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. From the assumption, we have $\frac{2}{1-\sqrt{\alpha_n \kappa}} \le \frac{2}{1-\sqrt{\kappa}} < \infty$. Easily, we can prove $\|x_0 - Tx_n\| + \|x_0 - x_{n+1}\|$ is bounded. Then, $\lim_{n\to\infty} \frac{\sqrt{1-\alpha_n}}{1-\sqrt{\alpha_n\kappa}}(\|x_0 - Tx_n\| + \|x_0 - x_{n+1}\|) = 0$. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \le \frac{2}{1-\sqrt{\alpha_n\kappa}}\|x_n - z\| + \frac{\sqrt{1-\alpha_n}}{1-\sqrt{\alpha_n\kappa}}(\|x_0 - Tx_n\| + \|x_0 - z\|)\}$, then using Lemma 7.2, $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, by Lemma 2.6 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. $\square$

Theorem 7.3. *Let C be a nonempty closed convex subset of a Hilbert space H and T a $\kappa-$strict pseudo-contraction of C into itself for some $0 \le \kappa < 1$ with $F(T) \neq \emptyset$.*

Suppose $x_0 \in C$ *chosen arbitrarily and* $\{x_n\}$ *is given by*

$$\begin{cases} v_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n = (1-\beta_n)x_0 + \beta_n T x_n \\ C'_n = \{z \in C : \|v_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n(\kappa - (1-\alpha_n))\|x_n - Tx_n\|^2\} \\ C''_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + 2(1-\beta_n)\langle x_n - x_0, z\rangle + \theta_n\} \\ C_n = C'_n \cap C''_n \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases} \tag{7.11}$$

where C_n^* *is a closed convex set with* $F(T) \subset C_n^* \subset C_n$, $\{\alpha_n\}$ *is chosen such that* $0 < \alpha \leq \alpha_n \leq 1$, $\{\beta_n\}$ *is a sequence in* $[0,1]$ *and* $\theta_n = (1-\beta_n)(\|x_0\|^2 - \|x_n\|^2) + \beta_n\kappa\|x_n - Tx_n\|^2 - \beta_n(1-\beta_n)\|x_0 - Tx_n\|^2$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Proof. It is obvious that $F(T) \subset C_n^* \subset C_n \subset C'_n$. Hence, by Theorem 5.3, $x_n \to P_{F(T)}x_0$. $\square$

The following theorem is a deduced result of Theorem 7.3.

Theorem 7.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$. *Suppose that* $\{\alpha_n\}$ *is a real sequence in* $[0,1]$ *satisfies that* $\lim_{n\to\infty} \alpha_n = 1$. *Let a sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = (1-\alpha_n)x_0 + \alpha_n T x_n \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + 2(1-\alpha_n)\langle x_n - x_0, z\rangle + \theta_n\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} \end{cases}$$

where C_n^* *is a closed convex set with* $F(T) \subset C_n^* \subset C_n$ *and* $\theta_n = (1-\alpha_n)(\|x_0\|^2 - \|x_n\|^2) - \alpha_n(1-\alpha_n)\|x_0 - Tx_n\|^2$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Theorem 7.5. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* T *be a nonexpansive mapping of* C *into itself such that* $F(T) \neq \emptyset$. *Assume that* $\{\alpha_n\}$ *is a sequences in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 1$. *Define a sequence* $\{x_n\}$ *in* C *by algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = (1-\alpha_n)x_0 + \alpha_n T x_n \\ C_n = \{z \in C : \|z - y_n\|^2 \leq \|z - x_n\|^2 + (1-\alpha_n)(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0\rangle \geq 0\} \\ x_{n+1} = P_{C_n^* \cap Q_n} x_0 \end{cases}$$

where C_n^* *is a closed convex set with* $F(T) \subset C_n^* \subset C_n$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Proof. Obviously, $4\|x_0\| + \|x_0 - Tx_n\| + \|x_n - x_0\| + \|x_0 - x_{n+1}\|$ is bounded. Then, $\lim_{n\to\infty} \sqrt{1-\alpha_n}[4\|x_0\| + \|x_0 - Tx_n\| + \|x_n - x_0\| + \|x_0 - x_{n+1}\|] = 0$. Let ${}^*C_n = \{z \in C : \|x_n - Tx_n\| \leq 2\|x_n - z\| + \sqrt{1-\alpha_n}[4\|x_0\| + \|x_0 - Tx_n\| + \|x_n - x_0\| + \|x_0 - z\|]\}$, then using Lemma 7.3, $F(T) \subset C_n^* \subset C_n \subset {}^*C_n$. Hence, by Lemma 2.6 and Theorem 3.1, $x_n \to P_{F(T)}x_0$. $\square$

Remark 7.1. In this theorem, let $C_n^* = C_n$, then we obtain an algorithm which is also proposed in [8]. And Theorem 7.4 is also the deduced result of Theorem 7.5.

Remark 7.2. In last three sections, C_n itself is closed and convex. So, setting $C_n^* = C_n$, we can yield normal CQ algorithms.

8. Relations of different algorithms

In [12], Takahashi, Takeuchi and Kubota obtained another strong convergence theorem for nonexpansive mappings, named monotone C method.

Theorem 8.1. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C_0 = C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \leq \|z - x_n\|\} \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases} \tag{8.1}$$

where $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

In [13], Su and Qin got a new hybrid method, named Monotone CQ iteration processes.

Theorem 8.2. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C_0 = C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_0 = \{z \in C : \|z - y_0\| \leq \|z - x_0\|\} \\ Q_0 = C \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|z - y_n\| \leq \|z - x_n\|\} \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle z - x_n, x_n - x_0 \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases} \tag{8.2}$$

where $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Remark 8.1. Theorem 1.1, 8.1 and 8.2 seem different from each other. However, the steps of their proof are more or less the same. So, they may share some properties or may have some relations.

In this section, we give the relations of the following four theorems.

Theorem 8.3. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0 \rangle \geq 0\} \\ x_{n+1} = P_{C_{n,3} \cap Q_n} x_0 \end{cases}$$

where $C_{n,3}$ is a closed convex set with $F(T) \subset C_{n,3} \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \le \alpha_n \le 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Theorem 8.4. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \\ Q_0 = C \\ Q_n = \{z \in Q_{n-1} : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_{n,4} \cap Q_n} x_0 \end{cases}$$

where $C_{n,4}$ is a closed convex set with $F(T) \subset C_{n,4} \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \le \alpha_n \le 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Theorem 8.5. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C_0 = C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_{n+1} = \{z \in C_{n,5} : \|y_n - z\| \le \|x_n - z\|\} \\ x_{n+1} = P_{C_{n+1,5}} x_0 \end{cases}$$

where $C_{n+1,5}$ is a closed convex set with $F(T) \subset C_{n+1,5} \subset C_{n+1}$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \le \alpha_n \le 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Theorem 8.6. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_0 = \{z \in C : \|y_0 - z\| \le \|x_0 - z\|\} \\ Q_0 = C \\ C_n = \{z \in C_{n-1,6} \cap Q_{n-1} : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C_{n-1,6} \cap Q_{n-1} : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_{n,6} \cap Q_n} x_0 \end{cases}$$

where $C_{n,6}$ is a closed convex set with $F(T) \subset C_{n,6} \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \le \alpha_n \le 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proposition 8.7. *Theorem 8.3 TRUE $\Rightarrow$ Theorem 8.4 TRUE $\Rightarrow$ Theorem 8.5 TRUE $\Leftrightarrow$ Theorem 8.6 TRUE, Where Theorem 8.3 TRUE indicates that Theorem 8.3 is valid.*

Proof. Clearly, Theorem 8.3 is valid.

(1) Theorem 8.3 TRUE $\Rightarrow$ Theorem 8.4 TRUE. Obviously, $F(T) \subset C_n \cap (\bigcap_{i=0}^{n-1} Q_i)$. So, there exists a closed convex set $C_{n,8}$ such that $F(T) \subset C_{n,8} \subset C_n \cap (\bigcap_{i=0}^{n-1} Q_i) \subset C_n$. By Theorem 8.3, we obtain the following theorem.

Theorem 8.8. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0 \rangle \geq 0\} \\ x_{n+1} = P_{C_{n,8} \cap (\bigcap_{i=0}^n Q_i)} x_0 \end{cases}$$

where $C_{n,8}$ is a closed convex set with $F(T) \subset C_{n,8} \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Let $C_{n,8} = C_{n,4}$. Then, Theorem 8.8 is equivalent to Theorem 8.4.

(2) Theorem 8.4 TRUE $\Rightarrow$ Theorem 8.5 TRUE. Actually, Theorem 8.5 can be rewritten as follows.

Theorem 8.9. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_0 = \{z \in C : \|y_0 - z\| \leq \|x_0 - z\|\} \\ C_n = \{z \in C_{n-1,9} : \|y_n - z\| \leq \|x_n - z\|\} \\ x_{n+1} = P_{C_{n,9}} x_0 \end{cases}$$

where $C_{n,9}$ is a closed convex set with $F(T) \subset C_{n,9} \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Since $x_n = P_{C_{n-1,9}} x_0$, then, $C_{n-1,9} \subset Q_n = \{z \in C : \langle z - x_n, x_n - x_0 \rangle \geq 0\}$. Together with $C_{n,9} \subset C_{n-1,9}$, we claim that $C_{n,9} \subset (\bigcap_{i=0}^n Q_i)$, i.e., $C_{n,9} \cap (\bigcap_{i=1}^n Q_i) = C_{n,9}$, where $Q_i = \{z \in C : \langle z - x_i, x_i - x_0 \rangle \geq 0\}$. Hence, Theorem 8.9 is equivalent to the following result.

Theorem 8.10. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$ chosen arbitrarily and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_0 = \{z \in C : \|y_0 - z\| \leq \|x_0 - z\|\} \\ C_n = \{z \in C_{n-1,10} : \|y_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C : \langle z - x_n, x_n - x_0 \rangle \geq 0\} \\ x_{n+1} = P_{C_{n,10} \cap (\bigcap_{i=0}^n Q_i)} x_0 \end{cases}$$

where $C_{n,10}$ is a closed convex set with $F(T) \subset C_{n,10} \subset C_n$ and $\{\alpha_n\}$ is chosen such that $0 < \alpha \leq \alpha_n \leq 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

It is easy to observe that Theorem 8.10 is equivalent to the following result.

Theorem 8.11. *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_0 \in C$*

chosen arbitrarily and $\{x_n\}$ *is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ C_0 = \{z \in C : \|y_0 - z\| \le \|x_0 - z\|\} \\ Q_0 = C \\ C_n = \{z \in C_{n-1,11} : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in Q_{n-1} : \langle z - x_n, x_n - x_0\rangle \ge 0\} \\ x_{n+1} = P_{C_{n,11} \cap Q_n} x_0 \end{cases}$$

where $C_{n,11}$ *is a closed convex set with* $F(T) \subset C_{n,11} \subset C_n$ *and* $\{\alpha_n\}$ *is chosen such that* $0 < \alpha \le \alpha_n \le 1$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Clearly, Theorem 8.11 can be deduced by Theorem 8.4. So, Theorem 8.4 TRUE $\Rightarrow$ Theorem 8.5 TRUE.

(3) Theorem 8.5 TRUE $\Leftrightarrow$ Theorem 8.6 TRUE. Obviously, let $C_{n,11} = C_{n,6}$, $C_{n,11} \cap Q_n$ in Theorem 8.11 is equal to $C_{n,6} \cap Q_n$ in Theorem 8.6. Hence, Theorem 8.11 is equivalent to Theorem 8.6.

□

Moreover, if we take $C_{n,5} = C_n$ and $C_{n,6} = C_n$, then, we can conclude that Theorem 8.1 is equivalent to Theorem 8.2.

Remark 8.2. From the proof of the Proposition 8.7, we observe that the proposition is independent of mapping T. If Theorem 8.3, 8.4, 8.5 and 8.6 represent CQ method, monotone Q method, monotone C method and monotone CQ method, respectively, then, we have the following relations:

CQ method TRUE $\Rightarrow$ monotone Q method TRUE $\Rightarrow$ monotone C method TRUE $\Leftrightarrow$ monotone CQ method TRUE.

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